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Abstract

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Subdifferential analysis of convex-like functions

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Preface

The need for studying non-smooth functions, non-smooth optimization problems and sets with a non-smooth boundary has emerge quite natural with the development of modern mathematics. *Operations Research* and *Variational Analysis* are good examples of fields of research that study these objects. Concepts like distance functions, solution-sets, projection-sets, indicator functions, normal cones, tangent cones and subdifferentials are in the heart of the development of this fields, but all of them generally are unavoidably non-smooth. As an example consider the real valued function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := |x| = \max\{x, -x\}.$$

This function f is the maximum of two differentiable functions but it is non-differentiable at $x = 0$. Nonetheless it's graph does not have a unique tangent line at $(0,0)$, but a whole family of tangent lines. The concept of subdifferentials comes quite in handy to characterize this family of tangent lines. In the sense of *Convex Analysis* the *convex subdifferential* $\partial f(x)$ for this function f at x is the set

$$\partial f(x) := \{p \in \mathbb{R} : f(x) + p(y - x) \leq f(y),\ \forall y \in \mathbb{R}\}.$$

One can interpret the latter as the set of all of the gradients of liner functions which are tangent to the point $(x, f(x))$ of the graph of the function f and are always under it. It can be shown that

$$\partial f(x) = \begin{cases} \{-1\}, & x < 0, \\ [-1, 1], & x = 0, \\ \{1\}, & x > 0. \end{cases}$$

and that if $p_0 \in \partial f(0)$, then the vector $(p_0, -1)$ is normal to one of the tangent lines to the graph of the function f at $(0,0)$.

This example is not an artificial one. The maximum of two linear functions appears in some types of *Cutting Stock Problems*, which are widely used in optimization problems coming from the manufacturing industry.

For a Banach space X the convex subdifferential of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$ is the set

$$\partial f(x) := \{p \in X^* : \langle p, y - x\rangle \leq f(y) - f(x),\ \forall y \in X\}.$$

Convex subdifferentials have many properties which resembles classical properties of derivatives. One can say that they are a continuation and generalization of derivatives. For example, one has that if $0 \in \partial f(x)$ then the point x is a minimzer of the function f, the set $\partial f(x)$ is reduced to the singleton $\{f'(x)\}$ when f is differentiable at the point x, sum rules of the forme

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$$

for appropriately chosen function f and g and many more.

Of course there are other subdifferentials except the convex subdifferential. For example the *Dini* subdifferential, the *Clarke* subdifferential and the *Michel-Penot* subdifferential which definitions relay respectively on the corresponding generalized derivatives of *Dini*, *Clarke* and *Michel-Penot*, see [16, Chapter 6]. One can also define a subdifferential axiomatically as an abstract subdifferential, see [50].

Let C be a convex subset of a real valued vector space. The function $f : C \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in C$ and $\lambda \in [0, 1]$.

The function f is said to be strictly convex if the latter inequality is strict for all $x, y \in C$ and $\lambda \in (0, 1)$.

The geometric interpretation of this definition is that for fixed x and y in C the graph of the function f on the segment $[x, y]$ lies below the segment between the point on the graph $(x, f(x))$ and $(y, f(y))$.

Notice that this definition does not rely on derivatives, although there are classical result of necessary and sufficient conditions for a differentiable function to be convex which include derivatives. One example is that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if its derivative f' is monotone increasing.

Another way to characterise a convex function $f : C \to \mathbb{R}$, where the set C is a convex subset of some real valued vector space, is by its epigraph which is the set

$$\operatorname{epi} f := \{(x, r) \in C \times \mathbb{R} : f(x) \leq r\}.$$

One has that f is convex if and only if it's epigraph $\operatorname{epi} f$ is a convex set in $C \times \mathbb{R}$, i.e.

$$\lambda(x_1, r_1) + (1 - \lambda)(x_2, r_2) \in \operatorname{epi} f$$

for all $(x_1, r_1), (x_2, r_2) \in \operatorname{epi} f$ and $\lambda \in [0, 1]$.

This necessary and sufficient conditions gives us an important connection between convex functions and their epigraphs and gives us yet another good geometric intuition.

