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Continuous Dependence on the Initial Functions and Stability Properties in Hyers–Ulam–Rassias Sense for Neutral Fractional Systems with Distributed Delays

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Abstract: We study several stability properties on a finite or infinite interval of inhomogeneous linear neutral fractional systems with distributed delays and Caputo-type derivatives. First, a continuous dependence of the solutions of the corresponding initial problem on the initial functions is established. Then, with the obtained result, we apply our approach based on the integral representation of the solutions instead on some fixed-point theorems and derive sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of the investigated systems. A number of connections between each of the Hyers–Ulam, Hyers–Ulam–Rassias, and finite-time Lyapunov stability and the continuous dependence of the solutions on the initial functions are established. Some results for stability of the corresponding nonlinear perturbed homogeneous fractional linear neutral systems are obtained, too.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; integral representation

MSC: 34A08; 34A12

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1. Introduction

The practical application of models based on fractional differential equations (systems) have shown that these models are very convenient for describing real-world phenomena. For reliable information with the required level of precision concerning fractional calculus and the fractional differential equations, we recommend the remarkable books by Kilbas et al. [1] and Podlubny [2]. For the important practical aspects devoted to the distributed order fractional differential equations and impulsive fractional equations, see Jiao et al. [3] and Stamova and Stamov [4], respectively.

Practically, the most convenient for use are the models of real-world phenomena that have the following properties: a small (in some appropriate sense) perturbation of the input parameters leads to a small (in some appropriate sense) difference in the output results. This means that a predictable process can be physically realized only if it is stable in some appropriate sense [5]. That is why the investigations of the stability problems are an ”evergreen” theme and a lot of articles are devoted to the study of stability problems. Information about works related to different aspects of stability problems for fractional differential equations published before 2011 can be found in the survey of Li and Zhang [6]. For the past decade, a historical overview is given in works [7–9] and the references therein. From the works published in the last few years concerning the Hyers–Ulam and Hyers–Ulam–Rassias stability for fractional differential equations with Caputo-type derivatives, we recommend Refs. [10,11]. The same theme for the delayed fractional equations is considered in Refs. [12–15]. For works devoted to the neutral case, see Refs. [16,17] and for fractional differential equations with Riemann–Liouville derivatives, we refer to [18]. Note that some works explore “neutral” equations that do not include the highest order of derivative for different values of the independent variable, i.e., they are not neutral equations (see, for example, [19]).

In this article, we study the stability properties in the Hyers–Ulam and Hyers–Ulam–Rassias sense on an arbitrary finite or infinite interval for neutral inhomogeneous linear fractional systems with distributed delays and Caputo-type derivatives. The motivation to study such systems with distributed delays is because this type of delay includes as partial cases all types of delays (it follows from the Riesz theorem applied for the Krasovskii functional) and, in this sense, it is most appropriate to establish the common properties of all types of delays. On the other hand, the motivation to study Hyers–Ulam and Hyers–Ulam–Rassias stability for such systems is because these types of stability play an important role in numerical analysis by approximation of the solutions.

The article is organized in the following way: Section 2 includes, as usual, the needed definitions, the problem statement, and some auxiliary results essentially used in our exposition. Section 3 considers the problem of the continuous dependence of the solutions of the initial problem (IP) (formulated in Section 2) on the initial functions. The obtained result allows, via the Weierstrass theorem, the use of polynomials as initial functions and, therefore, to extend the applicability and give a more useful form of the integral representation of the solutions obtained in [7] for the studied linear inhomogeneous system. In Section 4, we introduce new notions of Hyers–Ulam and Hyers–Ulam–Rassias local stability and by applying the results obtained in the previous Section 3, we also establish sufficient conditions for these local stabilities for the studied linear neutral inhomogeneous systems. Moreover, we prove that the Hyers–Ulam type local stability implies finite-time stability of the zero solution for the investigated homogeneous systems. Section 5 is devoted to the Hyers–Ulam and Hyers–Ulam–Rassias stability on an infinite interval of the studied inhomogeneous linear systems. It is proved also that the boundedness of the fundamental matrix of the investigated homogeneous systems is a necessary condition for the Lyapunov stability of the zero solution, as well as that, together with Hyers–Ulam stability, it leads to Lyapunov stability of the zero solution. In Section 6, applying the approach introduced in [18] (to use the integral representation of the solutions instead of some fixed-point theorem), we study the same problems for nonlinear perturbed neutral homogeneous systems and under some natural conditions concerning the nonlinear perturbation term we prove Hyers–Ulam and Hyers–Ulam–Rassias stability of these systems, too. In the case when the nonlinear system possesses a zero solution, it is proved that the Hyers–Ulam stability leads to finite-time Lyapunov stability of the zero solution for the perturbed system. Finally, Section 7 presents some comments and conclusions about the considered problems and the obtained results.

2. Preliminaries and Problem Statement

For clarity and to avoid eventual misunderstandings, we recall the definitions of the Riemann–Liouville (RL) fractional integral and the used fractional derivatives.

Let $a \in \mathbb{R}, \alpha \in (0,1)$ and $g \in L_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ be arbitrary. The left-sided RL fractional integral operator of order α for any $t > a$ is defined by

$$(I^{\alpha}_{a+}g)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g(s)\mathrm{d}s,$$

the left-sided Riemann–Liouville fractional derivative with

$$({}_{RL}D^{\alpha}_{a+}g)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(I^{1-\alpha}_{a+}g)(t)$$

and the corresponding Caputo fractional derivative with

$${}_{C}D^{\alpha}_{a+}g(t) = {}_{RL}D^{\alpha}_{a+}[g(t)-g(a)].$$

For all used formulas and details, we refer to [1].

Consider the nonlinear perturbed neutral linear delayed system in the following general form:

$$D_{a+}^{\alpha}\Big(X(t)+\int\limits_{-h}^{0}[\mathrm{d}_{\theta}V(t,\theta)]X(t+\theta)\Big)=\int\limits_{-h}^{0}[\mathrm{d}_{\theta}U(t,\theta)]X(t+\theta)+\mathbf{F}(t,X_t(\theta)), \tag{1}$$

where $h>0, J=[a,\infty)$, $a\in\mathbb{R}$, $X(t)=col(x_1(t),\dots,x_n(t)):J\times\mathbb{R}^n$ (the notation col means vector column), $\mathbf{F}(t,X_t)=col(f_1(t,X_t),\dots,f_n(t,X_t)):J\times PC([-h,0],\mathbb{R}^n)$, $X_t(\theta)=X(t+\theta)$, $t\in J$, $\theta\in[-h,0]$, $D_{a+}^{\alpha}X(t)=col(D_{a+}^{\alpha}x_1(t),\dots,D_{a+}^{\alpha}x_n(t))$, $D_{a+}^{\alpha}x_k(t)$ denotes the left-sided Caputo fractional derivative, $k\in\langle n\rangle=\{1,2,\dots,n\}$ and $\langle m\rangle_0=\langle m\rangle\bigcup\{0\}$, $U(t,\theta)=\sum\limits_{i\in\langle m\rangle_0}U^i(t,\theta)$, $V(t,\theta)=\sum\limits_{l\in\langle r\rangle}V^l(t,\theta)$, $U^i(t,\theta)=\left\{u_{kj}^i(t,\theta)\right\}_{k,j=1}^{n}:$ $J\times\mathbb{R}\to\mathbb{R}^{n\times n}$ and $V^l(t,\theta)=\left\{v_{kj}^l(t,\theta)\right\}_{k,j=1}^{n}:J\times\mathbb{R}\to\mathbb{R}^{n\times n}$.

The corresponding homogenous linear system of the system (1) (i.e., when $f_k(t,X_t)\equiv 0$, $t\in J$, $k\in\langle n\rangle$) described in detail has the form:

$$\begin{aligned} &D_{a+}^{\alpha}\left(x_k(t)+\sum_{l\in\langle r\rangle}(\sum_{j\in\langle n\rangle}\int\limits_{-h}^{0}x_j(t+\theta)\mathrm{d}_{\theta}v_{kj}^l(t,\theta))\right)\\ &=\sum_{i\in\langle m\rangle_0}(\sum_{j\in\langle n\rangle}\int\limits_{-h}^{0}x_j(t+\theta)\mathrm{d}_{\theta}u_{kj}^i(t,\theta)), \end{aligned} \tag{2}$$

where $k\in\langle n\rangle$, $l\in\langle r\rangle$, $n,r\in\mathbb{N}$.

We will use also the following notations: $J^{-h}=[a-h,\infty)$, $\mathbb{R}_+=(0,\infty)$, $\mathbf{0}\in\mathbb{R}^n$ and $I,\Theta\in\mathbb{R}^{n\times n}$ are the zero vector, the identity and the zero matrices, respectively. For $Y(t,\theta)=\{y_{kj}(t,\theta)\}_{k,j=1}^{n}:J\times\mathbb{R}\to\mathbb{R}^{n\times n}$, $|Y(t,\theta|=\sum\limits_{k,j\in\langle n\rangle}|y_k^j(t,\theta)|$. When $Y(t,\theta)$ for any fixed $t\in J$ has bounded variation in $\theta\in[a,b]$, $[a,b]\subset\mathbb{R}$ be arbitrary then $\mathrm{Var}_{[a,b]}Y(t,\cdot)=\{\mathrm{Var}_{[a,b]}y_k^j(t,\cdot)\}_{k,j=1}^{n}$, $\left|Var_{[a,b]}Y(t,\cdot)\right|=\sum\limits_{k,j=1}^{n}\mathrm{Var}_{[a,b]}y_k^j(t,\cdot)$ and will be denoted $Y(t,\cdot)\in BV_{\theta}^{loc}(J\times\mathbb{R},R^{n\times n})$.

As is standard by $BL_1^{loc}(J,\mathbb{R}^n)$, we denote the real linear space of the locally bounded functions $g\in L_1^{loc}(J,\mathbb{R}^n)$ and by $BC(J,\mathbb{R}^n)$ the real linear space of the bounded functions $g\in C(J,\mathbb{R}^n)$.

Consider the real linear spaces of initial functions $\Phi=col(\phi_1,\dots,\phi_n):[-h,0]\to\mathbb{R}^n$ as follows: piecewise continuous (**PC**); piecewise continuous with bounded variation (**PC***), continuous (**C**) and absolutely continuous (**AC**). All these linear spaces are endowed with the sup-norm $\|\Phi\|=\sum\limits_{k\in\langle n\rangle}\sup\limits_{s\in[-h,0]}|\phi_k(s)|<\infty$ are Banach spaces. S^{Φ} denotes the set of all jump points for any $\Phi\in\mathbf{PC}$ and, in addition, we will assume that they are right continuous at $t\in S^{\Phi}$.

