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Graded algebras and noncommutative invariant theory

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Abstract

of Ph.D. Thesis for acquisition of the educational and scientific degree "Doctor" in professional direction 4.5 Mathematics (doctor program "Algebra, Number Theory and Applications" - Topology)

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The Ph. D. Thesis occupies 84 pages and consists of an introduction, three chapters, conclusion, index and bibliography, which has 56 titles. The enumeration of all definitions, examples, propositions, lemmas and theorems in the abstract is the same as in the thesis.

Chapter 1

Introduction

Invariant theory is a branch of abstract algebra, which, broadly speaking, studies objects that remain unchanged under linear transformations. Its origin can be traced to the works of Lagrange and Gauss at the end of the 18-th and beginning of 19-th century. They studied quadratic binary forms and used discriminant to differentiate distinct forms. The real invariant theory, however, began with the works of George Boole and Otto Hessee. The early years of invariant theory saw a lot of effort invested into the study of binary forms (homogeneous polynomials in two variables). It is well known that a quadratic equation $ax^2 + 2bx + c$ has a double root if and only if the invariant $b^2 - ac$ of the quadratic binary form $ax^2 + 2bxy + y^2$ is 0. Perhaps inspired by results like this one, mathematicians early on were motivated by the idea that any property of invariant polynomials can be described by vanishing of some invariants. Results by Boole, Cayley and Einsenstein about invariants of quadratic forms from the period of 1840 to 1850 can be found in [18]. The efforts in this direction by Cayley, Aronhold, Clebsch and Gordan culminated in the development of the "symbolic" method[1]. This method allows reduction of computations of binary forms of degree n to n-th power of linear forms. The problem with that method, which allowed computation of invariants, is that it was enormously hard to actually realise it, except in some specific cases. That is one of the reasons research in the field shifted towards finding "fundamental systems" of invariants, i.e. finite sets, such that any invariant can be expressed as a polynomial of those fundamental invariants.

[1] https://en.wikipedia.org/wiki/Symbolic_method

An example is the fundamental theorem of symmetric polynomials. In 1868 James Gordan [23] proved that the set of invariants of binary forms of any degree n is finitely generated. In 1890, Hilbert [25], in one of the most fundamental papers in mathematics, generalised the result of Gordan to systems of several homogeneous forms. His proof was, however, noncostructive and was not widely accepted at the time. It even prompted Gordan to say the famous words:

"Das ist nicht Mathematik. Das ist Theologie."

Whether or not he actually said that is not exactly clear, as earliest quotes of it dates after his death in 1912. He was actually supportive of Hilbert's ideas and enforced some of his methods, so the widespread idea that he opposed his work in invariant theory is most probably a myth. Anyway, Hilbert [26] came back in 1893 with a constructive proof of the theorem.

The problem of finite generation was a central one for invariant theory. The question "Are all polynomials in d variables, invariant under the action of a subgroup G of the matrix group, finitely generated?" was one of the main motivations behind Hilbert's 14-th problem [27]. Emmy Noether gave affirmative answer [44, 45] for finite groups, and Nagata [43] constructed a counterexample for infinite groups.

Another important combinatorial question of invariant theory is how many invariants are there. It was answered by Molien's formula [41], by providing a way to calculate the number of generators of each degree.

After that, invariant theory was seemingly left with no big questions to answer, and was presumed dead. This turned out to not be the case. To quote Rota [48],

"Like the Arabian phoenix arising from its ashes, classical invariant theory, once pronounced dead, is once again at the forefront of mathematics."

The first revival of the theory was around 1935 with the works of Schur, Weyl and Cartan. About that time it was realized that classical invariant theory can be looked as a special case of theory of semi-simple groups. That was made evident by Weyl's [54] book *Classical Groups*, in which one of the main topics was the study of polynomial invariants in any number of variables under the action of classical groups. There is a funny remark on that book by Howe [28],

> *"Most people who know the book feel the material in it is wonderful. Many also feel the presentation is terrible. (The author is not among these latter.)"*

This still wasn't enough to offer interesting problem and attract the attention of the mathematical society. It was later, with the work of Mumford, who used elements of invariant theory to solve problems of "moduli" of algebraic curves. His newer approach to invariant theory was to study it in a more general setting - algebraic groups acting on algebraic varieties. In his book [42] he generalized and modernized the ideas of classical invariant theory.

In modern days, branches of mathematics such as Lie theory, algebraic geometry, differential algebra and others are influenced by invariant theory. To again quote Rota,

> *"Eventually, invariant theory was to become a victim of its own success: the very term "invariant theory" is nowadays understood in such a wide variety of senses that it has become all but meaningless."*

By that he probably meant that in order to talk about "invariant theory", we need a mathematical context.

But what about noncommutative invariant theory? It began in 1936 with Margarete Wolf's paper [55], in which she studied noncommutative symmetric polynomials. Naturally, mathematicians were interested in what clasical invariant theory results can be generalized to the noncommutative case. The answer was, not many. Emmy Noether's theorems [44, 45], for example, look nothing alike in the noncommutative case. It was proved independently in the early 1980's by Dicks and Formanek [17], and [33], that the noncommutative algebra of invariants is finitely generated only for finite groups, consisting of scalar matrices. The results were later generalized by Koryukin [35] in 1984 to infinite groups.

Seemingly that leaves “nothing to do” in noncommutative invariant theory. Perhaps noncommutative invariant theory, like commutative one, was also dead? The answer is, again, no.

In his 1984 paper [35], which was the main motivation behind our work, Koryukin defined an action, which he called S-action. This allowed to “simulate” commutativity by acting on the positions of the elements in homogeneous polynomials. He proceeded to prove that, equipped with this action, the algebra of invariants of noncommutative polynomials, under the action of reductive groups, is finitely generated. Koryukin’s result bring back one of the fundamental problems of invariant theory.

Problem 1.0.1. For fixed reductive group G, find a fundamental system of generators of the algebra of the noncommutative polynomials, invariant under the action of G.

We provide answer to this problem when G is the symmetric group for polynomials in any number of noncommuting variables d, and when G is the alternative group and $d = 3$.

The structure of the thesis is as follows.

The second chapter contains all the preliminary notations, definitions and results that we will need later on in the thesis. More specifically, section 2.2 contains the classical results of commutative invariant theory, written in a more “modern language”.

In section 2.3 we introduce some of the fundamental results in noncommutative invariant theory and compare them to their commutative counterparts. Section 2.4 is devoted to Koryukin’s paper [35], the importance of which we already stated. We formulated a result regarding finite generation of algebra of invariants with the additional S-action, this is Theorem 2.4.24. In order to present this result and proof to it, several technical lemmas, together with two other theorems, are necessary.

The last section 2.5 of the second chapter is devoted to Margarete Wolf’s results in symmetric noncommutative polynomials. We have tried to present her results from a more modern point of view and in language, consistent with everything established so far, while also staying as faithful to the original as possible.

The third and fourth chapter of the thesis contain our new results.

The former section 3.1 of chapter 3 is devoted to the results of our paper [10]. In it, we prove a noncommutative analogue 3.1.5 of the fundamental theorem of commutative symmetric polynomials by constructing a finite generating set of elementary symmetric noncommutative polynomials. We do so by first proving that, with Korykin's S-action, for base field of any characteristic, the algebra of symmetric noncommutative polynomials in any number of variables is generated by the power sums. We then prove an analogue 3.1.4 of Newton's identities, relating the power sums and the elementary symmetric polynomials. The main result 3.1.5 of this section is true under the assumption that the base field is of characteristic either 0 or greater than the number of variables of the symmetric polynomials. We illustrate our ideas with plenty of examples and provide alternative proofs of the main theorem in special cases of small number of variables ($d = 2$). The main techniques used in this section were generalization of the commutative results and "lifting" a fundamental set of the commutative algebra to the noncommutative.