There is no debate that convex functions play a key role in many optimization problems due to their wide range of convenient properties. For example if a strictly convex functions has a minimizer it is unique. However, finding new results for *convex-like* function is essential. By *convex-like* we mean functions which have properties similar to those of convex functions but are not necessarily convex. One such class of functions is the class of *primal lower-nice functions* which was introduced by Poliquin in 1991, see [41] and has been extensively studied ever since, see [42, 43, 27, 9, 52, 36]. In his work [41] Poliquin shows that *primal lower-nice functions* considered on a finite dimensional space can be fully characterized by their *Clarke* subdifferential. But there is not only one way to define *primal lower-nice functions*. For example Ivanov and Zlateva in [27] prove the equivalence of two different definitions for primal lower-nice functions defined on a β smooth Banach space. Afterwards Ivanov and Zlateva showed in [28] that the proximal subdifferential and the Clarke subdifferential of a *primal lower nice function* defined on a β smooth Banach space coincide. This result suggests that the class of *primal lower nice functions* does not depend on the subdifferential involved in their definition. After these results a very interesting question arises: Is it possible to characterize *primal lower nice functions* without using subdifferentials?

In Chapter 2 we show that *primal lower nice functions* defined on a Hilbert space satisfy the following property: For any $a, b \in \operatorname{dom} f$ such that

$$\sqrt{\|a-b\|^2 + (f(a) - f(b))^2} < 2r$$

and any $\lambda \in [0, 1]$ there is $u \in \operatorname{dom} f \cap B[\lambda a + (1-\lambda)b, \varphi(\lambda)]$ such that either

$$f(u) \le \lambda f(a) + (1-\lambda) f(b),$$

or

$$\lambda f(a) + (1-\lambda) f(b) < f(u) \le \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda),$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

In particular,

$$\inf_{B[\lambda a+(1-\lambda)b, \varphi(\lambda)]} f \le \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda).$$

Note that this property does not use subdifferential in any way. To achieve it we introduce and study a property we call *epi prox-regularity* of an epigraph set which slightly differs from the well-known prox-regularity property of a set. We take this approach because *primal lower nice* functions are strongly related to *prox-regular* sets. Indeed in a Hilbert space a set is prox-regular if and only if it's indicator function is primal lower nice, see [44, Proposition 2.1].

In Chapter 1 we provide a new proof of an intrinsic property of prox-regular sets in Hilbert spaces. The term prox-regularity was given by Poliquin and Rockafellar in [43]. They defined the term for sets and functions and unfold their properties first in $\mathbb{R}^n$. In [44, Theorem 4.1] Poliquin, Rockafellar and Thibault achieved various characterizations of a prox-regular set C defined on a Hilbert space. Note that all of this characterizations use the distance function, the projection set or the proximal normal cone in some way. The characterization which we prove does not use them and is the following:

For any $a, b \in C$ such that $\|a - b\| < 2r$ and any $\lambda \in (0, 1)$ for

$$x_\lambda := \lambda a + (1 - \lambda)b$$

there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}. \tag{1}$$

This characterization is well know, but the methods used in our proof help us to achieve the results in Chapter 2. This is because of the relationship between a prox-regular function and its epigraphs.

In Chapter 3 we give new proof of the Moreau-Rockafellar theorem which states that a proper, lower semicontinuous and convex function on a Banach space is determined up to a constant by its subdifferential. To this end we develop a novel Epsilon Subdifferential Method (ESM) which is similar to the classical Epsilon Subdifferential Method, see e.g. [12, 13].

Although in the classical Epsilon Subdifferential Method the function under consideration is defined on $\mathbb{R}^n$ there is no issue to continue the method to the infinite dimensional case. A key difference between the novel and the classical ESM is that the classical approximates a minimum of the function under consideration by making the epsilon in the method smaller and smaller every time when $0 \in \partial_\varepsilon f(x_{i_0})$, where x_{i_0} has been generated in the previous iteration of the method, while our novel ESM finds an ε-minimum for a fixed in advance positive ε, i.e. it stops when $0 \in \partial_\varepsilon f(x_{i_0})$ and x_{i_0} is the last point found by it. The latter is not a disadvantage of the novel EMS, because finding a ε minimum is more than enough to prove in a new way the famous *Rockafellar-formula*, (see [45],[46] and [29, Theorem 1.2]).

Historically the first complete proof of the famous Moreau-Rockafellar theorem in a Banach space is due to Rockafellar, see [46]. However this proof depends on duality arguments. A simpler proof which does not use any duality was done by Ivanov and Zlateva in [29] which resembles the proof that a monotone function is Riemann integrable (a classical result in Calculus). To this end they prove the *Rockafellar-formula* by using in [29, Lemma 3.3] proved by Ekeland variational principle. Since the proof of *Lemma 3.3* relays on Ekeland variational principle the relationship between x_i and p_i is not clear. One of the main features and merits of the novel ESM is to partially clarify and reveal the relationship between them. This is done without using a Variational principle.