For arbitrary $\Phi\in\mathbf{PC}$ we introduce the following initial condition for the system (1):

$$X(t)=\Phi(t-a),t\in[a-h,a],\ (X_a(\theta)=\Phi(\theta),\theta\in[-h,0]),h\in\mathbb{R}_+. \tag{3}$$

Definition 1. *([20,21]) We say that for the kernels $U^i,V^l:J\times\mathbb{R}\to\mathbb{R}^{n\times n}$ the hypotheses* (**S**) *hold, if the following conditions are fulfilled for any $i\in\langle m\rangle_0$ and $l\in\langle r\rangle$:*

(**S1**) *The functions $(t,\theta)\to U^i(t,\theta)$, $(t,\theta)\to V^l(t,\theta)$ are measurable in $(t,\theta)\in J\times\mathbb{R}$ and normalized so that $U^i(t,\theta)=0$, $V^l(t,\theta)=0$ for $\theta\ge 0$, $U^i(t,\theta)=U^i(t,-\sigma)$ for $\theta\le-\sigma$, $V^l(t,\theta)=V^l(t,-\tau)$ for $\theta\le-\tau$, $\sigma,\tau>0,h\ge\max(\sigma,\tau)$ and $t\in J$.*

(**S2**) *For any fixed* $t \in J$ *the kernels* $U^i(t,\theta)$ *and* $V^l(t,\theta)$ *are left continuous in* θ *on* $(-\sigma,0)$ *and* $(-\tau,0]$, $U^i(t,\cdot) \in BV_\theta^{loc}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$ *and* $\left|\mathrm{Var}_{[-h,0]} U^i(t,\cdot)\right| \in BL_1^{loc}(J,\mathbb{R}_+)$. *The kernels* $V^l(t,\cdot) \in BV_\theta^{loc}(J\times\mathbb{R},\mathbb{R}^{n\times n})$ *uniformly in* $t \in J$, $\left|\mathrm{Var}_{[-h,0]} V^l(t,\cdot)\right| \in BL_1^{loc}(J,\mathbb{R}_+)$ *and are uniformly nonatomic at zero [21] (i.e., for every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for each* $t \in J$ *we have that* $\mathrm{Var}_{[-\sigma,0]} V^l(t,\cdot) < \varepsilon$*).*

(**S3**) *For any fixed* $t \in J$ *the Lebesgue decomposition of the kernels* $U^i(t,\theta)$ *and* $V^l(t,\theta)$ *has the form:*
$U^i(t,\theta) = U_j^i(t,\theta) + U_{ac}^i(t,\theta) + U_s^i(t,\theta),\ V^l(t,\theta) = V_j^l(t,\theta) + V_{ac}^l(t,\theta) + V_s^l(t,\theta)$,
$U_j^i(t,\theta) = \{a_{kj}^i(t)H(\theta+\sigma_{kj}^i(t))\}_{k,j=1}^n,\ A^i(t) = \{a_{kj}^i(t)\}_{k,j=1}^n \in BL_1^{loc}(J,\mathbb{R}^n)$,
$V^l(t,\theta) = \{\overline{a}_{kj}^l(t)H(\theta+\tau_{kj}^l(t))\}_{k,j=1}^n,\ \overline{A}^l(t) = \{\overline{a}_{kj}^l(t)\}_{k,j=1}^n \in C(J_a,\mathbb{R}^n)$,
$\sigma_{kj}^i(t) \in C(J,[0,\sigma]),\ \tau_{kj}^l(t) \in C(J,[0,\tau]),\ \sigma_{kj}^0(t) \equiv 0,\ k,j \in \langle n\rangle$,
$H(t)$ *is the Heaviside function,*
$U_{ac}^i(t,\cdot), V_{ac}^l(t,\theta) \in AC([-h,0],\mathbb{R}^{n\times n})$ *and* $U_s^i(t,\cdot), V_s^l(t,\theta) \in C([-h,0],\mathbb{R}^{n\times n})$.

(**S4**) *The sets* $S_\Phi^i = \{t \in J \mid t - \sigma_{kj}^i(t) \in S_\Phi,\ k,j \in \langle n\rangle\}$, $S_\Phi^l = \{t \in J \mid t - \tau_{kj}^l(t) \in S_\Phi,\ k,j \in \langle n\rangle\}$ *do not have limit points and for any* $t_* \in J$ *the relationships* $\lim_{t\to t_*} \int_{-h}^0 |U^i(t,\theta) - U^i(t_*,\theta)|\mathrm{d}\theta = 0$ *and* $\lim_{t\to t_*}\int_{-h}^0 |V^l(t,\theta) - V^l(t_*,\theta)|\mathrm{d}\theta = 0$ *hold.*

Definition 2. *We say that for arbitrary* $\Phi \in \mathbf{PC}$ *the vector-valued functional* $\mathbf{F} : J \times \mathbf{PC} \to \mathbb{R}^n$ *satisfies the conditions* (**H**) *(modified Caratheodory conditions) in* $J \times \mathbf{PC}$ *if the following conditions hold:*

(**H1**)*For almost all fixed* $t \in J$ *the functional* $(t,\Phi) \to \mathbf{F}(t,\Phi)$ *is continuous in any* $\Phi \in \mathbf{PC}$ *and for any fixed* $\Phi \in \mathbf{PC}$ *we have that* $\mathbf{F}(t,\Phi) \in BL_1^{loc}(J,\mathbb{R}^n)$.

(**H2**)*(Local Lipschitz type condition) There exists a function* $\ell(t) \in BL_1^{loc}(J,\mathbb{R}_+)$ *, such that for any* $(t,\Phi^1),(t,\Phi^2) \in J\times\mathbf{PC}$ *the inequality*

$$|\mathbf{F}(t,\Phi_1) - \mathbf{F}(t,\Phi_2)| \le \ell(t)\|\Phi_1 - \Phi_2\| \tag{4}$$

holds for $t \in J$.

Consider the auxiliary system for any $\Phi \in \mathbf{PC}$ and $t \in J$

$$\begin{aligned} X(t) =& C_{\Phi(0)} + \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] X(t+\theta) \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau)\Big(\int_{-h}^{0} [\mathrm{d}_\theta U(\tau,\theta)]X(\tau+\theta) + \mathbf{F}(\tau, X_\tau^T)\Big)\mathrm{d}\tau \end{aligned} \tag{5}$$

where $C_{\Phi(0)} = \Phi(0) - \int_{-h}^{0}[\mathrm{d}_\theta V^l(a,\theta)]\Phi(\theta)$, $I_{-1}(\Gamma(\alpha)) = \Gamma^{-1}(\alpha)I$, $I_{\alpha-1}(t-a) = (t-a)^{\alpha-1}I$.

Definition 3. *([22]) The vector function* $col\,X(t) = (x_1(t),\ldots,x_n(t))$ *is a solution of IP (1), (3) or IP (5), (3) in* $[a,b](J)$ *if* $X|_{[a,b]} \in C([a,b],\mathbb{R}^n)(X|_J \in C(J,\mathbb{R}^n))$ *and satisfies the system (1), respectively, (5) for all* $t \in [a,b]$ $(t \in J)$ *and the initial condition (3) too.*

Definition 4. *([8]) For any initial function* $\Phi \in \mathbf{PC}$ *the low terminal* a *is called a regular or irregular jump point relative to the kernel* $V(t,\theta)$*, if* $a \in S_\Phi^\tau = \bigcup_{l\in\langle r\rangle} S_\Phi^l$, $\tau_{kj}^l(a) = 0$ *for at least one* $l \in \langle r\rangle$, $k,j \in \langle n\rangle$ *and then there exists a constant* $\varepsilon \in (0,h]$ *(eventually depending on* τ_{kj}^l*), such* *that $t - \tau^l_{kj}(t) \geq a$ for $t \in [a, a+\varepsilon]$ or, respectively, we have that $t - \tau^l_{kj}(t) < a$ and $\Phi(t - \tau^l_{kj}(t))$ is continuous for $t \in (a, a+\varepsilon]$.*

In our exposition below, we need the following results:

Theorem 1. *(Theorem 3 in [8]) Let the following conditions hold:*

1. *Conditions $(\mathbf{S})$ and $(\mathbf{H})$ hold.*
2. *There exists $\gamma > 0$ such that for $\theta \in \mathbb{R}$ and $l \in \langle r \rangle$ the kernels $V^l_j(\cdot,\theta), V^l_{ac}(\cdot,\theta), V^l_s(\cdot,\theta) \in C([a, a+\gamma], \mathbb{R}^{n\times n})$.*
3. *For every initial function $\Phi \in \mathbf{PC}$ with $a \in S^\tau_\Phi$, the low terminal a is at most a regular jump point relative to the kernel $V(t,\theta)$.*

Then, for every initial function $\Phi \in \mathbf{PC}$ the IP (1), (3) has a unique solution in J.

Remark 1. *Please note that if the conditions $(\mathbf{S})$ hold and the condition $(\mathbf{H1})$ holds in $J \times \mathbf{PC}$ then every solution of the IP (1), (3) is a solution of the IP (5), (3) and vice versa (see [8], Lemma 1). Moreover, if $\Phi \in \mathbf{C}$, Condition 3 is unnecessary for the validity of Theorem (1).*

Let $\Phi^l(t,s) = \{\varphi^l_{kj}(t,s)\}^n_{k,j=1} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$, $\Phi^l_j(t,s) = col(\varphi^l_{1j}(t,s), \ldots, \varphi^l_{nj}(t,s))$ for $l = 1,2$ are defined as follows:

$$\bar{\Phi}_1(t,s) = \begin{cases} I, & t = s \\ \Theta, & t < s \text{ or } t,s < a \end{cases}$$

for any fixed $s \in J$ and

$$\bar{\Phi}_2(t,s) = \begin{cases} I, & a-h \leq s \leq t \leq a \\ \Theta, & t < s \text{ or } s,t < a-h \end{cases}$$

for each fixed number $s \in [a-h, a]$.

Consider the following matrix system for arbitrary fixed $s \in J$

$$D^\alpha_{a+} W(t,s) = \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)] W(t+\theta, s), t > s, \tag{6}$$

with one of the following two initial conditions:

$$W(t,s) = \Phi_1(t,s), t \leq s; \tag{7}$$

$$W(t,s) = \Phi_2(t,s), s \in [a-h, a]. \tag{8}$$

Definition 5. *The matrix valued function $t \to C(t,s) = (C^1(t,s), \ldots, C^n(t,s)) = \{c_{kj}(t,s)\}^n_{k,j=1}$ is called a solution of the IP (6), (7) if for any fixed $s \in J$ the matrix valued function $t \to C(t,s)$ fulfills $C(\cdot,s) \in C([s,\infty), \mathbb{R}^{n\times n})$ and satisfies the matrix Equation (6) on $t \in (s,\infty)$, as well as the initial condition (7) too.*

As in the integer case, we call the matrix $C(t,s)$ a fundamental matrix of the system (2).

Definition 6. *The matrix valued function $t \to Q(t,s) = (Q^1(t,s), \ldots, Q^n(t,s)) = \{q_{kj}(t,s)\}^n_{k,j=1}$ is called a solution of the IP (6), (8) for arbitrary fixed $s \in [a-h, a]$, if $Q(\cdot,s) \in Q([s,\infty), \mathbb{R}^{n\times n})$ and satisfies the matrix Equation (6) for $t \in J$, as well as the initial condition (8) too.*

Remark 2. *Please note that according to Lemma 5 and Theorem 4 in [7], each one of both matrix problems— IP (6), (7) for any $s \in J$ and IP (6), (8) for any $s \in [a-h, a]$ —possess a unique solution and $C(\cdot,s) \in BV^{loc}(J, \mathbb{R}^{n\times n})$ for any $s \in J$.*

3. Continuous Dependence of the Solutions on the Initial Functions

The main goal of this section is to study the continuous dependence of the solutions of the IP (1) and (3) on the initial function in the sense of the definition below as well as to obtain some useful technical consequences.