The latter section 3.2 of chapter 3 contains the results of our paper [11]. We explore the algebra of symmetric noncommutative polynomials, when the base field is of non-zero characteristic, which is less than the number of variables. This is not covered in Koryukin's result 2.4.24. We give answers to two important problems. First, we prove in Theorem 3.2.10 that in this instance, the algebra of invaraints is not finitely generated. We do so by first reducing the problem for the case of the characteristic being equal to the number of variables, and then conveying the problem to the algebra of indecomposables, i.e. the augmentation ideal 3.2.3, factored by its square. The second question we answer in that section is the existence of minimal generating set for the algebra of the symmetric noncommutative polynomials. We prove in Theorem 3.2.12 that the power sums form a minimal generating set for the algebra of symmetric noncommutative polynomials. The idea behind the proof is illustrated by a concrete example 3.2.11.

In chapter 4 we again try to solve problem 1.0.1 for G being the alternating group. We do so for the polynomials in 3 variables, invariant under the action of the alternating group of order 3. We again try to "lift" results from the commutative case to the noncommutative, under the assumption that the noncommutative algebra is equipped with Koryukin's S-action. We obtain a generating set for said algebra for fields of characteristic 0 or greater than 3 and use our result 3.2.10 to prove that if the characteristic of the base field is 2 or 3, the algebra is not finitely generated.

Chapter 2

Preliminaries

2.1. Basic notations

Throughout the thesis, we use the following notations:

1. As usual, $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively.
2. For a set of variables $X_d = \{x_1, \ldots, x_d\}$ and field K, $K[X_d]$ is the algebra of the polynomials of d commuting variables with coefficients in K.
3. By $\mathrm{Sym}(d)$ and $\mathrm{Alt}(d)$ we denote the symmetric and alternative group of order d, respectively.
4. $\mathrm{GL}_d(K)$ denotes the general linear group of order d with matrix entries from the field K.
5. For vector spaces V, W over a field F, $\mathrm{Hom}(V, W)$ is the vector space of all linear maps $V \to W$.
6. If V is a vector space over a field F, $\mathrm{Hom}\, V$ is the vector space of all endomorphisms from V to V.
7. For a field K, KX_d is the vector space over K with basis $x_1, x_2, \ldots, x_d$.

2.2. Commutative invariant theory

In classical invariant theory results are usually over the field of complex numbers $\mathbb{C}$, however most of the results remain true over any field K of characteristic 0.

Let V_d be the d-dimensional vector space with basis $\{v_1, v_2, \ldots, v_d\}$ and

$$x_i : V_d \to \mathbb{C}, \quad i = 1, 2, \ldots, d$$

be the linear functions defined by

$$x_i(\xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_d v_d) = \xi_i, \quad \xi_1, \xi_2, \ldots, \xi_d \in \mathbb{C}.$$

These are called *coordinate functions.* The functions $x_1, x_2, \ldots, x_d$ generate a subalgebra of the algebra of all $\mathbb{C}$-valued functions on V. This algebra is denoted by $\mathbb{C}[X_d] = \mathbb{C}[x_1, x_2 \ldots, x_d]$ and is called the algebra of polynomial functions. Note that there is an isomorphism φ from $\mathbb{C}[x_1, x_2 \ldots, x_d]$ onto the polynomial algebra $\mathbb{C}[y_1, y_2, \ldots, y_d]$ defined by $\varphi(f_i) = y_i$, $i = 1, 2, \ldots, d$.

Let the group of invertible matrices $\mathrm{GL}_d(\mathbb{C})$ act on the vector space V_d. That action induces an action of $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{C}[X_d]$ by

$$g(f) : v \to f\big(g^{-1}(v)\big), \quad g \in \mathrm{GL}_d(\mathbb{C}),\ f(X_d) \in \mathbb{C}[X_d],\ v \in V_d. \qquad (2.1)$$

Definition 2.2.1. Let G be a subgroup of $\mathrm{GL}_d(\mathbb{C})$. *The algebra of G-invariants* are all the polynomials in $\mathbb{C}[X_d]$, which remain unchanged under the action of all elements of G, that is

$$\mathbb{C}[X_d]^G = \{f \in \mathbb{C}[X_d] \mid g(f) = f \text{ for all } g \in G\}.$$

It is more convenient to assume that $\mathrm{GL}_d(\mathbb{C})$ acts canonically on the vector space with basis $X_d = x_1, x_2, \ldots, x_d$ and to extend that action diagonally on $\mathbb{C}[X_d]$ by

$$g\big(f(x_1, x_2, \ldots, x_d)\big) = f\big(g(x_1), g(x_2), \ldots, g(x_d)\big), \quad g \in \mathrm{GL}_d(\mathbb{C}), f \in \mathbb{C}[X_d]. \qquad (2.2)$$

Both actions 2.1 and 2.2 yield the same algebra of invariants, as the groups are isomorphic.

Perhaps the earliest example of a result in invariant theory that a student encounters is the fundamental theorem of symmetric polynomials. This is a basic result in algebra, that illustrates some of the main questions in invariant theory.

Let K be a field of arbitrary characteristic. The symmetric group $\mathrm{Sym}(d)$ acts on the vector space X_d by

$$\sigma(x_i) = x_{\sigma(i)}, \quad \sigma \in \mathrm{Sym}(d), i = 1, \dots, d, \tag{2.3}$$

which means that σ permutes the variables. A polynomial $f \in K[X_d]$ is symmetric, if it remains unchanged under the action of all the permutations of $\mathrm{Sym}(d)$. Let

$$\begin{aligned} e_1 &= x_1 + x_2 + \cdots + x_d, \\ e_2 &= x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots + x_{d-1}x_d, \\ &\vdots \\ e_d &= x_1x_2 \dots x_d \end{aligned} \tag{2.4}$$

be the elementary symmetric polynomials.

Theorem 2.2.2 (Fundamental theorem of symmetric polynomials)**.** *Every symmetric polynomial $f \in K[X_d]^{\mathrm{Sym}(d)}$ can be written uniquely as a polynomial*

$$f = p(e_1, e_2, \dots, e_d)$$

in the elementary symmetric polynomials $e_1, e_2, \dots, e_d$.

In the language of invariant theory, this gives us that the algebra $K[X_d]^{\mathrm{Sym}(d)}$ is generated by the elementary symmetric polynomials, that is

$$K[X_d]^{\mathrm{Sym}(d)} = K[e_1, e_2, \dots, e_n].$$

The “unique” part of the theorem means that the elementary symmetric polynomials

$e_1, e_2, \ldots, e_d$ are algebraically independent. Note that the generating set $e_1, e_2, \ldots, e_d$ isn't unique. If we denote $p_k = x_1^k + x_2^k + \cdots + x_d^k$ to be the k-th power sum, then:

Lemma 2.2.3. *The algebra of symmetric polynomials $K[X_d]^{\mathrm{Sym}(d)}$ is generated by the first d power sums, that is*

$$K[X_d]^{\mathrm{Sym}(d)} = K[p_1, p_2, \ldots, p_d].$$

The question of finite generation has been fundamental to invariant theory from the very start.

Definition 2.2.4. Let K be a field and A - a (commutative) associative algebra over K. Then A is *finitely generated* if there exist elements $a_1, a_2, \ldots, a_n \in A$, such that each element of A can be written as polynomial in $a_1, a_2, \ldots, a_n$ with coefficients in K.

Remark 2.2.5. Note that the above definition is equivalent to the following. If

$$\varphi : K[X_n] \to A$$

is the evaluation homomorphism that maps x_i to a_i for $i = 1, 2, \ldots, n$, then A is finitely generated if φ is surjective. Applying the first isomorphism theorem, we obtain that

$$A \cong K[X_n]/\operatorname{Ker}(\varphi).$$

The converse also holds. If A is isomorphic to a factor algebra $K[X_n]/I$ for ideal I of $K[X_n]$, then each element $a \in A$ is polynomial in the cosets $x_i + I$ for $i = 1, 2, \ldots, n$ and thus A is finitely generated.