Chapter 1

An intrinsic property of prox-regular sets in Hilbert space

Prox-regularity has been introduced as an important new regularity property in Variational Analysis by Poliquin and Rockafellar in [43]. They defined the concept for functions and sets and developed the subject in $\mathbb{R}^n$. Numerous significant characterizations of prox-regularity of a closed set C in Hilbert space at point $\overline{x} \in C$ were obtained by Poliquin, Rockafellar and Thibault in [44] in terms of the distance function d_C and metric projection mapping P_C, e.g. d_C being continuously differentiable outside of C on a neighbourhood of $\overline{x}$, or P_C being single-valued and norm-to-weak continuous on this same neighbourhood. On a global level, in [44] the authors show that uniformly prox-regular sets are proximally smooth sets provided new insights on them.

Throughout this chapter H stands for a real Hilbert space endowed with the inner product

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R},$$

and with the associated with it norm

$$\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}.$$

The open (resp. closed) ball of H centered at $x \in H$ with radius $t > 0$ is denoted by

$$B(x, t) := \{y \in H : \|y - x\| < t\} \text{ (resp. } B[x, t] := \{y \in H : \|y - x\| \leq t\}).$$

In the particular case of the closed unit ball we use the notation

$$\mathbb{B} := B[0; 1].$$

In the following notations we consider C to be a nonempty subset of H.

The distance function

$$d_C : H \to \mathbb{R}_+,$$

which measures the distance of a point $x \in H$ to the set C is defined as

$$d_C(x) := \inf_{y \in C} \|x - y\|, \quad \text{for all } x \in H.$$

For $\varepsilon \geq 0$ the $\varepsilon-$argmin set of the distance function is defined as

$$\varepsilon\text{-}\operatorname{argmin} d_C(x) := \{y \in C : \|x - y\| \leq d_C(x) + \varepsilon\}.$$

For an extended real $r \in (0, +\infty]$ through the distance function, one defines the (open) r-tube of C as the set

$$T_C(r) := U_C(r) \setminus C,$$

where $U_C(r)$ is the (open) r-enlargement of C

$$U_C(r) := \{x \in H : d_C(x) < r\}.$$

The multi-valued mapping $P_C : H \rightrightarrows H$ which gives the set of all nearest points in C to a point $x \in H$ is defined by

$$P_C(x) := \{y \in C : d_C(x) = \|x - y\|\}, \quad \text{for all } x \in H.$$

The set $P_C(x)$ is called the projection set of the point x to the set C.

Whenever for some $\overline{x} \in H$ the latter set is reduced to a singleton, i.e.

$$P_C(\overline{x}) = \{\overline{y}\},$$

the vector $\overline{y} \in H$ is denoted by $p_C(\overline{x})$.

The proximal normal cone of C at $x \in H$, denoted by $N_C(x)$, is defined as,(see [47])

$$N_C(x) := \{p \in H : \text{ there exist } r > 0 \text{ such that } x \in P_C(x + rp)\}.$$

By convention, $N_C(x') = \varnothing$ for all $x' \notin C$. The elements of the proximal normal cone $N_C(x)$ are called *proximal normals* to the set C at x.

It is easy to see that $p \in N_C(x)$ if and only if there is a real $\sigma > 0$ such that

$$\langle p, x' - x\rangle \leq \sigma\|x' - x\|^2, \quad \text{for all } x' \in C. \tag{1.1}$$

It is key to point out that the $\sigma > 0$ in (1.1) depends on x as well as on p.

The following definition considers such nonempty closed subset of H for which the $\sigma > 0$ in (1.1) stays the same for all proximal normals taken at $x \in C$.

Definition 1.1.2. *Let C be a nonempty closed subset of H and $r \in (0, +\infty]$. One says that C is r-prox-regular (or uniformly prox-regular with constant r) whenever, for every $x \in C$ and $p \in N_C(x) \cap \mathbb{B}$ one has that $x = p_C(x + rp)$, i.e. x is the unique nearest point from $x + rp$ to C.*

1.1 Intrinsic property

This section contains the intrinsic characteristic properties of a r-prox-regular set which we prove in a new way:

Theorem 1.1.1. *Given a real $r > 0$, a non-empty closed set C in a Hilbert space H. The following are equivalent:*

(a) C is r-prox-regular.