Definition 7. *We say that the unique solution $\overline{X}(t)$ of the IP (1), (3) with initial function $\overline{\Phi} \in \mathbf{C}$ and vector-valued functional $\mathbf{F} : J \times \mathbf{PC} \to \mathbb{R}^n$ which satisfies the conditions $(\mathbf{H})$ depends continuously on the initial function if for any $\varepsilon > 0$ and $b \in \mathbb{R},\ b > a$ there exists $\delta = \delta(\varepsilon, b) \in (0, \varepsilon)$ such that for any $\Phi \in \mathbf{C}$ (or $\Phi \in \mathbf{PC}$) with $\|\Phi - \overline{\Phi}\| < \delta$ we have that $\sup_{t\in[a,b]} \left|X(t) - \overline{X}(t)\right| < \varepsilon$.*

Theorem 2. *Let the following conditions hold:*

1. *Conditions $(\mathbf{S})$ and $(\mathbf{H})$ hold.*
2. *There exists $\gamma > 0$ such that for any fixed $\theta \in \mathbb{R}$ the kernels $V^l \in (C[a, a+\gamma], \mathbb{R}^{n\times n})$ for $l \in \langle r \rangle$.*

Then for every initial function $\overline{\Phi} \in \mathbf{C}$, the IP (1), (3) has a unique solution $\overline{X}(t)$ in J, which depends continuously on the initial function.

Proof. Let $b \in \mathbb{R},\ b > a$ be arbitrary, $\overline{X}(t)$ and $X(t)$ be the solutions (unique) of the IP (1), (3) with initial functions $\overline{\Phi} \in \mathbf{C}$ and $\Phi \in \mathbf{C}$, respectively, and let denote for shortness $Y(t) = X(t) - \overline{X}(t)$. Substituting $\overline{X}(t)$ and $X(t)$ in (5), subtracting both equations we obtain for $t \in J$

$$
\begin{aligned}
Y(t) &= X(t) - \overline{X}(t) = C_{\Phi(0)} - C_{\overline{\Phi}(0)} + \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] Y(t+\theta) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_{-h}^{0} [\mathrm{d}_\theta U(\tau,\theta)] Y(t+\theta) + (\mathbf{F}(\tau, X_\tau) - (\mathbf{F}(\tau, \overline{X}_\tau)) \right) \\
&= \left(\Phi(0) - \overline{\Phi}(0)\right) + \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] (Y(t+\theta) - Y(a+\theta)) \\
&+ \int_{-h}^{0} [\mathrm{d}_\theta (V(t,\theta) - V(a,\theta))] Y(a+\theta) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_{-h}^{0} [\mathrm{d}_\theta U(\tau,\theta)] Y(\tau+\theta) \right) \mathrm{d}\tau \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \big(\mathbf{F}(\tau, X_\tau) - (\mathbf{F}(\tau, \overline{X}(t))\mathrm{d}\tau.
\end{aligned}
\tag{9}
$$

Let $\varepsilon > 0,\ \delta^* \in (0, \varepsilon)$ and $\Phi \in \mathbf{C}$ with $\|\Phi - \overline{\Phi}\| < \delta^*$ be arbitrary. Since the function $Y(t)$ is continuous at $t = a$, $Y_a(\theta) = \Phi(\theta) - \overline{\Phi}(\theta)$ then there exists $t^* > a$ such that $\sup_{t\in[a,t^*]} |Y(t) - Y(a)| < \delta^*$ and hence $\sup_{t+\theta\in[a,t^*]} |Y(t+\theta) - Y(a+\theta)| < \delta^*$. The condition **S** implies that $\sup_{t\in[a,t^*]} \mathrm{Var}_{\theta\in[-h,0]} V(t,\theta) \le V^* < \infty$ and $\sup_{t\in[a,t^*]} \mathrm{Var}_{\theta\in[-h,0]} U(t,\theta) \le U^* < \infty$. Denote $\sup_{t\in[a,b]} |\ell(t)| = L^*$ and then for the first three addends, on the right side of (9), we obtain the estimation

$$\begin{aligned}
&|(\Phi(0)-\overline{\Phi}(0))+\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)](Y(t+\theta)-Y(a+\theta))+\int_{-h}^{0}[\mathrm{d}_\theta(V(t,\theta)-V(a,\theta))]Y(a+\theta)| \\
&\le \|\Phi-\overline{\Phi}\|+\sup_{t+\theta\in[a-h,t^*]}|Y(t+\theta)-Y(a+\theta)|\sup_{t\in[a,t^*]}\mathrm{Var}_{\theta\in[-h,0]}V(t,\theta) \\
&+\|\Phi-\overline{\Phi}\|\sup_{t\in[a,t^*]}\mathrm{Var}_{\theta\in[-h,0]}(V(t,\theta)-V(a,\theta)) \\
&\le \delta^*+\delta^*V^*+2\delta^*V^*=\delta^*(1+3V^*)
\end{aligned} \tag{10}$$

For the fourth and the fifth addends, on the right side of (9), we have that

$$\begin{aligned}
&|I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)]Y(\tau+\theta)\right)\mathrm{d}\tau \\
&+I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)(\mathbf{F}(\tau,X_\tau)-\mathbf{F}(\tau,\overline{X}_\tau))\mathrm{d}\tau| \\
&\le \frac{n^2}{\Gamma(1+\alpha)}(\left|\int_a^t\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)]Y(\tau+\theta)\right)\mathrm{d}(t-\tau)^\alpha\right| \\
&+L^*\int_a^t|X(\tau+\theta)-\overline{X}(\tau+\theta)|\mathrm{d}(t-\tau)^\alpha) \\
&\le \frac{n^2}{\Gamma(1+\alpha)}(\left|\int_a^t\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)](Y(\tau+\theta)-Y(a+\theta)\right)\mathrm{d}(t-\tau)^\alpha\right| \\
&+L^*\int_a^t|Y(\tau+\theta)-Y(a+\theta)|\mathrm{d}(t-\tau)^\alpha) \\
&\le \frac{n^2}{\Gamma(1+\alpha)}(\left|\int_a^t\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)]Y(a+\theta)\right)\mathrm{d}(t-\tau)^\alpha\right| \\
&+L^*\int_a^t|Y(a+\theta)|\mathrm{d}(t-\tau)^\alpha)\le\frac{n^2(t-a)^\alpha}{\Gamma(1+\alpha)}(\delta^*U^*+\delta^*L^*) \\
&+\frac{n^2(t-a)^\alpha}{\Gamma(1+\alpha)}(\delta^*U^*+\delta^*L^*)\le\delta^*\frac{2n^2(b-a)^\alpha(U^*+L^*)}{\Gamma(1+\alpha)}
\end{aligned} \tag{11}$$

Denote $C_*=(1+3V^*+\frac{2n^2(b-a)^\alpha(U^*+L^*)}{\Gamma(1+\alpha)})$, choose $\delta^*<\varepsilon C_*^{-1}$ and from (10) and (11) we obtain that for any $\varepsilon>0$ we have that $|X(t)-\overline{X}(t)|<\varepsilon$ for $t\in[a,t^*]$.

Let assume that $t^*<b,\varepsilon>0$ and $b^*\in(t^*,b]$ are arbitrary. Then, there exist sequences $\{\delta_k\}_{k\in\mathbb{N}}\subset(0,\varepsilon)$ with $\lim_{k\to\infty}\delta_k=0$, $\{\Phi_k\}_{k\in\mathbb{N}}\subset\mathbf{C}$ with $\|\Phi_k-\overline{\Phi}\|<\delta_k$ such that for any solution $X^k(t)$ of the IP (1), (3) (with initial function Φ_k) there exist $t_k\in(t^*,b^*]$ for which the following relationships hold

$$\left|\overline{X}^k(t)-\overline{X}(t)\right|<\varepsilon,t\in[a,t_k);\left|\overline{X}^k(t_k)-\overline{X}(t_k)\right|=\varepsilon,k\in\mathbb{N}. \tag{12}$$

The sequence $\{t_k\}_{k\in\mathbb{N}}\subset(t^*,b^*]$ and hence is bounded. From (12) it follows that the sequence of the functions $\{\overline{X}^k(t)\}_{k\in\mathbb{N}}$ is uniformly bonded and equicontinuous. Then there exist a convergent subsequence $\{t_j\}_{j\in\mathbb{N}}\subset\{t_k\}_{k\in\mathbb{N}}$ with $\lim_{j\to\infty}t_j=\overline{b}\in(t^*,b^*]$

and $\{\overline{X}^j(t)\}_{j\in\mathbb{N}} \subset \{\overline{X}^k(t)\}_{k\in\mathbb{N}}$ with $\lim_{j\to\infty}\overline{X}^j(t) = X^*(t)$ uniformly on any subinterval $[a,\widetilde{t}] \subset [a,\overline{b})$. Since the function $X^*(t)$ is uniformly continuous on $[a,\overline{b})$ then it can be continuously prolonged on $[a,\overline{b}]$ as $\left|X_*(\overline{b}) - \overline{X}(\overline{b})\right| = \left|\overline{X}^j(t_j) - \overline{X}(t_j)\right| = \varepsilon$. Please note that the function $\overline{X}^j(t)$ for any $j \in \mathbb{N}$ and $t \in [a,t_j]$ is a solution of the IP (5), (3) and hence we obtain

$$\begin{aligned}
\overline{X}^j(t) &= C_{\Phi_j(0)} + \int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]\overline{X}^j(t+\theta) \\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)]\overline{X}^j(\tau+\theta)\right)\mathrm{d}\tau \\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\Big(\mathbf{F}(\tau,X^j_\tau) - \mathbf{F}(\tau,X^*_\tau)\Big)\mathrm{d}\tau \\
&\quad + \int_a^t I_{\alpha-1}(t-\tau)(\mathbf{F}(\tau,X^*_\tau)\mathrm{d}\tau, t \in [a,t_j], \\
(\overline{X}^j(t) - X^*(t)) + X^*(t) &= \Phi_j(t-a),\ t \in [a-h,a].
\end{aligned} \tag{13}$$

We will prove that the fourth addend, on the right side of (13), tends to zero as $j \to \infty$. Let $[a,\widetilde{t}] \subset [a,\overline{b})$ be an arbitrary subinterval, $\widetilde{t} > a$. Then for any $[a,\widetilde{t}]$ we have the following estimation

$$\begin{aligned}
&\left|I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)(\mathbf{F}(\tau,X^j_\tau) - \mathbf{F}(\tau,X^*_\tau))\mathrm{d}\tau\right| \\
&\le \frac{n^2L^*}{\Gamma(1+\alpha)}\int_a^t \left|X^j_\tau - X^*_\tau\right|\mathrm{d}(t-\tau)^\alpha \le \frac{n^2L^*(t-a)^\alpha}{\Gamma(1+\alpha)}\sup_{\tau\in[a,t]}\left|X^j_\tau - X^*_\tau\right| \\
&\le \frac{n^2L^*(b-a)^\alpha}{\Gamma(1+\alpha)}\sup_{\tau\in[a,t]}\left|X^j_\tau - X^*_\tau\right|
\end{aligned} \tag{14}$$

and hence the right side of (14) tends to zero as $j \to \infty$, since $\lim_{j\to\infty}\overline{X}^j(t) = X^*(t)$ uniformly on any subinterval $[a,\widetilde{t}] \subset [a,\overline{b})$. Then passing both equations in (13) to limit as $j \to \infty$ we obtain that $X^*(t)$ is a solution of the IP (5), (3), respectively, of the IP (5), (3) in $[a,\widetilde{t}]$ with the same initial function $\overline{\Phi} \in \mathbf{C}$ as $\overline{X}(t)$. Then $X^*(t) \equiv \overline{X}(t)$ on $[a,\widetilde{t}]$, since $\widetilde{t} \in [a,\overline{b})$ is arbitrary then $X^*(t) \equiv \overline{X}(t)$ on $[a,\overline{b})$ and hence we obtain $X^*(\overline{b}) = \overline{X}(\overline{b})$ which is a contradiction. Thus, the case $t^* < b$ is impossible, which completes the proof. □