Problem 2.2.6. Is the algebra $K[X_d]^G$ finitely generated for all subgroups G of $\mathrm{GL}_d(K)$?

This question was the motivation behind Hilbert's 14-th problem [27]. When the group G is finite and K has characteristic 0, an affirmative answer to Problem 2.2.6 was given by Emmy Noether [44] in 1916. When G is finite and K is a field of arbitrary characteristic, the answer is again yes, and it was also proved by Emmy Noether [45] in 1926. In the general case, however, this is not true - a counterexample was constructed by Nagata [43] in 1959. Bellow is Emmy Noether's first cited theorem.

Theorem 2.2.7 (Endlichkeitssatz of Emmy Noether [44]). *Let K be a field of characteristic 0 and G be a finite subgroup of $\mathrm{Gl}_d(K)$. Then the algebra of invariants $K[X_d]^G$ is finitely generated and has a system of generators $f_1, \ldots, f_m$, where each f_i is homogeneous polynomial of degree bounded by the order of the group G.*

A translation of the paper [44] by Colin McLarty [39] is very helpful towards understanding this cornerstone theorem of invariant theory.

Theorem 2.2.8 (Noether normalization lemma [45]). *Let K be a field of arbitrary characteristic and A be finitely generated commutative K-algebra. There exist algebraically independent elements $a_1, a_2, \ldots, a_n \in A$ such that A is finitely generated module over the polynomial ring $K[a_1, a_2, \ldots, a_n]$.*

In Remark 2.2.5 we noted that an K-algebra is finitely generated if and only if it is a factor algebra of polynomial algebra. We proceed to give an example why that is not the same as A being isomorphic to a polynomial algebra (i.e. the kernel of the evaluation homomorphism to be trivial).

Example 2.2.9. It is a basic result in the early algebra courses that

$$\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$$

and by 2.2.5, $\mathbb{C}$ is finitely generated $\mathbb{R}$-algebra. Considered as a linear space over the reals, $\mathbb{C}$ has dimension 2. If $\mathbb{C}$ was isomorphic to polynomial algebra, it would have an infinite dimension over $\mathbb{R}$, which is a contradiction.

Naturally, the following question in invariant theory arises.

Problem 2.2.10. For which subgroups G of $\mathrm{Gl}_d(K)$, the algebra of invariants $K[X_d]^G$ is isomorphic to a polynomial algebra?

Definition 2.2.11. Let V be a finite dimensional vector space over a field K with dimension n. A *pseudoreflection* is an invertible linear transformation

$$\varphi : V \to V,$$

such that φ is not the identity, φ has a finite multiplicative order and φ fixes a hyperplane.

Theorem 2.2.12 (Chevalley-Shephard-Todd [14, 49])**.** *For a finite group G and a field K of characteristic* $\mathrm{char}(K) = 0$*, the algebra of invariants $K[X_d]^G$ is isomorphic to a polynomial algebra, $K[X_d]^G \cong K[Y_d]$ if and only if G is generated by pseudoreflections.*

The most recent generalisation of 2.2.12 is by Abraham Broer [12] in 2007 for fields of positive characteristic.

From combinatorial point of view, the following question arises: how many invariants are there? To answer it, we introduce some definitions.

Definition 2.2.13. A ring R is said to be *graded*, if it can be decomposed as direct sum

$$R = \bigoplus_{i=0}^{\infty} R_i$$

of additive groups, such that $R_i R_j \subseteq R_{i+j}$.
An algebra A is said to be graded if it is graded as a ring.

There is a natural grading for the algebra of invariants $K[X_d]^G$ for any group G:

$$K[X_d]^G = K \oplus (K[X_d]^G)^{(1)} \oplus (K[X_d]^G)^{(2)} \oplus \dots,$$

where $(K[X_d]^G)^{(n)}$ is the vector space of the homogeneous invariants of degree n.

Definition 2.2.14 ([7], chapter 11)**.** Let λ be an integer valued function on the class of all finitely generated modules over a ring A and M be a finitely generated graded A-module, $M = M_0 \oplus M_1 \oplus \dots$. The *Poincaré (or Hilbert)* series of M with respect to λ is the generating function of $\lambda(M_n)$

$$H(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n.$$

In the case of algebra of invariants $K[X_d]^G$, the integer valued function is the dimension of the vector space of homogeneous invariants, and the Hilbert series is

$$H(K[X_d]^G, t) = \sum_{n=0}^{\infty} \dim(K[X_d]^G)^{(n)} t^n.$$

The coefficients in this power series gives the number of invariants of each degree.

Theorem 2.2.15 (Hilbert, Serre)**.** *The Hilbert series $H(M,t)$, defined in 2.2.14, is a rational function of t in the form*

$$\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})},$$

where $f(t) \in \mathbb{Z}[t]$.

A proof of that theorem can be found in [7], chapter 11, and it is by induction on the number of generators of the module A. However, that is not Hilbert's original proof. It made use of his syzygy theorem [25].

The next theorem is a formula from 1897 that gives the explicit form of the Hilbert series $H(K[X_d]^G, t)$.

Theorem 2.2.16 (Molien formula [41])**.** *Let* $\operatorname{char}(K) = 0$. *For a finite group G,*

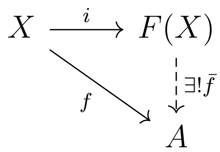
$$H(K[X_d]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1-gt)}.$$

The results listed above are just some of the fundamental ones in classical invariant theory. In the next chapter we will see how they "translate" to the noncommutative case.

2.3. Noncommutative invariant theory

The first step of going from commutative algebra of invariants to noncommutative is choosing a noncommutative alternative to the polynomial algbera $K[X_d]$. The natural candidate is the free associative algebra, as it has the same universal property as the polynomial algebra $K[X_d]$. Let us give a more general, categorical definition of "free".

Definition 2.3.1. Let $\mathcal{C}$ be an arbitrary category, X a set, $F(X)$ a $\mathcal{C}$-object and $i : X \to F(X)$ be a set injection. $F(X)$ is called a *free object* of X in $\mathcal{C}$, if for every $\mathcal{C}$-object A and each mapping between sets $f : X \to A$ exists unique $\mathcal{C}$-morphism $\bar{f} : F(X) \to A$, such that the following diagram



is commutative, that is $\bar{f} \circ i = f$. This is called *universal property.*

In the category of both unitary commutative and noncommutative associative algebras, morphism is a homomorphism between the algebras.

Proposition 2.3.2 ([19]). *For an arbitrary set X, the polynomial algebra $K[X]$ is free in the category of all unitary commutative associative algebras.*

We now define the free associative algebra $K\langle X_d\rangle$ to be the free object in the category of unitary associative algebras.

Proposition 2.3.3 ([19]). *Let X be a set and K a field. The algebra $K\langle X\rangle$ with basis all the words*

$$x_{i_1}x_{i_2}\dots x_{i_k}, \quad x_{i_k} \in X, k = 0, 1, \dots$$

and multiplication concatenation of words with respect to elements of K

$$(x_{i_1}x_{i_2}\dots x_{i_k})(x_{j_1}x_{j_2}\dots x_{j_s}) = x_{i_1}x_{i_2}\dots x_{i_k}x_{j_1}x_{j_2}\dots x_{j_s},$$

is free in the category of all unitary associative algebras.

To emphasise, our noncommutative analogue for monomials in d variables are words with the letters $x_1, \ldots, x_d \in X_d$, and polynomials are linear combinations of such words with coefficients in K.

In the commutative case, we have a natural way to compare monomials. There are multiple ways to order noncommutative monomials.

