

REVIEW

under the procedure for acquisition of the educational and scientific degree “Doctor”
by candidate Hristina Yordanova Belcheva,
of the PhD Thesis entitled: “Applications of the Variational Analysis Methods”, In the
Scientific field: 4. Natural Sciences, Mathematics and Informatics Professional field: 4.5.
Mathematics Doctoral program (Operations Research),
Department „ Probability, Operations Research and Statistics”,
Faculty of Mathematics and Informatics (FMI),
Sofia University “St. Kliment Ohridski” (SU),

The review has been prepared by:
Prof. Boyan Georgiev Zlatanov, Dr.Sci,
University of Plovdiv Paisiy Hillendarski,
Faculty of Mathematics and Informatics,
Department of Mathematical Analysis.

I was appointed as a member of the jury by the Rector of SU by official order № ПД-38-685/5.11.2025. At the first meeting of the jury, I was elected to present the review.

I will begin with a description of the documents submitted for participation in the procedure. List of Submitted Documents for the PhD Procedure:

1. Application for admission to the PhD defence procedure
2. PhD thesis
3. Abstract of the PhD thesis (in Bulgarian)
4. Abstract of the PhD thesis (in English)
5. Declaration under the regulations for obtaining the educational and scientific degree “Doctor”
6. Compliance report with the Minimal National Requirements
7. Report on readiness for defence
8. Similarity / plagiarism-detection report
9. Protocol confirming originality from the anti-plagiarism system
10. Supervisor’s statement on the dissertation
11. List of published or accepted scientific papers related to the dissertation
12. Report on the extension of the departmental committee
13. Extract from the Order for the Pre-defence
14. Order for enrolment in the PhD programme
15. Certificate of successfully passed exams from the individual study plan
16. Order for temporary termination (“detachment”) of PhD student status
17. Internal review of the dissertation: Protocol of the Departmental Council No. 7 (06.10.2025); Protocol from electronic voting (9–13 October 2025)
18. Diploma for higher education — Master’s degree
19. Diploma for higher education — Bachelor’s degree
20. Curriculum Vitae (Europass format)
21. Professional profile card
22. Certificate of identity of names (name equivalence document)
23. Extract from the Order for the pre-defence appointment

1. General characteristics of the dissertation thesis and the presented materials

The PhD Thesis, entitled “Applications of the Variational Analysis Methods” with an author Hristina Belcheva, consists of a Preface, Notations and Preliminaries, three chapters presenting original results, a Summary, Contributions, a list of related publications, and a Bibliography with 40 items. The full volume of the work is 62 pages of clearly structured scientific content. The topic is modern and relevant to current development in Variational Analysis and Optimization Theory.

2. Short CV and personal impressions of the candidate

Hristina Yordanova Belcheva was born on 18 September 1995 in Bulgaria and has completed her entire education in highly competitive mathematical institutions. During the course of her PhD studies, Hristina Belcheva changed her surname as a result of marriage; she appears as an author in the submitted publications and in the enrollment order under the surname Topalova. A document certifying the identity of the same person under different names has also been submitted. She graduated from the Sofia High School of Mathematics “Paisii Hilendarski” (2009–2014) and subsequently obtained a Bachelor’s degree in Applied Mathematics (2014–2018) and a Master’s degree in Computational Mathematics and Mathematical Modelling (2018–2020), both from Sofia University “St. Kliment Ohridski” (SU). Since February 2022 she has been a full-time PhD student in the doctoral program “Operations Research” at the Faculty of Mathematics and Informatics of SU. She has demonstrated strong research motivation, professional engagement in several scientific projects, and active participation in international scientific events, which reflects a high level of academic maturity and independence in conducting research.

I know the candidate personally from her presentations at scientific seminars and conferences, as well as from informal research discussions. I have very positive impressions of her presentation skills, scientific interests, and strong motivation to develop further in the field and continuously acquire new knowledge. She demonstrates independence, professional responsibility, and genuine enthusiasm for advanced mathematical research.

3. Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

The PhD presents a coherent and original contribution to the modern theory of Variational Analysis, a rapidly advancing field that integrates optimization and nonsmooth analysis. The central accomplishments of the thesis lie in the creation and development of

perturbation methods that ensure existence, stability, and continuity of strong minima for optimization problems in nonconvex and infinite-dimensional settings.