Remark 3. *Please note that if we study the Lyapunov stability of the zero solution (or some other constant equilibrium) we must introduce the condition* $\mathbf{F}(t,\mathbf{0}) \equiv \mathbf{0}$ *for* $t \in J$ *(which guarantees that the constant zero is an equilibrium). Thus, the linear inhomogeneous case (*$\mathbf{F}(t,X_j) \equiv F(t)$*, when* $F(t) \not\equiv \mathbf{0}, t \in J$*) is excluded.*

Consider the system (1) for $t \in J$ when the second addend in the right side has the form $\mathbf{F}(t,X_t(\theta)) \equiv F(t)$ i.e.,

$$D^\alpha_{a+}\left(X(t) + \int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X(t+\theta)\right) = \int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X(t+\theta) + F(t) \tag{15}$$

and in the equivalent integral form

$$X(t) = C_{\Phi(0)} + \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)]X(t+\theta) + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau)\left(\int_{-h}^{0}[\mathrm{d}_\theta U(\tau,\theta)]X(\tau+\theta) + F(\tau)\right)\mathrm{d}\tau. \tag{16}$$

Remark 4. *It must be noted that the IP* (15), (3) *possess a unique solution in J for any $\Phi \in \mathbf{PC}^*$ under the following weakened assumptions: in Theorem 1 the conditions* $(\mathbf{H})$ *can be replaced with the assumption that $F(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ only, and condition 3 is unnecessary (see Corollary1 in [7]). Moreover, for any initial function $\Phi \in \mathbf{C}$ the unique solution $X(t)$ of the IP* (15), (3) *possess the following integral representation (see Theorem 4 in [9]).:*

$$\begin{aligned} X(t) &= C(t,a)\Phi(0) + \int_a^t C(t,s)\mathrm{d}_s f_X(s), \\ f_X(t) &= C_{\Phi(0)} + \int_{-h}^{a-t} [\mathrm{d}_\theta V(t,\theta)]\Phi(t+\theta-a) \\ &\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\int_{-h}^{a-\tau}[\mathrm{d}_\theta U(\tau,\theta)]\Phi(\tau+\theta-a)\mathrm{d}\tau \\ &\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)F(\tau))\mathrm{d}\tau, \end{aligned} \tag{17}$$

where $I_{-1}(\Gamma(\alpha)) = \Gamma^{-1}(\alpha)I$ and $I_{\alpha-1}((t-a)) = (t-a)^{\alpha-1}I$.

First, we will prove that the unique solution $X(t)$ of IP (1), (3) depends continuously on small changes of the initial function and the inhomogeneous term in the sense of the next definition. The following theorem covers the case when $F(t) \not\equiv \mathbf{0}$, $t \in J$ according to the next definition of continuous dependence, which is an adapted version of Definition 7 for the linear inhomogeneous case.

Definition 8. *We say that the unique solution $\overline{X}(t)$ of the IP* (1), (3) *with initial function $\overline{\Phi} \in \mathbf{C}$ and inhomogeneous term $\mathbf{F}(t,\overline{X}_t) \equiv \overline{F}(t), t \in J, \overline{F}(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ depends continuously on the initial function and the inhomogeneous term $\overline{F}(t)$ if for any $\varepsilon > 0$ and $b \in \mathbb{R},\ b > a$ there exists $\delta = \delta(\varepsilon,b) \in (0,\varepsilon)$ such that for any $\Phi \in \mathbf{C}$ with $\|\Phi - \overline{\Phi}\| < \delta$ and for any $F(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ with $\sup\limits_{t\in[a,b]} |\overline{F}(t) - F(t)| < \delta$ we have that $\sup\limits_{t\in[a,b]} |X(t) - \overline{X}(t)| < \varepsilon$.*

Please note that if the inhomogeneous term has the form $\mathbf{F}(t,\overline{X}_t) \equiv \overline{F}(t), t \in J, \overline{F}(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ then every solution of the IP (15), (3) is a solution of the IP (16), (3) and vice versa (see [8], Lemma 1).

Theorem 3. *Let the following conditions hold:*

1. *Conditions* $(\mathbf{S})$.
2. $F(t) \in BL_1^{loc}(J,\mathbb{R}^n)$.
3. *There exists $\gamma > 0$ such that for any fixed $\theta \in \mathbb{R}$ the kernels $V^l \in C([a,a+\gamma],\mathbb{R}^{n\times n})$ for $l \in \langle r \rangle$.*

Then, for every initial function $\overline{\Phi} \in \mathbf{C}$, the IP (15), (3) *has a unique solution $\overline{X}(t)$ in J, which depends continuously on the initial function and the inhomogeneous term.*

Proof. The proof is very similar to the proof of Theorem 2, and because of this, we will only sketch the differences. As already mentioned in Remark (4) the IP (15), (3) has a unique solution $\overline{X}(t)$ in J for any initial function $\overline{\Phi} \in \mathbf{C}$ by virtue of Theorem 4 in [9]. Then the same way as in the proof of Theorem 2 we obtain instead (13) the following equalities

$$\begin{aligned}
\overline{X}^j(t) &= C_{\Phi_j(0)} + \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)]\overline{X}^j(t+\theta) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau)\left(\int_{-h}^{0} [\mathrm{d}_\theta U(\tau,\theta)]\overline{X}^j(\tau+\theta)\right)\mathrm{d}\tau \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau)(F^j(\tau) - \overline{F}(\tau))\mathrm{d}\tau + \int_a^t I_{\alpha-1}(t-\tau)\overline{F}(\tau))\mathrm{d}\tau, \\
&\left(\overline{X}^j(t) - X^*(t)\right) + X^*(t) = \Phi_j(t-a), t \in [a-h,a],
\end{aligned} \tag{18}$$

where the same notation as above is used: $\lim_{j\to\infty} \overline{X}^j(t) = X^*(t)$ uniformly on any subinterval $[a,\widetilde{t}] \subset [a,\overline{b})$ and the function $\overline{X}^j(t)$ for any $j \in \mathbb{N}$ and $t \in [a,t_j]$ is the unique solution of the IP (5), (3) for initial function $\Phi^j \in \mathbf{C}$ and inhomogeneous term $F^j(t)$, with $\|\Phi^j - \overline{\Phi}\| < \delta_j$, $\sup_{t\in[a,b]} \left|\overline{F}(t) - \overline{F}^j(t)\right| < \delta_j$ and $F^j(t) \in BL_1^{loc}(J,\mathbb{R}^n)$. Then, using (18), the proof can be finished the same way as in Theorem (2). □

The next simple but useful corollary allows, via the Weierstrass theorem, the use of polynomials as initial functions and so the extension of the applicability of the representation (18) and will be used essentially in the next sections.

Corollary 1. *Let the conditions of Theorem 3 hold and $b > a$ be an arbitrary fixed number.*

Then for any $\Phi \in \mathbf{C}$ and $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, b, \Phi) \in (0,\varepsilon)$ and $P_\Phi \in \mathbf{PC}^*$, such that if $\|\Phi - P_\Phi\| < \delta$ then $\left|X(t) - X^{P_\Phi}(t)\right| < \varepsilon$, where $X(t)$ and $X^{P_\Phi}(t)$ are solutions of the IP (2),(3) with initial functions Φ and P_Φ, respectively.

Proof. Let $\Phi \in \mathbf{C}$, $b > a$, $\varepsilon > 0$ be arbitrary and $X(t)$ be the solution of the IP (2),(3) with initial function Φ.

From Theorem 3 it follows that the solution $X_\Phi(t)$ of the IP (2),(3) with initial function Φ depends continuously on Φ and hence there exists $\delta = \delta(\varepsilon, b, \Phi) \in (0,\varepsilon)$, such that for any $\Psi \in \mathbf{C}$ with $\|\Phi - \Psi\| < \delta$ we have that $|X_\Psi(t) - X_\Phi(t)| < \varepsilon$ for $t \in [a,]$ ($X_\Psi(t)$ denote the solution of the IP (2),(3) with initial functions Ψ. Then, according to the Weierstrass theorem, there exists a vector function $P_\Phi(t-a) = col(p_\Phi^1(t), \ldots, p_\Phi^n(t)) : [a-h,a] \to \mathbb{R}^n$, where $p_\Phi^n(t), k \in \langle n\rangle$ are polynomials, such that $\|\Phi - P_\Phi\| < \delta$, $P_\Phi \in \mathbf{PC}^*$. Then we have that $\left|X^{P_\Phi}(t) - X_\Phi(t)\right| < \varepsilon$ for $t \in [a,b]$ where $X^{P_\Phi}(t)$ is the solution of the IP (2),(3) with initial function P_Φ. □

4. Local Hyers–Ulam and Hyers–Ulam–Rassias Stability

In this section, we introduce a new notion of Hyers–Ulam and Hyers–Ulam–Rassias local stability and apply a new approach based on the continuous dependence on the initial function of the solutions of the studied systems considered in the previous section. The continuous dependence will be the main tool in our investigation, and we will clarify the relationship between the continuous dependence on the initial functions and the local Hyers–Ulam stability of the linear systems.

Let $\varepsilon > 0$, $b > a$, $\varphi(t) \in C(J^{-h},\mathbb{R}_+)$ be arbitrary and consider for $t \in [a,b](t \in J)$ the inequalities

$$\left| D^{\alpha}_{a+}\left(Y(t) - \int_{-h}^{0} [\mathrm{d}_{\theta} V(t,\theta)] Y(t+\theta) \right) - \int_{-h}^{0} [\mathrm{d}_{\theta} U(t,\theta)] Y(t+\theta) - \mathbf{F}(t, Y_t) \right| \leq \varepsilon, \tag{19}$$

$$\left| D^{\alpha}_{a+}\left(Y(t) - \int_{-h}^{0} [\mathrm{d}_{\theta} V(t,\theta)] Y(t+\theta) \right) - \int_{-h}^{0} [\mathrm{d}_{\theta} U(t,\theta)] Y(t+\theta) - \mathbf{F}(t, Y_t) \right| \leq \varphi(t). \tag{20}$$

Definition 9. *([18]) The function $Y(t) \in C([a-h,b],\mathbb{R}^n)(C(J^{-h},\mathbb{R}^n))$ is a solution of (19) or (20) in J, if it satisfies the inequality (19), respectively (20) for $t \in [a,b](t \in J)$ with initial function $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ for $t \in [a-h,a]$.*

Definition 10. *([18]) The system (1) is said to be Hyers–Ulam (HU) stable on $t \in [a,b](t \in J)$, if there exists a constant $C > 0$, such that for any $\varepsilon > 0$ and any solution $Y(t) \in C([a-h,b],\mathbb{R}^n)(C(J^{-h},\mathbb{R}^n))$ with initial function $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ of (19), there exists an initial function $\Phi^{\varepsilon}(t) \in \mathbf{C}$ with $\left|\Phi^Y(t-a) - \Phi^{\varepsilon}(t-a)\right| \leq \varepsilon$ for $t \in [a-h,a]$ and a unique solution $X_{\Phi^{\varepsilon}}(t)$ of the IP (1), (3) with initial function $\Phi^{\varepsilon}(t)$, for which solution the inequality*

$$\left| Y(t)|_J - X_{\Phi^{\varepsilon}}(t) \right| \leq C\varepsilon, \tag{21}$$

holds for any $t \in [a,b](t \in J)$.