Definition 2.3.4 ([56]). An *admissible ordering* σ on the free monoid $\langle X_d \rangle$ is a relation on $\langle X_d \rangle \times \langle X_d \rangle$, such that:

- Any two monomials $w_1, w_2 \in \langle X_d \rangle$ are comparable, $w_1 \geq_\sigma w_2$ or $w_2 \geq_\sigma w_1$ (σ is total order on $\langle X_d \rangle$);
- Any monomial is comparable with itself, $w \geq_\sigma w$ (σ is reflexive);
- If for two monomials $w_1, w_2 \in \langle X_d \rangle$, $w_1 \geq_\sigma w_2$ and $w_2 \geq_\sigma w_1$, then $w_1 = w_2$ (σ is antisymetric).
- If $w_1, w_2, w_3 \in \langle X_d \rangle$, $w_1 \geq_\sigma w_2$ and $w_2 \geq_\sigma w_3$, then $w_1 \geq_\sigma w_3$ (σ is transitive);
- If $w_1, w_2 \in \langle X_d \rangle$ are such that $w_1 \geq_\sigma w_2$, for any $w_3, w_4 \in \langle X_d \rangle$, $w_3 w_1 w_4 \geq_\sigma w_3 w_2 w_4$ (σ is compatible with multiplication);
- Every descending chain of words $w_1 \geq_\sigma w_2 \geq_\sigma \ldots$ in $\langle X_d \rangle$ eventually stabilises (σ is a well-ordering).

From this definition it follows that if σ is admissible ordering of $\langle X \rangle$, then for any word $w \in \langle X \rangle$ $w \geq_\sigma 1$.

One of the most intuitive ways to order the monomials in $\langle X_d \rangle$ is the lexicographic order:

Example 2.3.5 ([56]). An example of an ordering on $\langle X_d \rangle$ is the lexicographic ordering, which we will denote with Lex. If w_1, w_2 are two words in $\langle X_d \rangle$, $w_1 \geq_{\text{Lex}} w_2$ if either $w_1 = w_2 w$ for some word $w \in \langle X_d \rangle$ or $w_1 = w x_i w'$ and $w_2 = w x_j w''$ for some words $w, w', w'' \in \langle X_d \rangle$ and letters $x_i, x_j \in \langle X_d \rangle$ with $i > j$.

Remark 2.3.6. The lexicographic order is total, reflexive, antisymmetric and transitive order of the monomials in $\langle X_d \rangle$. It is not, however, an admissible order, because it does not satisfy the last two conditions of the definition 2.3.4. To see that it isn't compatible with multiplication, it suffices to look at the free monoid, generated by two elements, $\langle x_1, x_2 \rangle$. We have that $x_2^2 \geq_{\text{Lex}} x_2$ but $x_2^2 x_1 \leq_{\text{Lex}} x_2 x_1$. We can also define an infinite descending chain in $\langle X_d \rangle$ by

$$x_2 x_1 \gneq_{\text{Lex}} x_2^2 x_1 \gneq_{\text{Lex}} x_2^3 x_1 \gneq_{\text{Lex}} \dots,$$

which proves that the lexicographic order isn't a well ordering.

Next is an example of admissible ordering of $\langle X_d \rangle$.

Definition 2.3.7 ([56]). The *degree-lexicographic ordering* (or *deg-lex ordering*) on $\langle X_d \rangle$ is the ordering of the monomials in $\langle X_d \rangle$ first by degree (or length), and then lexicographically.

This is the ordering we shall use, so we will just denote it by $\leq$ or $\geq$, without any subscript. The infinite descending chain we saw in Remark 2.3.6 looks different in the deg-lex ordering. We have that

$$x_2 x_1 \leq x_2^2 x_1 \leq \dots,$$

so it is ascending chain.

Having a way to compare monomials in $\langle X_d \rangle$ allows us to define leading monomial of a polynomial in $K\langle X_d \rangle$.

Now that we have established the algebra $K\langle X \rangle$, we can define algebra of invariants for subgroups G of the general linear group $\mathrm{GL}_d(K)$. The definition is similar to the commutative case 2.2.1.

Let the group $G \leq \mathrm{GL}_d(K)$ act canonically on the vector space over K with basis X_d and extend that action diagonally to $K\langle X_d \rangle$ by

$$g\big(f(x_1, x_2, \dots, x_d)\big) = f\big(g(x_1), g(x_2), \dots, g(x_d)\big), \quad g \in G, f \in K\langle X_d \rangle.$$

Definition 2.3.8. Let G be a subgroup of the general linear group $\mathrm{GL}_d(K)$ and $K\langle X_d\rangle$ be the free associative algebra. T*he algebra of G-invariants* $K\langle X_d\rangle^G$ consists of all polynomials in $K\langle X_d\rangle$ that are fixed by the action of all elements of G:

$$K\langle X_d\rangle^G = \{f \in K\langle X_d\rangle \mid g(f) = f \text{ for all } g \in G\}.$$

The start of the noncommutative invariant theory was given by Margarete Wolf [55] in 1936. Her work was primary concerned with the algebra of the symmetric polynomials in non commuting variables $K\langle X_d\rangle^{\mathrm{Sym}(d)}$.

The results of Dicks and Formanek and Kharchenko show that unlike the commutative case, where the algebra $K[X_d]^G$ is finitely generated for every finite group G, in the noncommutative this is only true for very specific groups.

Theorem 2.3.9 ([17, 33]). *Let G be a finite subgroup of $\mathrm{GL}_d(K)$. The algebra of invariants $K\langle X_d\rangle$ is finitely generated if and only if G is a cyclic group of scalar matrices.*

This theorem is obtained as a corollary in Koryukin's paper [35].

Theorem 2.3.10 ([35]). *Let G be an arbitrary (possibly infinite) subgroup of the matrix group $\mathrm{GL}_d(K)$. Let KY_m be a minimal (with respect to inclusion) vector subspace of X_d such that $K\langle X_d\rangle^G \subseteq K\langle Y_m\rangle$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on KY_m as a finite cyclic group of scalar matrices.*

We will return to Koryuikin's paper [35] shortly as it is central to our work and results.

Next is the analogue of Chevalley-Shephard-Todd theorem 2.2.12.

Theorem 2.3.11 ([32, 36]). *The algebra $K\langle X_d\rangle^G$ is free for any subgroup G of $\mathrm{GL}_d(K)$ and any field K.*

Theorem 2.3.12 ([32]). *For finite subgroups G of $\mathrm{GL}_d(K)$, there exists a Galois correspondence between the free subalgebras of $K\langle X_d\rangle$, containing the algebra of invariants $K\langle X_d\rangle^G$, and the subgroups of G. The subalgebra F of $K\langle X_d\rangle$ with $K\langle X_d\rangle^G \subseteq F$ is free if and only if $F = K\langle X_d\rangle^H$ for a subgroup H of G.*

Subalgebras of free algebras are not necessarily free, and here are two examples of that.

Example 2.3.13 ([15]). Let K be a field and $K[x]$ be the polynomial algebra in one variable. It is free, but the subalgebra $K[x^2, x^3]$, generated by x^2 and x^3, is not.

Proposition 2.3.14 ([15]). *Let A be a free associative algebra. If I is any non-zero ideal of A, such that the subalgebra B, generated by I is not equal to A, then B is not free.*

Very similar result is true for finitely generated algebras. A subalgebra of a finitely generated algebra need not be finitely generated. This is true even for commutative algebras. Here's an example of that:

Example 2.3.15. Let K be a field and $K[x, y]$ be the polynomial algebra in two commuting variables x and y. It is immediate that this is a finitely generated algebra by it's definition. Consider the subalgebra $K[x, xy, xy^2, \ldots]$. It is not finitely generated.

Finally, Molien's formula 2.2.16 has a direct analogue in the noncommutative case. It was proved by Dicks and Formanek [17] in the same paper as Theorem 2.3.9.

Theorem 2.3.16 ([17]). *If $G \subseteq \mathrm{GL}_d(K)$ is a finite group and the field K has characteristic 0, then the Hilbert series can be calculated by*

$$H\Big(K\langle X_d\rangle^G, t\Big) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{1 - tr(g)t}.$$

2.4. Koryukin's results

This section is devoted to Koryukin's paper [35]. We already saw the importance of it in Theorem 2.3.10. We will now delve into a comprehensive examination of all the results.