(b) For any $a, b \in C$ such that $\|a - b\| < 2r$ and any $\lambda \in (0, 1)$ for

$$x_\lambda := \lambda a + (1 - \lambda)b$$

there exists $u_\lambda \in C$ such that

$$\|x_\lambda - u_\lambda\| \leq r - \sqrt{r^2 - \lambda(1 - \lambda)\|a - b\|^2}. \tag{1.2}$$

(c) For any $a, b \in C$ with $\|a - b\| < 2r$ there is some $z \in C$ such that

$$\left\|\frac{a + b}{2} - z\right\| \leq r - \sqrt{r^2 - \frac{\|a - b\|^2}{4}}. \tag{1.3}$$

Chapter 2

Epigraphical characterization of uniformly lower regular functions

The concept of a primal lower nice function was introduced by Poliquin in [41] where it was proved that Clarke and proximal subdifferentials of a primal lower nice function defined on finite-dimensional space coincide. In particular this means that if the definition of primal lower nice property, see (2.1), is taken with respect to the Clarke subdifferential, this will produce the same class of functions. In [41] Poliquin proved that these functions in $\mathbb{R}^n$ are completely characterized by their Clarke subdifferential. This was the first large class of non-convex lower semicontinuous functions with this property.

The coincidence of proximal and Clarke subdifferentials of a primal lower nice function defined on Hilbert space was proved by Levy, Poliquin and Thibault in [35]. Later Ivanov and Zlateva in [28] showed that Clarke and proximal subdifferential of a primal lower nice function defined on a β smooth Banach space coincide. The result obtained in [28] shows that the class of primal lower nice functions does not depend on what reasonable subdifferential is used in defining the class.

Since the pioneering work of Poliquin [41], primal lower nice functions are studied in a series of publications, see e.g. [42, 43, 27, 9, 52, 36]. These functions are closely related to prox-regular sets, a term due to Poliquin, Rockafellar and Thibault [44]. Indeed, a set in a Hilbert space is prox-regular exactly when its indicator function is primal lower nice, see [44, Proposition 2.1].

The definition of an uniformly prox-regular set in H is well-known, see e.g. [44, 10, 11]. A nonempty closed subset C of H is *uniformly prox-regular* if there is $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}$ one has

$$\langle p, x' - x \rangle \leq \frac{1}{2r} \|x' - x\|^2, \quad \forall x' \in C.$$

It is not difficult to see that it is equivalent to the following

Definition 2.1.1. *A nonempty closed subset C of H is* uniformly prox-regular *if there is $r > 0$ such that for any $x \in C$ and $p \in N_C(x) \cap \mathbb{B}_H$ one has*

$$\langle p, x' - x \rangle \leq \frac{1}{2r} \|x' - x\|^2, \quad \forall x' \in B(x, 2r) \cap C.$$

If a set $C \subset H$ satisfies Definition 2.1.1, we will say that C is r prox-regular (omitting "uniformly" for brevity).

We will consider the space

$$\overline{H} := H \times \mathbb{R}$$

with the norm

$$|||(x, r)||| := \sqrt{\|x\|^2 + r^2},$$

for $(x, r) \in \overline{H}$. Then $(\overline{H}, |||\cdot|||)$ is a Hilbert space.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function. The *domain* of f is the set

$$\operatorname{dom} f := \{x \in H : f(x) \in \mathbb{R}\}$$

and the *epigraph* of f is the set

$$\operatorname{epi} f := \{(x, r) \in \overline{H} : r \geq f(x)\}.$$

The function f is *proper* exactly when $\operatorname{dom} f \neq \varnothing$ and f is lower semicontinuous on H exactly when $\operatorname{epi} f$ is closed in $\overline{H}$.

The *proximal subdifferential* of f at $x \in \operatorname{dom} f$ is defined as the set

$$\partial^p f(x) := \{p \in H | (p, -1) \text{ is a proximal normal to } \operatorname{epi} f \text{ at } \big(x, f(x)\big)\},$$

while $\partial^p f(x) = \varnothing$ for $x \notin \operatorname{dom} f$, see e.g. [11, p. 2216].

A proper lower semicontinuous function $f : H \to \mathbb{R}$ is said to be *uniformly primal lower nice* if there exist $\rho > 0$ and $\theta > 0$ such that for any $t \geq \theta$, any $p \in \partial^p f(x)$ with $\|p\| \leq \rho t$,

$$f(x') \geq f(x) + \langle p, x' - x \rangle - \frac{t}{2} \|x' - x\|^2, \quad \text{for all } x' \in H, \tag{2.1}$$

see [11, p. 2226].

From the very definition, it is clear that if f is uniformly primal lower nice with some positive constants ρ, and θ, then it is so for any $\rho' < \rho$ and $\theta' > \theta$. Hence, taking small ρ, and then $\theta = \rho^{-1}$ one comes to the following equivalent definition: a proper lower semicontinuous function $f : H \to \mathbb{R}$ is uniformly primal lower nice if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, and any $p \in \partial f(x)$ with $\|p\| \leq \rho t$, (2.1) holds. When the latter holds for f for some $\rho > 0$ one says that the function f is ρ primal lower nice (omitting "uniformly" for brevity).