Definition 11. *([18]) The system (1) is said to be Hyers–Ulam–Rassias (HUR) stable on $t \in [a,b](t \in J)$ with respect to $\varphi(t)$ if there exists a constant $C_{\varphi} > 0$, such that for any solution $Y(t) \in C([a-h,b],\mathbb{R}^n)(C(J^{-h},\mathbb{R}^n))$ with $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ of (20) for which there exists a function $\Phi^{\varphi}(t) \in \mathbf{C}$ with $\left|\Phi^Y(t-a) - \Phi^{\varphi}(t-a)\right| \leq C_{\varphi}\varphi(t-a)$, $t \in [a-h,a]$ and a unique solution $X^F_{\Phi^{\varphi}}(t)$ of the IP (1), (3) with initial function $\Phi^{\varphi}(t)$, for which solution the inequality*

$$\left| Y(t)|_J - X^F_{\Phi^{\varepsilon}}(t) \right| \leq C_{\varphi}\varphi(t), \tag{22}$$

holds for any $t \in [a,b](t \in J)$.

Definition 12. *The system (1) (or (15)) is said to be Hyers–Ulam (HU) locally stable or Hyers–Ulam–Rassias (HUR) locally stable if the system (1) (or (15)) is HU or HUR stable in any finite interval $[a,b]$ where $b > a$ is arbitrary.*

It is clear that Definitions 9–12 are applicable even in the linear case, i.e., when $F(t, X_t(\theta)) \equiv F(t)$.

First, we will study the Hyers–Ulam and Hyers–Ulam–Rassias local stability in the sense of Definition 12. The main tool in our investigation will be the continuous dependence on the initial functions considered in the previous section.

Theorem 4. *Let the conditions of Theorem 3 hold.*

Then, the system (15) is HU locally stable.

Proof. Let $\varepsilon > 0$, $b > a$ be arbitrary and $Y(t) \in C([a-h,b],\mathbb{R}^n)$ be a solution of (19) for $t \in [a,b]$. Denote

$$Z(t) = D^{\alpha}_{a+}\left(Y(t) - \int_{-h}^{0} [\mathrm{d}_{\theta} V(t,\theta)] Y(t+\theta) \right) - \int_{-h}^{0} [\mathrm{d}_{\theta} U(t,\theta)] Y(t+\theta) - F(t),\ t \in [a,b]. \tag{23}$$

Considering IP (15), (3) with initial function $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ and inhomogeneous term $\Phi^Y(t) = F(t) + Z(t)$ and taking into account that from (19) and (23) it follows that $|Z(t)| \le \varepsilon$ for any $t \in [a,b]$, we obtain that $F^Y(t) \in BL_1^{loc}([a,b],\mathbb{R}^n)$. Since all conditions of Theorem 3 hold we conclude that the unique solution $Y(t)|_{[a,b]}$ depends continuously on the initial function and the inhomogeneous term and let $\delta = \delta(\varepsilon, b, \Phi^Y) \in (0,\varepsilon)$ be the number existing according to this theorem. For $t \in [a-h,a]$ define $\Phi^\delta(t-a) = \Phi^Y(t-a) + col\,(\underbrace{\frac{\delta}{2n},\ldots,\frac{\delta}{2n}}_{n})$ and hence $\Phi^\delta(t-a) \in \mathbf{C}$ and $\left|\Phi^Y(t-a) - \Phi^\delta(t-a)\right| < \delta$. Consider the IP (15), (3) with initial function Φ^δ and $F^Y(t)$ as an inhomogeneous term (the same inhomogeneous term as in the previous IP (15), (3)). Then, according to Theorem 3 in [9], there exists a unique solution $X_{\Phi^\delta}(t) \in C([a,b],\mathbb{R}^n)$. Then, by virtue of Theorem 2, we obtain that $\left|Y(t)|_{[a-h,a]} - X_{\Phi^\delta}(t)\right| < \varepsilon$ for any $t \in [a,b]$, which completes the proof. □

Theorem 5. *Let the following conditions be fulfilled:*

1. *The conditions of Theorem 4 hold.*
2. *The function $\varphi(t) \in C([a-h,b],\mathbb{R}_+)$ and the relation $0 < \varphi_a = \inf_{t\in[a-h,a]} \varphi(t) \le \varphi_b = \inf_{t\in[a,b]} \varphi(t)$ holds.*

Then, the system (15) is HUR locally stable with respect to this type $\varphi(t)$.

Proof. The idea of the proof is the same as in the previous theorem, and that is why we will only sketch the proof.

Let $\varepsilon \in (0,\varphi_b)$, $b > a$ be arbitrary and $Y(t) \in C([a-h,b],\mathbb{R}^n)$ be an arbitrary solution of (20) for $t \in [a,b]$. We define $Z(t)$ via (23) and then, from (20) and (23), it follows that $|Z(t)| \le \varphi(t)$ for any $t \in [a,b]$. Therefore, $F^Y(t) \in BL_1^{loc}([a,b],\mathbb{R}^n)$ and consider as above the IP (15), (3) with initial function $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ and inhomogeneous term $F^Y(t) = F(t) + Z(t)$. From Theorem 3, it follows that that the unique solution $Y(t)|_{[a,b]}$ depends continuously on the initial function and the inhomogeneous term and let $\delta = \delta(\varepsilon, b, \varphi, \Phi^Y) \in (0,\varphi_a)$ be the existing number according to this theorem. For $t \in [a-h,a]$ we define $\Phi^\delta(t-a) = \Phi^Y(t-a) + col\,(\underbrace{\frac{\delta}{2n},\ldots,\frac{\delta}{2n}}_{n})$ and hence $\Phi^\delta(t-a) \in \mathbf{C}$ and then $|\Phi^Y(t-a) - \Phi^\delta(t-a)| \le \delta \le \varphi_a \le \varphi(t)$ for any $t \in [a-h,a]$. The IP (15), (3) with initial function Φ^δ and $F^Y(t)$ as inhomogeneous term by virtue of Theorem 3 in [9] possess a unique solution $X_{\Phi^\delta} \in C([a,b],\mathbb{R}^n)$. Then, Theorem 3 implies that for any $t \in [a,b]$ we have $\left|Y(t)|_{[a-h,a]} - X_{\Phi^\delta}(t)\right| < \varepsilon \le \varphi_b \le \varphi(t)$, which completes the proof. □

Remark 5. *The results of Theorems 4 and 5 are new, even in the delayed (not neutral) case. Please note that the standard assumption for HUR stability, even in the case of the compact interval, is that the function $\varphi(t) \in C([a-h,b],\mathbb{R}_+)$ must be non-decreasing. It is clear that all non-decreasing functions $\varphi(t) \in C([a-h,b],\mathbb{R}_+)$ satisfy the relations in condition 2 of Theorem 5 and since the functions $\varphi(t) \equiv \varepsilon$ satisfy the same for any $\varepsilon > 0$ then from HUR local stability it follows HU local stability for (15).*

Theorem 6. *Let the conditions of Theorem 3 hold.*

Then, for the system (15), the following statements are equivalent:

(a) The system (15) is HU locally stable.

(b) For any initial function $\overline{\Phi} \in \mathbf{C}$ and arbitrary $a > b$ the corresponding unique solution $\overline{X}(t) \in C([a,b],\mathbb{R}^n)$ of the IP (15), (3) depends continuously on the initial function.

Proof. The results follow from Theorems 3 and 4. □

Definition 13. *([22,23]) The zero solution of the IP (1), (3) (if it exists) is said to be finite-time stable with respect to $\{a, h, [a, b], \delta, \varepsilon\}$, for $0 < \delta \leq \varepsilon$, $t \in [a, b]$ if and only if the inequality $\|\Phi\| < \delta, \Phi \in \mathbf{C}$ implies that $|X_\Phi(t)| < \varepsilon$ for any $t \in [a, b]$, where $X_\Phi(t)$ is the corresponding unique solution of IP (1), (3).*

The next theorem clarifies for the system (2) the relationship between its HU local stability and the finite-time stability of the zero solution in any interval $[a, b]$, $b > a$.

Theorem 7. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 3 of Theorem 3 hold.*
2. *The IP (2) and (3) is HU locally stable.*

Then, the zero solution of IP (2), (3) is finite-time stable in any interval $[a, b]$, $b > a$.

Proof. Let $\varepsilon > 0$, $b > a$, $\Phi \in \mathbf{C}$ be arbitrary and denote with $X_\Phi(t) \in C([a, b], \mathbb{R}^n)$ the corresponding unique solution of the IP (2), (3). Then, from Condition 2 and Theorem 6, it follows that the zero solution $Z(t) \equiv \mathbf{0}$, $t \in [a-h, b]$ depends continuously on the initial function. Then, there exists $\delta \in (0, \varepsilon)$ such that for any $\Phi \in \mathbf{C}$ with $\|\Phi\| < \delta, t \in [a-h, b]$ for the corresponding solution $X_\Phi(t)$ we have that $|X_\Phi(t) - Z(t)| < \varepsilon$ and hence $|X_\Phi(t)| < \varepsilon$, which completes the proof. □

Remark 6. *The introduced approach (based on the continuous dependence on the initial function) allows the clarification of the relationship between the continuous dependence on the initial function and the HU local stability for the system (15). It is established that for the studied linear systems, the HU local stability and the continuous dependence on the initial function are equivalent when the conditions of Theorem 4 hold. We emphasize that Condition 1 of Theorem 7 guarantees only the existence and the uniqueness of the solution of the IP (15), (3).*

5. Hyers–Ulam and Hyers–Ulam–Rassias Stability on Infinite Intervals

The classical Hyers–Ulam and Hyers–Ulam–Rassias stability of linear systems is studied via an approach introduced in [18]. Our point of view concerning this approach, based on the integral representations of the solutions of the studied systems, is that it is applicable in more cases in comparison with the standard fixed-point approach. Our approach allows the establishment of the existence of the solutions of the IP (1), (3) with arbitrary proof techniques (not only with fixed-point theorems) and then use of the several results devoted to the integral representation of the solutions too.

We will study first the HUR stability of linear systems, and the HU stability will follow as a corollary.

Theorem 8. *Let the following conditions be fulfilled:*

1. *Conditions $(\mathbf{S})$ and condition 2 of Theorem (1) hold.*
2. *The relation $C_\alpha^\infty = \sup_{t\in J} t^\alpha C(t) < \infty$ holds, where $C(t) = \sup_{s\in[a,t]} |C(t, s)|$.*
3. *The function $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ is non-decreasing and $F(t) \in BL_1^{loc}(J, \mathbb{R}_+)$ with $F(t) \not\equiv \mathbf{0}$.*

Then, the system (15) is HUR stable on J with respect to $\varphi(t)$.

Proof. Let $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ be an arbitrary non-decreasing function and $C(t, s)$ be the fundamental matrix of (2). Please note that the columns of the matrix $C(t, s)$ are solutions of the system (2) and hence, they do not depend on the choice of the inhomogeneous term $F(t)$ in system (1).