A key conceptual innovation is the introduction of a strengthened notion of uniform epi-convergence formulated via conditions (1.4) and (1.5) in Definition 1.2.1. Traditional epi-convergence, while well-known and broadly applicable, often does not guarantee stability of strong minimizers. The refined version developed by the author enables such stability without convexity, which is a major conceptual shift. This new structure serves as a unifying foundation for all subsequent results.

Based on this strengthened convergence concept (uniform epi-convergence), the author establishes a general variational principle for strong minima in complete metric spaces (Theorem 1.3.1). Unlike many classical principles that only ensure approximate minimum or rely heavily on convexity, this result ensures genuine attainment of minima after small perturbations. The construction of an appropriate perturbation space (Definition 1.1.1) is central to the method and is handled with clarity and rigor.

A major highlight of the dissertation is the simultaneous perturbation theorem (Theorem 1.4.1), which shows that a single perturbing function can be chosen so that every function in a uniformly epi-convergent sequence attains a strong minimum, while the minimizers converge to the minimizer of the limit function. This creates an elegant and powerful analytic mechanism with significant implications for dynamical optimization and sensitivity theory.

In Chapter 2, the techniques are generalized to parameter-dependent optimization, introducing a perturbation space on the product $P \times X$ (Definition 2.1.1). To support stability in this more complex environment, the author develops the concept of uniform epi-continuity (conditions (2.1) and (2.2)) - effectively extending uniform epi-convergence to families of functions indexed by parameters. This leads to a central parametric stability theorem (Theorem 2.4.3) asserting that for a dense G_δ set of perturbations, strong minimizers exist for the perturbed problems on a dense G_δ set of parameters, with continuity of minimizers.

One of the most impactful applications concerns geometric functional analysis: the author proves a Stečkin-type well-posedness theorem (Theorem 2.5.4) for metric projection operators in separable Banach spaces. The result shows that after a suitable equivalent renorming, the metric projection becomes well-posed on a prescribed dense subset. This is original and of independent significance, offering new insights into renorming theory and stability of convex projections.

In Chapter 3, the author transfers the developed perturbation methods into the framework of Orlicz sequence spaces l_M , which represent an important extension of

classical l_p spaces. A variational principle tailored to constrained minimization (Theorem 3.2.2) is obtained, guaranteeing existence of minimizers under appropriate perturbations. Furthermore, the PhD thesis provides structural smoothness restrictions via nonexistence results for smooth bump functions (Propositions 3.4.1–3.4.3) under certain analytic conditions on the Orlicz function. These findings link variational behavior to the geometric smoothness of function spaces - a sophisticated and nontrivial contribution.

Overall, the PhD thesis exhibits a high level of mathematical rigor, internal coherence, and strong conceptual unity. It convincingly demonstrates that perturbation principles - when properly extended - can serve as a robust analytical framework applicable to a broad spectrum of optimization problems, including nonconvex and infinite-dimensional cases. The results show clear potential for future applications in other fields of mathematics.

The publication of the main results in peer-reviewed journals confirms their originality, correctness, and scientific relevance. The dissertation successfully advances the theoretical foundations of modern Variational Analysis and contributes new methodologies that may serve as a basis for further research breakthroughs.

4. Approbation of the results

a) The PhD thesis is based on three publications, all indexed in the Web of Science database - one in a Q1 journal and the other two in Q2 journals. This results in 195 points under Group Γ indicators (with a minimum requirement of 30 points). A PhD thesis has been submitted as part of the procedure, which satisfies Indicator A and contributes 50 points. This brief evaluation confirms that the PhD thesis submitted for defense meets the minimal national requirements for awarding the educational and scientific degree “Doctor”. It should also be noted that all three publications are indexed in the SCOPUS database as well. The research topic is advanced and highly specialized. Therefore, it is understandable that citations cannot yet be expected for publications from 2024–2025. Furthermore, the author has chosen to publish her research in non-APC (non-paid / diamond or traditional subscription-based) journals, which typically operate with significantly longer review and production timelines. For example, one of the accepted articles follows the timeline: Received: 31 October 2023; Accepted: 17 July 2024; Published online: 31 July 2024; final publication pending. Such publication strategy demonstrates a preference for high-quality and rigorously reviewed academic venues rather than accelerated publication facilitated by article-processing charges.