Further we will consider a slightly more general definition for uniform epi lower regularity of a function.

Definition 2.1.2. *A proper lower semicontinuous function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be epi uniformly lower regular if there exists $\rho > 0$ such that for any $t \geq \rho^{-1}$, any $p \in \partial^p f(x)$ with $\|p\| \leq \rho t$, it is true that*

$$\alpha' \geq f(x) + \langle p, x' - x\rangle - \frac{t}{2}\|x' - x\|^2,$$

$$\textit{for all } (x', \alpha') \in B((x, f(x)), 2\rho) \cap \operatorname{epi} f.$$

If a function f satisfies Definition 2.1.2 for some $\rho > 0$, we will say that f is epi ρ lower regular (again omitting "uniformly"). It is clear that any ρ primal lower-nice function is epi ρ lower regular.

A non-empty closed set $C \subset \overline{H}$ will be called an *epigraph set* if $C \equiv \operatorname{epi} f$ for a proper lower semicontinuous function $f : H \to \mathbb{R} \cup \{+\infty\}$.

For an epigraph set in $\overline{H}$ we will introduce the notion of epi uniform prox-regularity which slightly differs from well-known uniform prox-regularity of a set in $\overline{H}$.

Definition 2.1.3. *Let C be an epigraph set in $\overline{H}$. One says that C is uniformly epi prox-regular if there is $r > 0$ such that for any $(x, \alpha) \in C$, and $(q, \eta) \in N_C\big(x, \alpha\big) \cap \mathbb{B}_{\overline{H}}$ one has*

$$\big\langle (q, \eta), (x' - x, \alpha' - \alpha)\big\rangle \leq \frac{1}{2r}\|x' - x\|^2, \tag{2.2}$$

$$\textit{for all } (x', \alpha') \in B((x, \alpha), 2r) \cap C.$$

If an epigraph set C satisfies Definition 2.1.3 for some $r > 0$, we will say that C is epi r prox-regular (omitting "uniformly").

2.1 Relation between epi lower regular functions and their epigraphs

Theorem 2.2.1 and Theorem 2.2.2 give the relation between the epi lower regular functions and their epigraphs.

Theorem 2.2.1. *If $f : H \to \mathbb{R} \cup \{+\infty\}$ is epi ρ lower regular function, then $C \equiv \operatorname{epi} f$ is epi ρ prox-regular set in $\overline{H}$.*

Theorem 2.2.2. *If the epigraph set $C \equiv \operatorname{epi} f$ in $\overline{H}$ is epi r prox-regular, then the corresponding $f : H \to \mathbb{R} \cup \{+\infty\}$ is epi ρ lower regular function for*

$$\rho = \frac{r}{\sqrt{2}}.$$

2.2 Properties of epi uniformly prox-regular sets

Theorem 2.3.1. *Let $C \subset \overline{H}$ be an epi r prox-regular set in $\overline{H}$. Let $(a,\alpha),(b,\beta) \in C$ be such that*

$$|||(a,\alpha)-(b,\beta)||| < 2r.$$

Then for any $\lambda \in [0,1]$ and $(x_\lambda,\gamma_\lambda)$, where

$$x_\lambda := \lambda a + (1-\lambda)b \text{ and } \gamma_\lambda := \lambda\alpha + (1-\lambda)\beta$$

there exists $(u_\lambda,\xi_\lambda) \in C$ such that

$$d_C(x_\lambda,\gamma_\lambda) = |||(x_\lambda,\gamma_\lambda)-(u_\lambda,\xi_\lambda)||| \le \varphi(\lambda), \tag{2.3}$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Theorem 2.3.2. *Let $C \subset \overline{H}$ be an epigraph set. Then the following are equivalent:*
(a) *C is epi r prox-regular;*
(b) *For any $(a,\alpha),(b,\beta) \in C$ such that $|||(a,\alpha)-(b,\beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \le r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2};$$

(c) *For any $(a,\alpha),(b,\beta) \in C$ such that $|||(a,\alpha)-(b,\beta)||| < 2r$, it holds that*

$$d_C\big(\lambda a + (1-\lambda)b, \lambda\alpha + (1-\lambda)\beta\big) \le \frac{1}{2r}\min\{\lambda, 1-\lambda\}\|a-b\|^2.$$