For any solution $Y(t) \in C(J^{-h}, \mathbb{R}^n)$ with $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ of (20) in J, define for any $t \in [a-h, a]$ the initial function via $\Phi^X(t-a) \equiv \Phi^Y(t-a)$. Then, for $t \in [a-h, a]$ we have that $\left|\Phi^Y(t-a) - \Phi^\varphi(t-a)\right| \equiv 0 < \varphi(t-a)$. We introduce for $t \in J$ the vector function $R(t)$ via

$$R(t) = D^{\alpha}_{a+}\left(Y(t) - \int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]Y(t+\theta)\right) - \int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]Y(t+\theta) - F(t) \tag{24}$$

and hence from (20) it follows that $|R(t)| \leq \varphi(t)$ for $t \in J$. For the IP (15), (3) with inhomogeneous term $F_R(t) = R(t) + F(t)$, $F_R(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ on the right side of (15) and initial function $Y(t)|_{[a-h,a]} = \Phi^Y(t-a)$ denote the corresponding unique solution by $X_R(t) \in C(J,\mathbb{R}^n)$, existing according to Theorem 1. Then, from (20), (23) and Theorem 1, it follows that $X_R(t) \equiv Y(t)|_J$ for $t \in J$ and hence possess the following integral representation (see Theorem 4 in [9])

$$\begin{aligned}
Y(t)|_J &= C(t,a)\Phi^Y(0) + \int_a^t C(t,s)\mathrm{d}_s f_Y(s),\\
f_Y(t) &= C_{\Phi^Y(0)} + \int_{-h}^{a-t}[\mathrm{d}_\theta V(t,\theta)]\Phi^Y(t+\theta-a)\\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\int_{-h}^{a-\tau}[\mathrm{d}_\theta U(\tau,\theta)]\Phi^Y(\tau+\theta-a)\mathrm{d}\tau\\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)F_R(\tau))\mathrm{d}\tau
\end{aligned} \tag{25}$$

for $t \in J$. Analogically, according to Theorems 3 and 4 in [9], we obtain that IP (15), (3) with the inhomogeneous term $F(t)$ and initial function $\Phi^X(t-a)$ has a unique solution $X(t) \in C(J,\mathbb{R}^n)$ which possess the integral representation

$$\begin{aligned}
X(t) &= C(t,a)\Phi^X(0) + \int_a^t C(t,s)\mathrm{d}_s f_X(s),\\
f_X(t) &= C_{\Phi^X(0)} + \int_{-h}^{a-t}[\mathrm{d}_\theta V(t,\theta)]\Phi^X(t+\theta-a)\\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\int_{-h}^{a-\tau}[\mathrm{d}_\theta U(\tau,\theta)]\Phi^X(\tau+\theta-a)\mathrm{d}\tau\\
&\quad + I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)F(\tau))\mathrm{d}\tau.
\end{aligned} \tag{26}$$

From (25) and (26) for $t \in J$, it follows that

$$\begin{aligned}
f^Y(t) - f^X(t) &= I_{-1}(\Gamma(\alpha))\left(\int_a^t I_{\alpha-1}(t-\tau)F_R(\tau)\mathrm{d}\tau - \int_a^t I_{\alpha-1}(t-\tau)F(\tau)\mathrm{d}\tau\right)\\
&= I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)R(\tau)\mathrm{d}\tau
\end{aligned}$$

and hence, using Condition 2 of the theorem, we obtain

$$
\begin{aligned}
|Y(t)|_J - X(t)| &= \left| C(t,a)\left(\Phi^Y(0) - \Phi^X(0)\right) + \int_a^t C(t,s)\mathrm{d}_s(f_Y(s) - f_X(s)) \right| \\
&= \left| \int_a^t C(t,s)\mathrm{d}_s(f_Y(s) - f_X(s)) \right| \\
&= \left| I_{-1}(\Gamma(\alpha)) \int_a^t C(t,s)\mathrm{d}_s \left(\int_a^s I_{\alpha-1}(s-\tau)R(\tau))\mathrm{d}\tau \right) \right| \\
&\le \frac{n^2}{\alpha\Gamma(\alpha)} \int_a^t |C(t,s)|\mathrm{d}_s Var_{\eta\in[a,s]} \left(\int_a^\eta R(\tau)\mathrm{d}(\eta-\tau)^\alpha \right) \\
&\le \frac{n^2\varphi(t)}{\Gamma(1+\alpha)} C(t) Var_{s\in[a,t]} \left(\int_a^s R(\tau)\mathrm{d}(s-\tau)^\alpha \right) \\
&\le \frac{n^2\varphi(t)}{\Gamma(1+\alpha)} C(t)(t-a)^\alpha \le \frac{n^2 C_\alpha^\infty}{\Gamma(1+\alpha)}\varphi(t) \le C_\varphi \varphi(t)
\end{aligned}
\tag{27}
$$

where $C_\phi = \frac{n^2 C_\alpha^\infty}{\Gamma(1+\alpha)}$. Therefore, from (27), it follows that the system (15) is HUR stable. □

Remark 7. *It is clear that the condition $F(t) \not\equiv \mathbf{0}$ allows the avoidance of the obstacles generated from the neutral term, which essentially simplifies the proof of Theorem 8, but excludes the important case of the homogeneous systems.*

The next theorem overcomes this obstacle.

Theorem 9. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem 8 hold.*
2. *$F(t) \equiv \mathbf{0}$ for $t \in J$.*
3. *The function $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ is non-decreasing.*

Then, the system (15) is HUR stable on J with respect to $\varphi(t)$.

Proof. Let, as in Theorem 8, $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ be an arbitrary non-decreasing function, and $C(t,s)$ be the fundamental matrix of (2) and $\varepsilon > 0$ be an arbitrary number. Then, for any solution $Y(t) \in C(J^{-h}, \mathbb{R}^n)$ with $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ of (20) in J, we define $\Phi^X(t-a) \equiv \Phi^Y(t-a) - \frac{1}{2}\varphi(t-a)$ and hence, $\left|\Phi^X(t-a) - \Phi^Y(t-a)\right| = \frac{1}{2}\varphi(t-a) < \varphi(t-a)$ for any $t \in [a-h,a]$. Then, by virtue of Theorems 3 and 4 in [9], we obtain that IP (2), (3) has unique solutions $Y(t)|_J, X(t) \in C(J, \mathbb{R}^n)$ for the initial functions $\Phi^Y(t-a)$ and $\Phi^X(t-a)$, respectively, which have the integral representations (25) and (26), and hence

$$
\begin{aligned}
f_Y(t) - f_X(t) &= C_{\Phi^Y(0)} - C_{\Phi^X(0)} + \int_{-h}^{a-t} [\mathrm{d}_\theta V(t,\theta)]\left(\Phi^Y(t+\theta-a) - \Phi^X(t+\theta-a)\right) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)]\left(\Phi^Y(\tau+\theta-a) - \Phi^X(\tau+\theta-a)\right)\mathrm{d}\tau
\end{aligned}
\tag{28}
$$

Please note that the initial function $\Phi(t-a)$ is defined only in the interval $[-h,0]$ and since $-h \le t+\theta-a \le 0$ for $\theta \in [-h,0]$ then, we have that $t \in [a, a+h]$. For convenience, we can prolong $\Phi(t-a)$ for $t \in \mathbb{R}$ as $\Phi(t-a) = \mathbf{1}_{[a,a+h]}(t)\Phi(t-a)$, where $\mathbf{1}_{[a,a+h]}(t)$ is the indicator function of $[a, a+h]$. These restrictions follow from the conditions for splitting off in the system (16) that part that explicitly depends on the initial data which splitting is used in the proof of the integral representation (17) (see Lemma 1 in [8] for details) Denoting $V_h = \sup_{t\in[a,a+h]} Var_{\theta\in[-h,0]} V(t,\theta)$ and $U_h = \sup_{t\in[a,a+h]} Var_{\theta\in[-h,0]} U(t,\theta)$, from (25), (26) and (28) it follows

$$\begin{aligned} |Y(t)|_J - X(t)| &\le \left|C(t,a)\left(\Phi^Y(0) - \Phi^X(0)\right)\right| + \left|\int_a^t C(t,s)\mathrm{d}_s(f_Y(s) - f_X(s))\right| \\ &= \frac{1}{2}C(t,a)\varphi(0) + \frac{1}{2}\left|\int_a^t C(t,s)\mathrm{d}_s \int_{-h}^{a-s} [\mathrm{d}_\theta V(s,\theta)]\varphi(s+\theta-a)\right| \\ &+ \frac{n^2}{2\alpha\Gamma(\alpha)}\left|\int_a^t C(t,s)\mathrm{d}_s \int_a^s \left(\int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)]\varphi(\tau+\theta-a)\right)\mathrm{d}(s-\tau)^\alpha\right| \end{aligned} \tag{29}$$

The third addend, on the right side of (29), can be estimated as follows:

$$\begin{aligned} &\frac{n^2}{2\alpha\Gamma(\alpha)}\left|\int_a^t C(t,s)\mathrm{d}_s \int_a^s \left(\int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)]\varphi(\tau+\theta-a)\right)\mathrm{d}(s-\tau)^\alpha\right| \\ &\le \frac{n^2}{2\Gamma(1+\alpha)}\left|\int_a^t C(t,s)\mathrm{d}_s \int_a^s \left(\int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)]\varphi(\tau+\theta-a)\right)\mathrm{d}(s-\tau)^\alpha\right| \\ &\le \frac{n^2 C(t)}{2\Gamma(1+\alpha)} Var_{s\in[a,t]}\left(\int_a^s \left(\int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)]\varphi(\tau+\theta-a)\right)\mathrm{d}(s-\tau)^\alpha\right) \\ &\le \varphi(t)\frac{n^2 U_h C(t)(t-a)^\alpha}{2\Gamma(1+\alpha)} \le \varphi(t)\frac{n^2 U_h C_\alpha^\infty}{2\Gamma(1+\alpha)} \end{aligned} \tag{30}$$

For the second addend, on the right side of (29) substituting $\theta = \eta + a - s$ we obtain

$$\begin{aligned} &\frac{1}{2}\left|\int_a^t C(t,s)\mathrm{d}_s \int_{-h}^{a-s} [\mathrm{d}_\theta V(s,\theta)]\varphi(s+\theta-a)\right| \\ &= \frac{1}{2}\left|\int_a^t C(t,s)\mathrm{d}_s \int_{s-(h+a)}^{0} [\mathrm{d}_\theta V(s,\eta+a-s)]\varphi(\eta)\right| \\ &\le \frac{C(t)}{2} Var_{s\in[a,t]} \int_{-h}^{0} [\mathrm{d}_\theta V(s,\eta+a-s)]\varphi(\eta) \\ &\le \frac{V_h C(t)}{2}\varphi(t) \le \frac{V_h C_0^\infty}{2}\varphi(t) \end{aligned} \tag{31}$$

Then, from (29), (30) and (31), it follows that

$$|Y(t)|_J - X(t)| \le \left(\frac{V_h C_0^\infty}{2} + \frac{n^2 U_h C_\alpha^\infty}{2\Gamma(1+\alpha)}\right)\varphi(t) \tag{32}$$

for $t \in J$ which completes the proof. □

Corollary 2. *Let Conditions 1 and 2 of Theorem 8 hold.*
Then, the system (15) is HU stable on J.

Proof. Since from (27) and (32) it follows that the constant

$$C = \max\left(\frac{n^2 C_\alpha^\infty}{\Gamma(1+\alpha)}, \left(\frac{V_h C_0^\infty}{2} + \frac{n^2 U_h C_a^\infty}{2\Gamma(1+\alpha)}\right)\right)$$

does not depend on the choice of the function $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$. Then, for arbitrary $\varepsilon > 0$ choosing the non-decreasing function $\varphi(t) \equiv \varepsilon$ from Theorems 8 and 9, it follows that the system (15) is HU stable on J. □

The next simple but useful theorem clarifies for the system (2) the relationship between the Lyapunov stability of the zero solution and the boundedness of the fundamental matrix of (2).