In the paper [35], all considerations are for the tensor algebra $F\langle V\rangle$. This algebra is isomorphic to the free associative algebra.

Definition 2.4.1. Let A be a set of noncommutative polynomials in the free associative algebra $F\langle X_d\rangle$. The least, with respect to inclusion, subspace of FX_d, containing A, is called the *support space* of A.

Lemma 2.4.2 (Koryukin's Lemma 1)**.** *If a set of polynomials in the free associative algebra $F\langle X_d\rangle$ is invariant relative to the action of a group $G \leq \mathrm{GL}_d(F)$, meaning $A^G = A$, then the support space of A is also invariant under the action of G.*

Definition 2.4.3. Let $M \subseteq \langle X_d\rangle$ be a set of monomials, $A \subseteq K\langle X_d\rangle$ be a set of polynomials in d noncommuting variables and $x_i \in \{x_1, \ldots, x_d\}$ be a letter. We say that x_i *has an occurance* in M (in A), if x_i is used in a monomial in M (in the expression of a polynomial in A). We say that a sequence of letters $x_{i_1} \ldots, x_{i_n}, \ldots$ is *compatible* with M (with A), if there is at least one monomial of the form $x_{i_1} \ldots x_{i_n}$ in M (a monomial of the form $x_{i_1} \ldots x_{i_n}$ is used in the expression of a polynomial in A).

Definition 2.4.4. Let the symmetric group $\mathrm{Sym}(n)$ acts on the homogeneous component of $\langle X_d\rangle$ of order n (that is the monomials of degree n) by the formula

$$(y_1 \ldots y_n) \circ \sigma = y_{\sigma^{-1}(1)} \ldots y_{\sigma^{-1}(n)}.$$

This action is called *S-action.*

It is important to note that this action is not the same as we defined in the beginning of 2.2 and extended to noncommutative algebras in 2.3.8. This action is not on the elements of the algebra itself, but rather the position of said elements. For example, consider the monomial $x_1x_2x_1 \in \langle x_1, x_2\rangle$ and let $\sigma \in \mathrm{Sym}(3)$ be the permutation (12). Then

$$x_1x_2x_1^{(12)} = x_2x_1x_2$$

is the usual action, but the $S-$action of (12) is

$$x_1x_2x_1 \circ (12) = x_2x_1x_1.$$

Lemma 2.4.5 (Koryukin's Lemma 2). *Let $M \subseteq \langle X_d \rangle$ be a finite set of monomials. If the multiplicative closure of M is closed under the S-action of the symmetric group on the homogeneous components, then any infinite sequence of letters $x_{j_1}, \ldots, x_{j_n}, \ldots$, having an occurance in M, is compatible with M.*

Lemma 2.4.6 (Koryukin's Lemma 3). *Let R be finitely generated algebra, $R = K\langle f_1, \ldots, f_n \rangle$ with $f_1, \ldots, f_n \in K\langle X_d \rangle$. If R is closed under the S-action of symmetric groups, then any infinite sequence of letters $x_{j_1}, \ldots, x_{j_n}, \ldots$ having an occurrence in the set of generators $f_1, \ldots, f_n$, is compatible with it.*

Definition 2.4.7. Let V be a vector space and $\varphi \in \operatorname{Hom} V$ be an automorphism. We call φ *semisimple*, if it is diagonalizable. It is called *scalar*, if it's matrix is scalar in a basis of V (meaning it's diagonizable and it's eigenvalues are all equal).

Lemma 2.4.8 (Koryukin's Lemma 4). *Let V be a finite-dimensional vector space over F - algebraically closed field, and $g \in \operatorname{Hom} V$ be an automorphism. If g is not scalar, there exists a basis $x_1, x_2, \ldots, x_n$ of V and an infinite letter sequence (of elements of X) which is not compatible with the free associative algebra $F\langle X \rangle^g$ of the invariants of g.*

The next formulated Lemma in [35] is the following:

Lemma 5. *Let K be extension of a field F, G a group of automorphisms of V over F, $W = V \otimes_F K$. Then the algebra $F\langle V \rangle^G$ is finitely generated if and only if the algebra $F\langle W \rangle^G$ is finitely generated.*

Instead of the tensor product $W = V \otimes_F K$, since $V \cong F^d$, we have that $W = V \otimes_F K \cong K^n$ (see, for example, [20], page 363).

With said remarks, Lemma 5 becomes:

Lemma 2.4.9 (Koryukin's Lemma 5). *Let K be an extension of the field F, $G \leq \mathrm{GL}_d(F)$, $V = FX_d$ and $W = KY_d$. Then the algebra $F\langle X_d \rangle^G$ is finitely generated if and only if $K\langle Y_d \rangle^G$ is finitely generated.*

Theorem 2.4.10 ([35]). *Let G be an arbitrary (possibly infinite) subgroup of the matrix group $\mathrm{GL}_d(K)$. Let KY_m be a minimal (with respect to inclusion) vector* *subspace of X_d such that $K\langle X_d\rangle^G \subseteq K\langle Y_m\rangle$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on KY_m as a finite cyclic group of scalar matrices.*

Definition 2.4.11. Let $\mathrm{GL}_d(K)$ be the matrix group and $G \leq \mathrm{GL}_d(K)$ its subgroup. G is called *almost special group*, if the index $[G : \mathrm{SL_d(K)}]$ of G over the group of special matrices is finite.

Corollary 2.4.12 ([35]). *Let $G \leq \mathrm{GL}_d(K)$ be an almost special group. If the algebra $K\langle X_d\rangle^G$ is finitely generated, then G is a finite cyclic group of scalar matrices.*

Remark 2.4.13. Dicks - Formanek - Kharchenko's Theorem 2.3.9 follows directly from Corollary 2.4.12 by applying it to a finite group G.

Definition 2.4.14. [30] Let G be a group and F be a field. A *representation* of G over F is a homomorphism

$$\rho : G \to \mathrm{GL}_n(F)$$

from G to the general linear group of order n for some $n \in \mathbb{N}$.

Definition 2.4.15. [30] Let $\rho : G \to \mathrm{GL}_n(F)$ be a representation. We say that ρ is *irreducible*, if ρ has no nontrivial subrepresentation. A group is called *irreducible*, if it has no reducible representation.

It is clear that if G is an irreducible group of matrices, it has no invariant eigen subspaces.

Another Corollary of Theorem 2.4.10 is:

Corollary 2.4.16 ([35]). *Let $G \leq \mathrm{GL}_n(K)$ be an irreducible group. Then the algebra of $G-$ invariants $K\langle X_d\rangle^G$ is either trivial, or not finitely generated.*

Definition 2.4.17. The algebra $K\langle X_d\rangle$, together with the S-action of the symmetric group on the homogeneous components, is called S-algebra and is denoted by $(K\langle X_d\rangle, \circ)$.

If F is a homogeneous subalgebra (ideal) of $K\langle X_d\rangle$, closed under the S-action of $\mathrm{Sym}(n)$ on the homogeneous components of F, then F is called a $S-$*subalgebra*

(S-ideal). If F is $S-$(sub)algebra, it is called *finitely generated S-algebra*, if there exists a finite subset $W \subseteq F$, such that the support space of W is F.

The action of $\mathrm{Sym}(n)$ on $(K\langle X_d\rangle)^{(n)}$, the homogeneous component of degree n of $K\langle X_d\rangle$ and the action of $G \leq \mathrm{GL}_d(K)$ on $K\langle X_d\rangle$ commute and $\left(K\langle X_d\rangle^G, \circ\right)$ is a S-algebra.

The second part of Koryukin's paper answers the question if $\left(K\langle X_d\rangle^G, \circ\right)$ is finitely generated as a S-algebra for reductive groups G.