2.3 Epigraphical characterization

Theorem 2.4.1. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. If f is epi r lower regular, then*
(i) *for any (a,α), $(b,\beta) \in \operatorname{epi} f$ such that*

$$|||(a,\alpha)-(b,\beta)||| < 2r$$

and any $\lambda \in [0,1]$ there is $(u_\lambda,\xi_\lambda) \in \operatorname{epi} f$ such that

$$\|u_\lambda - (\lambda a + (1-\lambda)b)\|^2 + |\xi_\lambda - (\lambda\alpha + (1-\lambda)\beta)|^2 \le \varphi^2(\lambda), \tag{2.4}$$

where

$$\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}.$$

Conversely, if (i) *holds, then f is epi ρ lower regular for $\rho = \frac{r}{\sqrt{2}}$.*

Taking $\alpha = f(a)$, $\beta = f(b)$ in Theorem 2.4.1 we obtain

Corollary 2.4.2. *If $f : H \to \mathbb{R} \cup \{+\infty\}$ is ρ primal lower regular function, then for any $a, b \in \operatorname{dom} f$ such that $|||(a, f(a)) - (b, f(b))||| < 2r$ and any $\lambda \in [0,1]$ there is $u \in \operatorname{dom} f \cap B[\lambda a + (1-\lambda)b), \varphi(\lambda)]$ such that either*

$$f(u) \leq \lambda f(a) + (1-\lambda) f(b),$$

or

$$\lambda f(a) + (1-\lambda) f(b) < f(u) \leq \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda),$$

where $\varphi(\lambda) := r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$. In particular,

$$\inf_{B[\lambda a + (1-\lambda)b), \varphi(\lambda)]} f \leq \lambda f(a) + (1-\lambda) f(b) + \varphi(\lambda).$$

Theorem 2.4.1 says that *Convex-like* functions f such that for some $r > 0$ it holds that for any $(a, \alpha), (b, \beta) \in \operatorname{epi} f$ with $||||(a, \alpha) - (b, \beta)|||| < 2r$ and $\lambda \in (0,1)$ there exist $(u_\lambda, \xi_\lambda) \in \operatorname{epi} f$ such that

$$\|u_\lambda - (\lambda a + (1-\lambda)b)\|^2 + |\xi_\lambda - (\lambda\alpha + (1-\lambda)\beta)|^2 \leq \varphi^2(\lambda)$$

with

$$\varphi(\lambda) = r - \sqrt{r^2 - \lambda(1-\lambda)\|a-b\|^2}$$

are exactly the functions which proximal subdifferential for some $r > 0$ has the property

$$\alpha' \geq f(x) + \langle p, x' - x \rangle - \frac{t}{2}\|x' - x\|^2$$

for all $(x', \alpha') \in \mathbb{B}((x, f(x)), 2\rho) \cap \operatorname{epi} f$ where $t \geq r^{-1}$ and $p \in \partial^p f(x)$.

Chapter 3

Epsilon Subdifferential Method and Integrability

The notations used throughout the chapter are standard. $(X, \|\cdot\|)$ denotes a real Banach space, that is, a complete normed space over $\mathbb{R}$. The dual space X^* of X is the Banach space of all continuous linear functionals p from X to $\mathbb{R}$. The natural norm of X^* is again denoted by $\|\cdot\|$. The value of $p \in X^*$ at $x \in X$ is denoted by $\langle p, x\rangle$. Let us recall that for $\varepsilon \geq 0$, the ε-subdifferential of a proper, lower semicontinuous and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial_\varepsilon f(x) := \{p \in X^* \; : \; -\varepsilon + \langle p, y - x\rangle \leq f(y) - f(x), \quad \forall y \in X\},$$

and $\partial_\varepsilon f = \varnothing$ on $X \setminus \operatorname{dom} f$. Of course, for $\varepsilon = 0$, $\partial_0 f(x)$ coincides with the subdifferential of f at x in the sense of Convex Analysis $\partial f(x)$. The *domain* $\operatorname{dom} \partial_\varepsilon f$ consists of all points $x \in X$ such that $\partial_\varepsilon f(x)$ is non-empty. But while $\partial f(x)$ could be empty, for $\varepsilon > 0$, the sets $\partial_\varepsilon f(x)$ are non-empty for any $x \in \operatorname{dom} f$.

The Epsilon Subdifferential Method is well known and widely used for minimizing convex functions, see e.g. [12, 13]. In this chapter we develop a novel Epsilon Subdifferential Method (ESM).

We will outline it here:

ESM applies to a given proper, convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined on a Banach space X, such that

$$0 = f(0) = \min_{x \in X} f(x)$$

with fixed in advance parameters $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$.