Theorem 10. *Let the following conditions be fulfilled:*

1. *Condition 1 of Theorem 8 holds.*
2. *The zero solution of (2) is stable in Lyapunov sense for any solutions of IP (2), (3) with initial function $\Phi \in \mathbf{PC}^*$.*

Then, the relations $C_0^\infty = \sup_{t\in J} C(t) < \infty$ and $Q_0^\infty = \sup_{t\in J} Q(t) < \infty, Q(t) = \sup_{s\in[a-h,a]} |Q(t,s)|$ hold.

Proof. According to Condition 2 for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that each solution of IP (2), (3) $X_\Phi(t)$ with arbitrary initial function $\Phi \in \mathbf{PC}^*$ with $\|\Phi\| < \delta$ satisfies the estimation $X_\Phi(t) < \varepsilon$ for $t \in J$. For any $j \in \langle n\rangle$ for the j-th column $Q^j(t,s)$ of $Q(t,s)$, we have that the function $\delta Q^j(t,s)$ is the unique solution of IP (2), (3) with the initial function the j-th column of the function $\delta\Phi^2(t,s)$ (see IP (6),(8)). Then, for any $t \in J$ and $s \in [a-h, a]$, we obtain that $\left|\delta Q^j(t,s)\right| < \varepsilon$ and hence the function $Q^j(t) = \sup_{s\in[a-h,a]} |Q^j(t,s)| \le \frac{\varepsilon}{\delta}$. Thus, we obtain that each column of $Q(t,s)$ is a bounded function and hence $Q_0^\infty = \sup_{t\in J} Q(t) = \sum_{j\in\langle n\rangle} |Q^j(t)| \le \frac{n\varepsilon}{\delta} < \infty$. The proof of the other relation is the same. □

Theorem 11. *Let the following conditions be fulfilled:*

1. *Condition 1 of Theorem 8 and the relation $Q_0^\infty = \sup_{t\in J} Q(t) < \infty$ hold.*
2. *The system (2) is HU stable.*

Then, the zero solution of (2) is stable in Lyapunov sense for all solutions of IP (2), (3) with initial function $\Phi \in \mathbf{C} \cap BV([-h,0], \mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$ be arbitrary and $C > 0$ be the constant in (21) existing since the system (2) is HU stable on J. Denote for any $\Phi \in \mathbf{C} \cap BV([-h,0], \mathbb{R}^n)$, $\overline{\Phi}(s-a) \equiv \Phi(s-a), s \in (a-h, a]$ and $\overline{\Phi}(-h) = \mathbf{0}$. Consider the zero solution $Z(t) \equiv \mathbf{0}$ for $t \in J$ of the IP (2), (3) with initial function $\Phi_Z(t-a) \equiv \mathbf{0}$ for $t \in [a, a+h]$ and then since $Z(t) \equiv \mathbf{0}$ satisfies (19), then there exists $\delta \in (0, \varepsilon)$ and initial function $\Phi^\delta \in \mathbf{C} \cap BV([-h,0], \mathbb{R}^n)$ with $\sup_{t\in[a,a+h]} |\Phi^\delta(t-a) - \Phi_Z(t-a)| = ||\Phi^\delta|| < \delta$, such that for the corresponding solution of IP (2), (3) according to Corollary 1 in [7] from (17) for $t \in J$, we have the following estimation

$$|X_{\Phi^\delta}(t)| = \left|\int_{a-h}^{a} Q(t,s)\mathrm{d}\overline{\Phi}^\delta(s-a)\right| \le C\varepsilon.$$

Without loss of generality, we can assume that $Q_0^\infty, C \ge 1$ and hence for the corresponding solution $X_C(t) = (4CC_0^\infty)^{-1} X_{\Phi^\delta}(t)$ of the initial function $\Phi_C(s) = (4CC_0^\infty)^{-1}\Phi^\delta(s)$ we obtain that

$$|X_C(t)| = \left|(4CC_0^\infty)^{-1}X_{\Phi^\delta}(t)\right| = \left|\int\limits_{a-h}^{a} Q(t,s)\mathrm{d}(4CC_0^\infty)^{-1}\overline{\Phi}^\delta(s-a)\right| \\ \le C\varepsilon(4CC_0^\infty)^{-1} \le \frac{\varepsilon}{4C_0^\infty} \le \frac{\varepsilon}{4}. \tag{33}$$

Taking into account that $||\overline{\Phi}|| \le |\overline{\Phi}(-h)| + Var_{s\in[a-h,a]}\overline{\Phi}(s-a) \le 2||\overline{\Phi}||$ and choose $\delta^* = \frac{\delta}{4CC_0^\infty}$. Then for any $\Phi \in \mathbf{C} \cap BV([-h,0],\mathbb{R}^n)$ with $||\Phi|| < \delta^*$ we have $||\Phi_C - \Phi|| < \delta^*$ and then from (33), it follows that

$$\begin{aligned}|X_\Phi(t) - X_C(t) + X_C(t)| &\le |X_\Phi(t) - X_C(t)| + |X_C(t)| \\ &\le \frac{\varepsilon}{4} + \left|\int\limits_{a-h}^{a} Q(t,s)\mathrm{d}\big(\overline{\Phi}_C(s-a) - \overline{\Phi}(s-a)\big)\right| \\ &\le \frac{\varepsilon}{4} + \int\limits_{a-h}^{a} |Q(t,s)|\mathrm{d}\Big(Var_{\eta\in[a-h,s]}(\overline{\Phi}_C(\eta-a) - \overline{\Phi}(\eta-a))\Big) \\ &\le \frac{\varepsilon}{4} + Q_0^\infty Var_{s\in[a-h,a]}(\overline{\Phi}_C(s-a) - \overline{\Phi}(s-a)) \\ &\le \frac{\varepsilon}{4} + 2\delta^* Q_0^\infty < \frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} < \varepsilon\end{aligned}$$

which completes the proof. □

Remark 8. *We emphasize that the HU and HUR stability in the case of an infinite interval essentially depends on the type of the Functional Solutions Space (FSS), where all solutions of the inequalities* (19) *and* (20) *that we seek belong. The FSS, as a rule, is determined from the type of the derivatives in the studied systems. For example, for equations with first-order derivatives and without delays as FSS is used the linear space of the differentiable functions (more often $AC(J,\mathbb{R}^n)$ or $C^1(J,\mathbb{R}^n)$). In the case of equations with first-order derivatives with delays as FSS the same spaces are used, but in addition, the space of the initial functions must also be specified (mainly $\mathbf{C}, C^1(J,\mathbb{R}^n)$ or $\mathbf{PC}^*$).*

6. Hayers–Ulam and Hayers–Ulam–Rassias Stability of a Nonlinear Perturbed Linear Fractional System

In this section, we provide stability analysis of the nonlinear problem IP (1), (3). Mainly, we study the stability in the HUR sense and its relationship with the finite-time stability in the Lyapunov sense for nonlinear systems in the form of (1), which can be considered to be a nonlinear perturbed homogeneous system (2) with nonlinear perturbing term $\mathbf{F}(t, X_t(\theta))$. Formally, almost all nonlinear systems can be written in this form, which is convenient for clarifying the relationship between the linear case and the impact of the nonlinear perturbation.

As in Section 4, we start our investigations with the case of compact interval $[a,b]$ with arbitrary $b > a$.

Theorem 12. *Let the following conditions be fulfilled:*

1. *The conditions* $(\mathbf{S})$ *and* $(\mathbf{H})$ *hold.*
2. *There exists $\gamma > 0$ such that for fixed $\theta \in \mathbb{R}$ the kernels $V^l \in C([a, a+\gamma], \mathbb{R}^{n\times n})$ for all $l \in \langle r \rangle$.*
3. *The function $\varphi(t) \in C([a-h,b],\mathbb{R}_+)$ and the relation $0 < \varphi_a = \inf\limits_{t\in[a-h,a]} \varphi(t) \le \varphi_b = \inf\limits_{t\in[a,b]} \varphi(t)$ holds.*

Then, the system (1) *is HUR locally stable with respect to this type $\varphi(t)$.*

Proof. Let $b > a$ and $\varepsilon \in (0, \varphi_b)$ be arbitrary. For any solution $Y(t) \in C([a-h,b],\mathbb{R}^n)$ of (20) and for $t \in [a,b]$ define the function $Z(t)$ via the equality

$$Z(t) = D^{\alpha}_{a+}\left(Y(t) - \int_{-h}^{0}[\mathrm{d}_{\theta}V(t,\theta)]Y(t+\theta)\right) - \int_{-h}^{0}[\mathrm{d}_{\theta}U(t,\theta)]Y(t+\theta) - \mathbf{F}(t, Y_t(\theta)) \quad (34)$$

and then from (20) and (34) it follows that for any $t \in [a,b]$ the inequality $|Z(t)| \le \varphi(t)$ holds. Thus, the function $Y(t)|_{[a,b]} \in C([a,b],\mathbb{R}^n)$ is the unique solution of the IP (1), (3) with initial function $\Phi^Y(t-a) = Y(t)|_{[a-h,b]} \in \mathbf{C}$ and inhomogeneous term $F^Y(t) = \mathbf{F}(t, Y_t(\theta)) + Z(t)$. Then, according to Theorem 2, it follows that $Y(t)|_{[a,b]}$ depends continuously on the initial function and the inhomogeneous term and let $\delta = \delta(\varepsilon, b, \varphi, \Phi^Y) \in (0, \varphi_a)$ be the number existing according to this theorem. Let $\Phi^{\delta}(t-a) = \Phi^Y(t-a) + col\,(\underbrace{\frac{\delta}{2n}, \ldots, \frac{\delta}{2n}}_{n})$ and hence for any $t \in [a-h,a]$ we have that $\Phi^{\delta}(t-a) \in \mathbf{C}$ which implies that Condition 3 of Theorem 1 holds. Thus, by virtue of Theorem 1, we obtain that the IP (1), (3), with initial function Φ^{δ} and $F^Y(t) = \mathbf{F}(t, Y_t(\theta)) + Z(t)$ as an inhomogeneous term has a unique solution $X_{\Phi^{\delta}}(t) \in C([a,b],\mathbb{R}^n)$. Since for any $t \in [a,b]$, we have that $\left|\Phi^Y(t-a) - \Phi^{\delta}(t-a)\right| < \delta \le \varphi_a \le \varphi(t)$ then according to Theorem 2, we have that $\left|Y(t)|_{[a-h,a]} - X_{\Phi^{\delta}}(t)\right| < \varepsilon \le \varphi_b \le \varphi(t)$ for any $t \in [a,b]$, which completes the proof. □

Corollary 3. *Let Conditions 1 and 2 of Theorem 12 hold.*

Then, the system (1) is HU locally stable on J.

Proof. Choosing for any $\varepsilon > 0$ the function $\varphi(t) \equiv \varepsilon$ for $t \in [a-h,b]$ which satisfies the condition 3 of Theorem 12 we conclude that the statement of Corollary 3 follows from Theorem 12. □

Theorem 13. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem 12 hold.*
2. $\mathbf{F}(t, \mathbf{0}) \equiv \mathbf{0}$.

Then the system (1) is HU locally stable if and only if for any initial function $\overline{\Phi} \in \mathbf{C}$ and arbitrary $b > a$ the corresponding unique solution $\overline{X}(t) \in C([a,b],\mathbb{R}^n)$ of the IP (1), (3) depends continuously on the initial function.