Definition 2.4.18. [40] If $G \leq GL_n(K)$, G is called *reductive*, if all its rational representations are completely reducible.

Lemma 2.4.19 (Highman's Lemma [24])**.** *Let X_d be a finite set of letters and $w_1, w_2, \ldots,$ be an infinite sequence of words in $\langle X_d\rangle$. There exists a pair of natural numbers $i, j \in \mathbb{N}$, $i < j$ and the word w_i is a subsequence of the word w_j (meaning w_i is obtained from w_j by omitting some letters).*

Theorem 2.4.20 (Koryukin's Theorem 2, [35])**.** *Let $R = (K\langle X_d\rangle, \circ)$ be a S-algebra. Any increasing sequence of S-ideals $I_1 \subseteq I_2 \subseteq \ldots$ in R stabilizes.*

Definition 2.4.21. Let R be a S-algebra, D a S-subalgebra of R and $f_1, \ldots, f_m$ homogeneous elements of D. The S-ideal $I_D\langle f_1, \ldots, f_m\rangle$ is the minimal homogeneous S-ideal of D, which contains $f_1, \ldots, f_m$.

If G is a fixed reductive group and $h \in K\langle X_d\rangle$, denote by M_h the minimal (by inclusion) vector space, containing h and invariant to G (meaning M_h has a basis $\{h^g \mid g \in G\}$. Denote by N_h the subspace of M_h wih basis $\{h^g - h \mid g \in G\}$. N_h has a codimension either 0 or 1. That means either $M_h = N_h$ or (by reductivity) there exist $h^* \in M_h^G$, such that $M_h = Kh^* + N_h$. In either case, by reductivity [1],

$$h = h^* + h', \quad h^* \in M_h^G, h' \in N_h. \tag{2.5}$$

[1] The theory of reductive groups is, by nature, algebraically-geometrical. In order to see why the decomposition is true, we need to go more deeply in a theory, which is for the most part, beyond the scope of the thesis. It is recommended to check the literature on that topic, for example [40,51].

Lemma 2.4.22 (Koyukin's Lemma 6)**.** *Let* $(K\langle X_d\rangle, \circ)$ *be a* S*-algebra and* G *a reductive group. Let* $f_1, \dots, f_m$ *be homogeneous elements of the* S*-algebra of* G*-invariants* $K\langle X_d\rangle^G$*. Then*

$$K\langle X_d\rangle^G \cap I_{K\langle X_d\rangle}\langle f_1, \dots, f_m\rangle = I_{K\langle X_d\rangle^G}\langle f_1, \dots, f_m\rangle.$$

Lemma 2.4.23 (Koryukin's Lemma 7)**.** *Let* $(K\langle X_d\rangle, \circ)$ *be a* S*-algebra and* R *be its* S*-subalgebra. Let* $f_1, \dots, f_m$ *be homogeneous elements of* R *and* $R = I_R\langle f_1, \dots, f_m\rangle$*. Then* $R = K\langle f_1, \dots, f_m\rangle$*.*

The last two lemmas easily prove the main result:

Theorem 2.4.24 (Koryukin's Theorem 3)**.** *Let* K *be any field and* $G \leq \mathrm{GL}_d(K)$ *be a reductive group. Then, the* S*-algebra of invariants* $\left(K\langle X_d\rangle^G, \circ\right)$*, is finitely generated.*

This theorem is the main motivation behind our paper [10]. The following question immediately follows from it:

Problem 2.4.25. Let G be a fixed reductive subgroup of the general linear group $\mathrm{GL}_d(K)$. Find a finite generating set for the S-algebra $\left(K\langle X_d\rangle^G, \circ\right)$.

We will answer it in the case of G being the symmetric group $\mathrm{Sym}(d)$ in Section 3.1.

2.5. The results of Margarete Wolf on symmetric noncommutative polynomials

Just like in the commutative case 2.3, the symmetric group $\mathrm{Sym}(d)$ acts on the free associative algebra $K\langle X_d\rangle$ by

$$\sigma : f(x_1, x_2, \dots, x_d) \mapsto f\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)}\right) \tag{2.6}$$

for all $\sigma \in \mathrm{Sym}(d)$ and $f \in K\langle X_d\rangle$.

Definition 2.5.1. A noncommutative polynomial $f \in K\langle X_d \rangle$ is said to be *symmetric*, if f remains unchanged under the action 2.6 of all the elements $\sigma \in \mathrm{Sym}(d)$.

As before, we will use the deg-lex 2.3.7 ordering of the monomials in $\langle X_d \rangle$. If $f \in K\langle X_d \rangle$ is a noncommutative polynomial, the deg-lex order allows us to define leading monomial of f, and we denote it by $\overline{f}$. The deg-lex ordering is admissible, so for any two ponynomails $f, g \in K\langle X_d \rangle$, the leading monomial $\overline{fg}$ of their product fg is the product $\overline{f}\overline{g}$ of their leading monomials.

The action of $\mathrm{Sym}(d)$ on the set of monomials $\langle X_d \rangle$ splits into orbits. If $f \in \langle X_d \rangle$ is a monomial, we adopt the standart notation

$$\sum f \tag{2.7}$$

to be the summ of all monomials, obtainable by the action of $\mathrm{Sym}(d)$ on f. That means we take sum over all the permutations $\sigma \in \mathrm{Sym}(d) \setminus \mathrm{St}(f)$, which are not in the stabilizer $\mathrm{St}(f)$ of the monomial f under the action of $\mathrm{Sym}(d)$. If we fix a monomial h in each orbit, then the set $\sum h$ forms a basis for the algebra of invariants $K\langle X_d \rangle^{\mathrm{Sym}(d)}$. In her paper [55], Margarete Wolf calls such polynomials *simple symmetric polynomials*. She also provides a way to count the number of such polynomials.

Margarete Wolf also provided a table with the number of simple symmetric polynomials of degrees up to 8:

Degree

Number of distinct elements in a term	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127
3			1	6	25	90	301	966
4				1	10	65	350	1701
5					1	15	140	1050
6						1	21	266
7							1	28
8								1
Total	1	2	5	15	52	203	877	4140

Theorem 2.5.2 (Margarete Wolf's Main Theorem [55]).

(i) The algebra of invariants $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *of the symmetric group of order d (the algebra of symmetric noncommutative polynomials in d variables) is free and has a system of homogeneous (simple symmetric polynomials) generators, such that of each degree there's atleast one generator.*

(ii) Each free generating system of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *has the same number of generators of each degree.*

(iii) If $\{e_i \mid i \in I\}$ *is a free generating set of simple symmetric polynomials for the algebra* $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *and* $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *is symmetric noncommutative polynomial,*

$$f = \sum \beta_j e_{i_1} \dots e_{i_k}, \quad \beta_j \in K,$$

then the coefficients β_j *in that representation are uniquely determined linear combinations with integer coefficients of the coefficients of the polynomial* $f(x_1, \dots, x_d)$.

In [55] are also included the polynomials in the generating set for lower degrees. If we denote by $H_k^{(j)}$ to be the j-th (in the deg-lex ordering) simple symmetric generating polynomial of degree k, then

$$
\begin{aligned}
H_1 &= \sum x_1, \\
H_2 &= \sum x_1 x_2, \\
H_3^{(1)} &= \sum x_1 x_2^2, & H_3^{(1)} &= \sum x_1 x_2 x_3, \\
H_4^{(1)} &= \sum x_1 x_2 x_1 x_3, & H_4^{(2)} &= \sum x_1 x_2^3, & H_4^{(3)} &= \sum x_1 x_2^2 x_3, \\
H_4^{(4)} &= \sum x_1 x_2 x_3 x_2, & H_4^{(5)} &= \sum x_1 x_2 x_3^2, & H_4^{(6)} &= \sum x_1 x_2 x_3 x_4.
\end{aligned}
$$

Margarete Wolf also calculated the number of free generators $H_k^{(j)}$ for degrees up to 6:

Degree

Number of distinct elements in a term	1	2	3	4	5	6
1	1					
2		1	1	1	1	1
3			1	4	12	33
4				1	8	44
5					1	13
6						1
Total	1	1	2	6	22	92

Her proof of the following Theorem makes use of [38].