Starting at an arbitrary $x_0 \in \operatorname{dom} f$, for $i = 0, 1, \ldots$

- if $0 \in \partial_\varepsilon f(x_i)$, then STOP;

- if $0 \notin \partial_\varepsilon f(x_i)$, for
$$\varphi_{x_i}(K) := \inf_{x \in X} F_{x_i}(K, x),$$
where
$$F_{x_i}(K, x) := f(x) - f(x_i) + \varepsilon + K\|x - x_i\|,$$
find $K_i > 0$ such that $\varphi_i(K_i) = 0$.

  Take any x_{i+1} satisfying
$$0 \leq f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \leq \delta.$$

In the finite dimensional case $\delta = 0$ works and our novel ESM is much more simple, i.e. x_{i+1} is the unique solution to the equation
$$f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| = 0.$$

Returning to the Banach space case if for some $c > 0$ the function f satisfies
$$f(x) \geq c\|x\| \text{ for all } x \in X,$$
the parameter δ is appropriately chosen, and the starting point $x_0 \in \operatorname{dom} \partial f$, then the number of iterations n of our novel ESM can be estimated by
$$n\sqrt{\varepsilon} \leq \text{const}.$$
The proof of this estimate relies on Lemma 3.1.5.

Note that the immediate estimate provided by the classical method is
$$n\varepsilon \leq \text{const}.$$

So, in our case $n\varepsilon$ tends to 0 as ε tends to 0, which allows us to present a new prove of the Moreau-Rockafellar Theorem, see e.g. [45, 46]:

Theorem 3.1.1. *Let X be a Banach space. Let g and h be proper, lower semicontinuous and convex functions from X to $\mathbb{R} \cup \{+\infty\}$. If*
$$\partial g \subset \partial h,$$
then
$$h = g + \text{const}.$$

To this end we use the novel ESM to prove in a different way the following

Theorem 3.1.2 (Rockafellar [45, 46], see also [29] Theorem 1.2)**.** *Let*

$$g : X \to \mathbb{R} \cup \{+\infty\}$$

be a proper, lower semicontinuous and convex function. Let $\bar{x} \in \operatorname{dom} \partial g$ *and* $\bar{p} \in \partial g(\bar{x})$*. Then for all* $x \in X$

$$g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x),$$

where

$$R_{\partial g,(\bar{x},\bar{p})}(x) := \sup \Bigg\{ \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle :$$

$$x_0 = x,\ x_n = \bar{x},\ q_n = \bar{p},\ q_i \in \partial g(x_i), n \in \mathbb{N} \Bigg\}.$$

To estimate the number of iteration of the novel ESM we prove the following result of its own interest.

Lemma 3.1.5. *Let* $n \in \mathbb{N}$ *and* $A > 0$, $B > 0$, $\varepsilon > 0$ *be real numbers. If there exist reals* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *which satisfy the following conditions*

$$a_i > 0 \text{ and } b_i > 0 \text{ for all } i \in \{1, \ldots, n\},$$
$$a_i\, b_i \geq \varepsilon \text{ for all } i \in \{1, \ldots, n\},$$
$$\sum_{i=1}^{n} a_i \leq A, \sum_{i=1}^{n} b_i \leq B,$$

then the inequality

$$n \leq \sqrt{\frac{AB}{\varepsilon}}$$

holds.

3.1 Novel Epsilon Subdifferential Method

We have outlined the method in the beginning of the chapter. In this section we will consider some of its properties. Throughout this section we work with a proper, lower semicontinuous and convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, such that

$$\min_{x \in X} f(x) = f(0) = 0,$$

and fixed $\varepsilon > 0$, and $\varepsilon > \delta > 0$.

The next result describes what happens at one of our Novel ESM iterations.

Lemma 3.2.1. *Let $x_0 \in \operatorname{dom} f$. The function $\varphi_{x_0} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_{x_0}(K) := \inf_{x \in X} F_{x_0}(K, x),$$

where

$$F_{x_0}(K, x) := f(x) - f(x_0) + \varepsilon + K\|x - x_0\|,$$

is strictly monotone increasing and locally Lipschitz on $(0, \infty)$.