Proof. The necessity follows from Theorems 2, and the proof is almost the same as of Theorem 14. The proof of the sufficiency is similar to the proof of Theorem 6 and, because of this, will be omitted. □

Theorem 14. *Let the following conditions be fulfilled:*

1. *The conditions of Theorem 13 hold.*
2. *The system (1) is HU locally stable on J.*
3. $\mathbf{F}(t, \mathbf{0}) \equiv \mathbf{0}$.

Then, the zero solution of IP (1), (3) is finite-time stable in any interval $[a,b], b > a$.

The proof is almost the same as that of Theorem 7 and will be omitted.

The next theorem establishes sufficient conditions that guarantee the HUR stability of (1) on J.

Theorem 15. *Let the following conditions be fulfilled:*

1. *The conditions (**S**), (**H**) and condition2 of Theorem 1 hold.*
2. *The relations $C^{\infty}_{\alpha} = \sup_{t \in J} t^{\alpha} C(t) < \infty$ and $L^{\infty} = \sup_{t \in J} \ell(t) < \infty$ hold.*

3. *The function $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ is non-decreasing.*

   Then, the system (1) is HUR stable on J with respect to $\varphi(t)$.

Proof. Let $\varphi(t) \in C(J^{-h}, \mathbb{R}_+)$ be an arbitrary non-decreasing function and $C(t,s)$ be the fundamental matrix of (2).

As in Theorem 8, for any solution $Y(t) \in C(J^{-h}, \mathbb{R}^n)$ in J of (20) with $\Phi^Y(t-a) = Y(t)|_{[a-h,a]} \in \mathbf{C}$ we define for any $t \in [a-h,a]$ the initial function $\Phi^X(t-a) \equiv \Phi^Y(t-a)$ and then $\left|\Phi^Y(t-a) - \Phi^X(t-a)\right| \equiv 0 < \varphi(t-a)$ and for $t \in J$ define the function $Z(t)$ via (34). Then, from (20) and (34) we obtain that the inequality $|Z(t)| \le \varphi(t)$ holds for any $t \in J$. The function $Y^*(t) \equiv Y(t)|_J \in C(j, \mathbb{R}^n)$ is the unique solution of the IP (1), (3) with initial function $Y(t)|_{[a-h,a]} = \Phi^Y(t-a)$ and inhomogeneous term $F^Y(t) = \mathbf{F}(t, Y^*(\theta)) + Z(t)$, where $F^Y(t) \in BL_1^{loc}(J, \mathbb{R}^n)$. According to Theorem 1, the IP (1), (3) possess a unique solution with initial function $\Phi^X(t-a)$ and inhomogeneous term $F^X(t) = \mathbf{F}(t, X^*(\theta))$. By virtue of Theorem 4 in [9], both solutions have the integral representation (17) with functions

$$
\begin{aligned}
f_X(t) &= C_{\Phi(0)} + \int_{-h}^{a-t} [\mathrm{d}_\theta V(t,\theta)] \Phi^X(t+\theta-a) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)] \Phi^X(\tau+\theta-a)\mathrm{d}\tau \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) F^X(\tau)\mathrm{d}\tau, \\
f_Y(t) &= C_{\Phi^Y(0)} + \int_{-h}^{a-t} [\mathrm{d}_\theta V(t,\theta)] \Phi^Y(t+\theta-a) \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \int_{-h}^{a-\tau} [\mathrm{d}_\theta U(\tau,\theta)] \Phi^Y(\tau+\theta-a)\mathrm{d}\tau \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) F^Y(\tau)\mathrm{d}\tau,
\end{aligned}
$$

respectively, and hence, we obtain

$$
\begin{aligned}
f_Y(t) - f_X(t) &= I_{-1}(\Gamma(\alpha)) \left(\int_a^t I_{\alpha-1}(t-\tau) F^Y(\tau)\mathrm{d}\tau - \int_a^t I_{\alpha-1}(t-\tau) F^X(\tau)\mathrm{d}\tau \right) \\
&= I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) Z(\tau)\mathrm{d}\tau \\
&+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau)(\mathbf{F}(t, Y_t^*(\theta)) - \mathbf{F}(t, X_t(\theta)))\mathrm{d}\tau.
\end{aligned} \tag{35}
$$

Then, since $\Phi^Y(\tau+\theta-a) = \Phi^X(\tau+\theta-a)$ for $t \in [a-h,a]$, by (17) and (35) it follows

$$
\begin{aligned}
|Y^*(t)-X(t)| &= \left|\int_a^t C(t,s)\mathrm{d}_s(f_Y(s)-f_X(s))\right| \\
&= \left|I_{-1}(\Gamma(\alpha))\int_a^t C(t,s)\mathrm{d}_s\left(\int_a^s I_{\alpha-1}(s-\tau)Z(\tau)\mathrm{d}\tau\right)\right| \\
&+ \left|I_{-1}(\Gamma(\alpha))\int_a^t C(t,s)\mathrm{d}_s\left(\int_a^s I_{\alpha-1}(s-\tau)(\mathbf{F}(\tau,Y^*_\tau)-\mathbf{F}(\tau,X_\tau))\mathrm{d}\tau\right)\right| \\
&\leq \frac{n^2}{\alpha\Gamma(\alpha)}\int_a^t |C(t,s)|\mathrm{d}_s Var_{\eta\in[a,s]}\left(\int_a^\eta Z(\tau)\mathrm{d}(\eta-\tau)^\alpha\right) \\
&+ \frac{n^2C(t)}{\alpha\Gamma(\alpha)}\int_a^t \sup_{\tau\in[a,s]}|\mathbf{F}(\tau,Y^*_\tau)-\mathbf{F}(\tau,X_\tau)|\mathrm{d}_s\left(Var_{\eta\in[a,s]}\int_a^\eta \mathrm{d}(\eta-\tau)^\alpha\right) \\
&\leq \frac{n^2\varphi(t)}{\Gamma(1+\alpha)}C(t)Var_{s\in[a,t]}\left(\int_a^s \mathrm{d}(s-\tau)^\alpha\right) \\
&+ \frac{n^2L^\infty}{\Gamma(\alpha)}C(t)\int_a^t (t-s)^{\alpha-1}\sup_{\tau\in[a,s]}|Y^*(\tau)-X(\tau)|\mathrm{d}s \\
&\leq \frac{n^2\varphi(t)}{\Gamma(1+\alpha)}C(t)(t-a)^\alpha + \frac{n^2L^\infty}{\Gamma(\alpha)}C(t)\int_a^t (t-s)^{\alpha-1}\sup_{\tau\in[a,s]}|Y^*(\tau)-X(\tau)|\mathrm{d}s.
\end{aligned} \tag{36}
$$

From (36), for $t\in J$ it follows the estimation

$$
\sup_{\tau\in[a,t]}|Y^*(\tau)-X(\tau)| \leq c\varphi(t)+gC(t)\int_a^t (t-s)^{\alpha-1}\sup_{\tau\in[a,s]}|Y^*(\tau)-X(\tau)|\mathrm{d}s \tag{37}
$$

where $c=\frac{n^2C^\infty_\alpha}{\Gamma(1+\alpha)}$ and $g=\frac{n^2L^\infty}{\Gamma(1+\alpha)}$.

Then, applying Corollary 2 in [24] to (37), we obtain the estimation

$$
|Y^*(t)-X(t)| \leq c\varphi(E_\alpha(g\Gamma(\alpha)C(t)t^\alpha)) \leq c\varphi E_\alpha(n^2L^\infty C^\infty_\alpha) \leq \varphi(t)C_\varphi,
$$

where $C_\varphi = aE_\alpha(n^2L^\infty C^\infty_\alpha)$ and $E_\alpha(z)=\sum\limits_{k\in\mathbb{R}}\frac{z^k}{(\alpha k+1)}$ is the one parameter Mittag–Leffler function. □

Corollary 4. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem* (15) *hold.*
2. *The relations* $C^\infty_\alpha = \sup\limits_{t\in J} t^\alpha C(t) < \infty$ *and* $L^\infty = \sup\limits_{t\in J}\ell(t) < \infty$ *hold.*

Then the system (1) *is HU stable on* J.

The proof is almost the same as of Corollary 3 and will be omitted.

7. Conclusions and Comments

This article is devoted to the study of Hyers–Ulam and Hyers–Ulam–Rassias stability for neutral inhomogeneous linear fractional systems with Caputo-type derivatives and distributed delays in both cases—on compact interval and on the half-axis of the type $[a,\infty)$ for arbitrary $a\in\mathbb{R}$.

First, we established for the linear case, that the conditions which guarantee the existence and the uniqueness of the solution of the studied IP (1), (3) also lead to the continuous dependence of the solution on the initial function and the inhomogeneous term. The proved Corollary 1 allowed in the case of compact interval for all considerations to use, without loss of generality, an initial function that is continuously differentiable instead of a continuous initial function.

Then, we introduced the notion of Hyers–Ulam local stability of the half-axis $[a, \infty)$, $a \in \mathbb{R}$ and established that on any compact subinterval $[a, b], b > a$ of it, the Hyers–Ulam stability is equivalent to the continuous dependence on the initial functions. Furthermore, we obtained that Hyers–Ulam local stability implies finite-time stability on these subintervals of the half-axis.

For the infinite case, a new approach was used as proposed by the co-authors in their former work [18], which is based on the integral representation of the solutions to the initial problem for the linear fractional systems. From our point of view, the applied approach, in comparison with the standard fixed-point approach, allows the obtaining of better sufficient conditions for stability in Hyers–Ulam and Hyers–Ulam–Rassias sense for the studied inhomogeneous delayed systems. The main advantage of the proposed approach is that establishing an integral representation of the studied system (mainly linear) is one very popular task, and we have a good chance to find the needed representation in some work from other authors or make an appropriate modification of one existing integral representation. Moreover, this task is significantly based on the existence of a fundamental matrix (mainly in the linear case), which is a more standard problem and can be more easily solved. Of course, finding a suitable integral representation can be also seen as a limitation of this approach.

Furthermore, we established that the boundedness of the fundamental matrix of the investigated homogeneous systems is a necessary condition for the Lyapunov stability of the zero solution, as well as that, together with HU stability, it leads to Lyapunov stability for this system.

Using the same approach as in the linear case, we presented some stability results for a corresponding nonlinear perturbed neutral homogeneous system. Under some natural conditions concerning the nonlinear perturbation term, we proved the Hyers–Ulam and Hyers–Ulam–Rassias stability of these systems. In the case where $\mathbf{F}(t, \mathbf{0}) \equiv \mathbf{0}$ for $t \in J$, i.e., the nonlinear system possesses a zero solution, it was proved that, as in the linear case, the Hyers–Ulam local stability (on any compact subinterval $[a, b], b > a$) implies finite-time stability on these subintervals of the half-axis $[a, \infty)$.

We emphasize that the conclusions concerning the necessity of part of the used sufficient conditions are still true in the nonlinear perturbed case, too.

Regarding some possibilities for practical applications, we can mention that the systems studied in our article are a generalization of the ones used in the control-theory models of closed-feedback systems with proportional plus derivative regulator (PD regulator), which are described either by first-order retarded or neutral differential systems. These systems are also a generalization of the systems used in the model of coexistence of competitive micro-organisms, which describes competing micro-organisms surviving on a single nutrient with delays in birth and death processes. For more details, see the book [5].

Some ideas for future works are to study the same neutral systems for different types of fractional derivatives, e.g., Riemann–Liouville, Caputo–Fabrizio, Atangana–Baleanu, or others, and to compare the obtained results.

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