Theorem 2.5.3 ([55]). *The algebra of the symmetric noncommutative polynomials in two variables $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ has exactly one generator of each degree in any homogeneous free generating set.*

The results of Margarete Wolf's paper [55] were generalized more than 30 years later by Bergman and Cohn [9] in 1969. Different aspects of the theory of symmetric function were studied in [1, 2, 3, 4, 5, 8, 13, 16, 21, 22, 29, 31, 34, 46, 47, 52, 53].

Chapter 3

Noncommutative symmetric polynomials

3.1. Symmetric noncommutative polynomials as an S-algebra

This section is based on our paper [10]. In it we describe our results about the finite generation of the S-algebra of the symmetric noncommutative polynomials in d variables $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$. More specifically, we give answer to question 2.4.25, which was posed at the end of Section 2.4 by constructing a finite generation set.

Definition 3.1.1 ([6]). Let $n \in \mathbb{N}^+$ be a non-zero integer. An *(integer) partition* of n is a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-zero integers $\lambda_1, \ldots, \lambda_k$, such that

$$n = \lambda_1 + \lambda_2 + \ldots \lambda_k \text{ and } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

If λ is a partition of n, we denote it by $\lambda \vdash n$.

Recall from the previous section 2.5 that the action of the symmetric group $\mathrm{Sym}(d)$ splits the set of monomials $\langle X_d\rangle$ into orbit and the homogeneous component $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)^{(n)}$ of degree n of the algebra of symmetric polynomials, as a vector space, has a basis $\sum v$, where $v \in \langle X_d\rangle^{(n)}$. We can chose such an element in the orbit $G(v)$, so that

$$\deg_{x_1}(u) \geq \deg_{x_2}(u) \geq \cdots \geq \deg_{x_d}(u)$$

and $\sum u = \sum v$. We attach to it the partition $\lambda = \Big(\deg_{x_1}(u), \ldots, \deg_{x_1}(u)\Big)$.

The S-action 2.4.4 $\circ$ allows us to further improve how u looks. There exists a permutation $\sigma \in \mathrm{Sym}(n)$, such that

$$\sum u = \sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_d^{\lambda_d} \circ \sigma.$$

We denote $p_\lambda = \sum x_1^{\lambda_1} \ldots x_d^{\lambda_d}$ and so

$$\sum u = \sum p_\lambda \circ \sigma.$$

For partition $\lambda = (n)$, we call

$$p_{(n)} = \sum x_1^n + x_2^n + \cdots + x_d^n, \quad n = 1, 2, \ldots$$

the (n-th) power sum and for $\lambda = (1^n)$, $n \leq d$,

$$p_{(1^n)} = \sum_{\sigma \in \mathrm{Sym}(d)} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$$

are the noncommutative elementary symmetric polynomials. The next result is true for field K of any characteristic.

Lemma 3.1.2. *Let K be any field. The S-algebra of the symmetric noncommutative polynomials in d variables is generated by the power sums $p_{(n)}$, $n = 1, 2, \ldots$.*

Lemma 3.1.2 helps us shrink the generating set of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$, but still leaves us with an infinite generating set $\{p_{(n)} \mid n \in \mathbb{N}\}$. In order to shrink it further, we need a way to obtain all the noncommutative power summs from a finite set. Recall that in the commutative case, this is done by the Newton identities (see, for example, [50], or the Wikipedia page [1]). In $K[X_d]^{\mathrm{Sym}(d)}$, if we denote $e_1, \ldots, e_d$ to

[1] https://en.wikipedia.org/wiki/Newton%27s_identities

be the elementary symmetric polynomials 2.4, and p_i, $i = 1, 2, \ldots,$ to be the power summs, we have that

$$\begin{aligned} ke_k &= \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i, \qquad & k \leq d, \\ 0 &= \sum_{i=k-n}^{k} (-1)^{i-1} e_{k-i} p_i, \qquad & k > d. \end{aligned} \tag{3.1}$$

Before we introduce our noncommutative analogue for Newton's identities, recall that a riffle shuffle[2] is a permutation, originating in shuffling playing cards, where a deck of cards is split into two decks and then the two smaller decks are interleaved. Similarly, we define

Definition 3.1.3. For $k \leq d$, we denote by Sh_i, $i = 1, 2, \ldots, k$ the set, consisting of all the "shuffle" permutations $\sigma \in \mathrm{Sym}(k)$, meaning permutations σ, such that σ^{-1} fix the order of $1, 2, \ldots, k-i$ and the order of $k-i+1, k-i+2, \ldots, k$. For $k > d$, Sh_i, $i = 0, \ldots, d$ consists of all the permutations $\sigma \in \mathrm{Sym}(k)$, which fix $d+1, \ldots, k$, and σ^{-1} preserves both the orders of $1, 2, \ldots, d-i$ and $d-i+1, d-i+2, \ldots, d$.

Recall that we defined S-action in 2.4.4 by $\sigma \in \mathrm{Sym}(n)$ to be

$$y_1 y_2 \ldots y_n \circ \sigma = y_{\sigma^{-1}(1)}\sigma = y_{\sigma^{-1}(2)} \ldots \sigma = y_{\sigma^{-1}(n)} \text{ for } y_i \in X_d, i = 1, \ldots, n.$$

The right action was by acting on the positions with σ^{-1} and that's why we put restraints on σ^{-1} in definition 3.1.3.

Lemma 3.1.4. *In the free associative S-algebra $\left(K\langle X_d\rangle, \circ\right)$, we have the following two identities:*

$$k! p_{(k)} + (-1)^k k p_{\left(1^k\right)} + \sum_{i=1}^{k-1} (-1)^{k-i} i! \left(p_{\left(1^{k-i}\right)} p_{(i)} \circ \sum_{\sigma \in Sh_i} \sigma \right) = 0, \quad k \leq d,$$

and

$$d! p_{(k)} + (-1)^d d p_{\left(1^d\right)} p_{(k-d)} + \sum_{i=1}^{d-1} (-1)^{d-i} i! \left(p_{\left(1^{d-i}\right)} p_{(k-d+i)} \circ \sum_{\sigma \in Sh_i} \sigma \right) = 0$$

[2] https://en.wikipedia.org/wiki/Riffle_shuffle_permutation

for $k > d$.

With the two lemmas 3.1.2 and 3.1.4, we can finally give the promised answer to 2.4.25 in the case of G being the symmetric group of order d.

Theorem 3.1.5. *For fields K of characteristic either 0 or greater than the number of variables, the S-algebra of the symmetric noncommutative polynomials in d variables $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ is freely generated by the elementary symmetric polynomials $p_{(1^i)}$, $i = 1, 2, \ldots, d$.*

In our paper we included several proofs for the special case of $d = 2$.

Theorem 3.1.6. *Let K be a field with characteristic $\mathrm{char}(K) \neq 2$. The S-algebra $\left(K\langle X_2\rangle^{\mathrm{Sym}(2)}, \circ\right)$ of the symmetric noncommutative polynomials in two variables is finitely generated.*

At the end of our paper [10], we formulated the following conjecture:

Conjecture 3.1.7. *Let $\mathrm{char}\, K = p \leq d$. Then the S-algebra of the symmetric noncommutative polynomials in d variables $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ is not finitely generated.*

We gave prove to it in our paper [11] and will see it in the next section 3.2.

3.2. Infinite generation and minimal generating set for the S-algebra of noncommutative symmetric polynomials in the case $p \leq d$

This section contains the results of our paper [11], where goal is to prove Conjecture 3.1.7 and to go further by constructing a minimal generating set for the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$.