Assume in addition that $0 \notin \partial_\varepsilon f(x_0)$. Then

(i) there exists $K_0 > 0$ such that $\varphi_{x_0}(K_0) = 0$;

(ii) for any $x_1 \in X$ such that

$$0 \leq f(x_1) - f(x_0) + \varepsilon + K_0\|x_1 - x_0\| \leq \delta,$$

there is $p_1 \in \partial_\delta f(x_1)$ such that

$$K_0 \geq \|p_1\| - \delta,$$

and,

$$\langle p_1, x_1 - x_0 \rangle \leq f(x_1) - f(x_0) + \varepsilon + \delta.$$

Moreover,

$$K_0 \leq \min\{\|p\| : p \in \partial_\varepsilon f(x_0)\},$$

and if $p_0 \in \partial_\delta f(x_0)$, then

$$\varepsilon \leq (\|p_0\| - \|p_1\|)\|x_1 - x_0\| + \delta \left(2 + \frac{f(x_0)}{K_0}\right).$$

In the context of the ESM, Lemma 3.2.1 ensures the existence of $K_i > 0$. As x_{i+1} can be taken any point of δ-minimum, i.e. such that

$$0 \leq f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \leq \delta.$$

From the lemma we also have the existence of $p_{i+1} \in \partial_\delta f(x_{i+1})$ such that

$$K_i \geq \|p_{i+1}\| - \delta, \; i \geq 0,$$

$$\langle p_{i+1}, x_{i+1} - x_i \rangle \leq f(x_{i+1}) - f(x_i) + \varepsilon + \delta, \; i \geq 0,$$

as well as,

$$\varepsilon \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\| + \delta \left(2 + \frac{f(x_i)}{K_i}\right), \; i \geq 1.$$

The next Lemma shows that our Novel ESM is finite.

Lemma 3.2.2. *The novel ESM ends after a finite number of iterations n such that $n \le \dfrac{f(x_0)}{\varepsilon - \delta} + 1$ at point x_{n-1} of ε-minimum of f.*

Lemma 3.2.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function satisfying $f(x) \ge 2c\|x\|$ for all $x \in X$ and some $c > 0$.*

Applied for f with $\varepsilon > 0$ and $\delta > 0$ such that

$$\delta \le \frac{c}{2}, \quad \delta \le 1, \quad \delta \left(1 + \frac{f(x_0)}{c}\right) \le \frac{\varepsilon}{4},$$

ESM ends after n iterations, and

$$\sum_{i=0}^{n-2} \|x_{i+1} - x_i\| \le \frac{2f(x_0)}{c}.$$

Moreover, for the number of iterations n we have the estimation

$$n \le 2\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c\varepsilon}} + 2, \tag{3.1}$$

where $p_0 \in \partial_\varepsilon f(x_0)$ is arbitrary.

Let us note that p_0 in (3.1) as an arbitrary element in $\partial_\varepsilon f(x_0)$ depends on ε. But when $x_0 \in \operatorname{dom} \partial f$, then p_0 could be taken in $\partial f(x_0)$ and in this case, the estimation (3.1) is of the type $n\sqrt{\varepsilon} \le \text{const}$.

Thanks to Lemma 3.2.3 we were able to prove in a new way Theorem 3.1.2.

Conclusion

Main contributions

In Chapter 1 we give a new proof of an intrinsic characterization of prox-regular sets in Hilbert spaces. Our prove avoids the use of properties of weakly convex sets and relies only on methods of classical analysis.
In Chapter 2 we provide a characterization of uniformly lower regular functions defined on a Hilbert space. To this end we introduce and study a property of epi prox-regularity of the epigraph set which slightly differs from the well known prox-regularity property of a set. This characterization is not a subdifferential one, but uses the properties of the epigraph.
In Chapter 3 we develop a novel variant of the classical epsilon subdifferential method in which the epsilon is fixed. We use our method to give a new prove of *Moreau-Rockafellar theorem* which states that a proper, lower semicontinuous and convex function defined on a Banach space is determined up to a constant by its subdifferential.

Publications related to the thesis

- M. Konstantinov and N. Zlateva, *Epsilon subdifferential method and integrability*, Journal of Convex Analysis **29** (2021), 571–582.
- M. Konstantinov, N. Zlateva, *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Journal of Applied Analysis (2023)(to appear).
- M. Konstantinov, N. Zlateva, *Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*, Journal of Convex Analysis (2023)(to appear).

Approbation

Some of the results contained in the thesis have been presented by the author at the following conferences:

- *Epsilon Subdifferential Method And Integrability*, 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, June 28–July 2, 2021, Borovets, Bulgaria, http://www.math.bas.bg/~bio/WP21/;
- *Direct proofs of intrinsic properties of prox-regular sets in Hilbert spaces*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", 26 March, 2022, Sofia, Bulgaria, https://www.fmi.uni-sofia.bg/bg/proletna-nauchna-sesiya-na-fmi-2022/;
- *Epsilon Subdifferential Method and Integrability*, 10-th International Conference on Numerical Methods and Applications, August 22–26, 2022, Borovets, Bulgaria, http://www.math.bas.bg/~nummeth/nma22/index.html;

Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his PhD supervisor. The results of other scientists were properly cited.

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