b) The results of the PhD thesis have been widely disseminated through a series of scientific presentations at both national and international forums. The candidate has

reported her work at the FMI Spring Science Sessions in 2023 and 2024 in Sofia, where she presented results concerning perturbation methods in Orlicz sequence spaces and the simultaneous perturbation of uniformly epi-convergent sequences of functions. She further presented the same line of research at the 16th International Workshop on Well-Posedness of Optimization Problems and Related Topics held in Borovets in 2023, as well as at the International Conference on Optimization: Challenges and Applications in Alicante, Spain in 2024, and during the Week of Mathematics and Informatics in Duni in 2024, ensuring broader international visibility of the work. The next stage of results - related to parametric families and generic continuity of strong minima has been presented at the 4th International Conference on Variational Analysis and Optimization in Santiago, Chile in January 2025. Moreover, the author is actively involved in advanced doctoral educational initiatives and has been invited to give several research talks connected to the thesis at the Doctoral Schools, both in Sofia and in Japan in 2025, covering continuity results for parameterized minimization problems and their applications. These appearances reinforce the continuous development of the candidate as an emerging researcher and show recognition of her contributions within specialized academic communities. Overall, the presented talks convincingly demonstrate that the scientific achievements of the thesis have been thoroughly communicated to experts in Variational Analysis, Optimization and Functional Analysis, thus meeting the highest standards of approbation for a PhD thesis.

c) This is the candidate's first procedure for academic career development, and therefore it is not possible for any of the submitted results or publications to repeat those from previous procedures. All scientific contributions are newly obtained and are presented here for the first time within such an evaluation process.

d) In addition, the dissertation has undergone examination by officially approved plagiarism-detection software, and the generated report confirms the originality of the results. There is no plagiarism detected in any of the submitted dissertation materials or related publications.

I have not found “plagiarism” in the works of the candidate in the sense of the “Law on the Development of the Academic Staff in the Republic of Bulgaria” in the Republic of Bulgaria.

5. Qualities of the abstract

The abstract of the PhD thesis provides a clear, accurate, and well-structured presentation of the scientific contributions, the methodology, and the main results. All essential definitions, theoretical achievements, and applications are faithfully reflected and

correctly attributed to the corresponding chapters of the dissertation. The key theorems are properly highlighted, and the logical progression of the research work is easy to follow.

The abstract contains:

- a concise motivation and background of the research area
- a well-organized summary of the contributions from all three chapters
- a complete list of publications supporting the PhD thesis
- detailed documentation of scientific presentations and dissemination of results
- a statement of originality confirming the author's contributions

Moreover, the terminology is used correctly and consistently, and the exposition style maintains scientific precision while remaining accessible to the broader optimization and variational analysis community. The structure adheres fully to the formal requirements of Sofia University for a PhD thesis abstract. In summary, the abstract correctly and professionally reflects both the scope and the depth of the scientific results presented in the dissertation.

6. Critical notes and recommendations

Minor typographical errors are observed, which occur only sporadically. They do not affect the scientific value or the readability of the work and therefore will not be discussed.

The author has chosen not to provide bibliographic citations directly after the numbering of definitions or theorems, but instead to introduce the relevant background and references within the preceding explanatory paragraphs. This stylistic choice does not diminish the scientific value or the clarity of the exposition, since all sources are properly acknowledged in the narrative and the reader is sufficiently guided to the corresponding literature. Moreover, the dissertation is written in a style consistent with the author's three published journal articles, ensuring a unified presentation of the research results. The structure and referencing format therefore meet the accepted standards for mathematical writing and do not affect the scientific rigor or correctness of the results.

The candidate is encouraged to extend the developed perturbation techniques in directions related to practical optimization models and operator theory.

7. Conclusion

Having become acquainted with the PhD thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, **I confirm** that the presented PhD

thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University “St. Kliment Ohridski” (FMI-SU) for acquisition by the candidate of educational and scientific degree “Doctor” in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics Doctoral program (Operations Research). In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, **I strongly recommend** the scientific jury to award **Hristina Yordanova Belcheva**, the educational and scientific degree “Doctor”/ in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics Doctoral program (Operations Research).

Date: 18.12.2025

Reviewer:

/Boyan Zlatanov, Professor, Dr.Sci. /