Remark 3.2.1. If $d' > d$, the projection $K\langle X_{d'}\rangle \to K\langle X_d\rangle$ which maps the extra generators $x_{d+1}, \ldots, x_{d'}$ to 0 induces a surjective map between the S-algebras of the symmetric polynomials. Because of that, it is enough to only prove that the S-algebra of the symmetric noncommutative polynomials $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ is not finitely genereated for $\mathrm{char}(K) = p = d$ only. So we assume that $p = d$.

We start by introducing some definitions that we will need in order to establish our goal.

Definition 3.2.2 ([37]). Let K be a field and A be an unitary associative algebra over K. We say that the algebra A is *augmented*, if there is a homomorphism of algebras $\varepsilon : A \to K$, called *augmentation map*. The kernel $\mathrm{Ker}(\varepsilon)$ is called *augmented ideal.*

Example 3.2.3 ([37]). If G is a group and $K[G]$ is the group algebra (the free module over K with basis G), then the map

$$\varepsilon : \sum r_i g_i \mapsto \sum r_i$$

is augmentation map and its kernel is augmentation ideal.

Example 3.2.4. If A is graded algebra over a field K, $A = A_0 \oplus A_1 \oplus$ and $A_0 = K$, the homomorphism $\varepsilon : A \to K$ which maps an element into its homogeneous component of degree 0 is augmentation.

The last example can be applied to the associative algebra $K\langle X_d \rangle$ and in that case a polynomial $f \in K\langle X_d \rangle$ maps to its constant term,

$$f = \sum a_s x_{i_1}^{j_1} \dots x_{i_s}^{j_s} \mapsto a_0.$$

This can be used in the case of S-algebra $\big(K\langle X_d\rangle, \circ\big)$ and that's exactly how we will use it.

If A is augmented algebra and we denote I^+ to be the augmentation ideal of A, we will also be interested in studying $I^+/(I^+)^2$. In [37] the authors call $I^+/(I^+)^2$ the *space of indecomposables* of A.

Example 3.2.5. Let G be a group and $G' = [G, G]$ be its commutator subgroup. Let I^+ be the augmentation ideal of the integral group ring $\mathbb{Z}[G]$. Then

$$I^+/(I^+)^2 \cong G/G'.$$

The group G/G' is called the *abealization* of G.

$$M_d \coloneqq \left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)^+ / \circ \left(\left(\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)^+\right)^2\right)$$

denotes the factor of the augmentation ideal by it's square. We have that

$$M_d = \bigoplus_{n\in\mathbb{N}} M_d^{(n)}$$

and thus M_d is naturally graded. Each of its homogeneous components $M_d^{(n)}$ is a $\mathrm{Sym}(n)$-module and so there is a natural S-action $\circ$ on M.

Lemma 3.2.6. *The vector space M_d is generated both as a $\circ$-module and vector space, by the images of the power sums*

$$p_i = x_1^i + \cdots + x_d^i \text{ for } i = 1, 2, \ldots.$$

Note that this doesn't imply the infinite generateness of $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ as some power sums might be projected to zero. Theorem 3.1.5 shows that for $p > d$ the power sums p_i for $i > d$ are projected to 0.

We now consider the abelianization map $\pi : K\langle X_d\rangle \to K[X_d]$ and the map it induces on the subalgebra of noncommutative polynomials

$$\pi : K\langle X_d\rangle^{\mathrm{Sym}(d)} \to \pi\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}\right).$$

Lemma 3.2.7. *The abelization map π sends a generating set of the S-algebra $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ to a generating set of its image - the commutative algebra*

$$\pi\left(\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)\right) \subset K[X_d]^{\mathrm{Sym}(d)}.$$

We have that

$$\sum u = \sum_{\sigma\in\mathrm{Sym}(d)\setminus H_u} u^g = \sum_{\sigma\in\mathrm{Sym}(d)\setminus H_u} g(u).$$

We need to be careful where u lies, as the stabilizer of u in $K[X_d]$ and $K\langle X_d\rangle$ is different.

Lemma 3.2.8. *For any monomial $u \in K\langle X_d\rangle$, there exist a integer constant $c_u \in \mathbb{N}$, such that*

$$\pi\Big(\sum u\Big) = c_u\Big(\sum \pi(u)\Big).$$

In the case of $p = d$, c_u is 0 if and only if $\pi(u) = x_1^s x_2^s \dots x_p^s$ for some $s \geq 1$.

Lemma 3.2.9. *Let* $\mathrm{char}(K) = p = d$*. Then, the commutative algebra*

$$\pi\Big(K\langle X_d\rangle^{\mathrm{Sym}(d)}\Big) \subset K[X_d]^{\mathrm{Sym}(d)} = K[e_1, \dots, e_d]$$

is spanned (as a K-vector space) by all the products of elementary symmetric polynomials $e_1^{m_1} \dots e_d^{m_d}$, except all the powers e_p^m of e_p, $m \geq 1$.

We can apply all the Lemmas to prove the main result we stated in the beginning of the section.

Theorem 3.2.10. *The S-algebra of the symmetric noncommutative polynomials $\Big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\Big)$ is not finite generated for fields K of non-zero characteristic, less or equal to the number of variables d.*

Example 3.2.11. We will show that p_3 does not belong to the S-algebra F of $\Big(K\langle X_2\rangle^{\mathrm{Sym}(2)}, \circ\Big)$, generated by the first two power sums p_1 and p_2.

Theorem 3.2.12. *If $0 < p = \mathrm{char}(K) \leq d$, the set $\{p_i \mid i = 1, 2, \dots\}$ of all the power sums is a minimal generating set for the S-algebra $\Big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\Big)$.*

Chapter 4

Noncommutative alternating polynomials

This section contains yet unpublished results. Our goal here is to extend our results for symmetric polynomials to alternative ones.

Lemma 4.0.1. *Any noncommutative alternative polynomial $f \in K\langle X_d\rangle^{\mathrm{Alt}(d)}$ can be written as $f = f_1 + f_2$, where f_1 is symmetric polynomial in d non commuting variables and f_2 is alternating, i.e. f_2 changes sign whenever we exchange any two variables.*

If $u \in \langle X_3\rangle$ is a monomial in 3 noncommuting variables, by $\sum\limits_{\mathrm{Alt}} u$ we denote the alternating sum

$$\sum_{\sigma\in \mathrm{Alt}(3)} (-1)^{\sigma} u^{\sigma}.$$

It is obvious that every alternating polynomial can be expressed in terms of such sums. If $u \in \langle X_3\rangle$ is a monomial of degree n, $u = x_{i_1}^{\mu_1} i_2{}^{\mu_2} \dots i_k{}^{\mu_k}$, where $\mu_1 + \mu_2 + \dots \mu_k = n$ and $i_1, i_2, \dots, i_k \in 1, 2, 3$, there exists a permutation $\rho \in \mathrm{Sym}(d)$, such that $u = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \circ \rho$, where $\lambda_1 + \lambda_2 + \lambda_3 = n$, $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and

$$\sum_{\mathrm{Alt}} u \circ \rho = \sum_{\sigma\in \mathrm{Alt}(3)} (-1)^{\sigma} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} x_{\sigma(3)}^{\lambda_3} \circ \rho.$$

Note that he leading monomial of any alternating polynomial (with Koryukin's $S-$ action) is either $x_1^{\lambda_1} x_2^{\lambda_2}$ or $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}$, where $\lambda_1 \geq \lambda_2 \geq 1$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, respectively.

If alternating polynomial has a leading monomial of the form $\sum\limits_{\text{Alt}(3)} x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}$ for which $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, then

$$\sum_{\text{Alt}(3)} x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3} = \frac{1}{3}(-1)^{n-1}\Big(\sum_{i=1}^{n} \sum_{\substack{k+s+t=i\\ k\leq\lambda_1, s\leq\lambda_2, t\leq\lambda_3}} \sum_{\tau\in Sh_{k,s,t}} (-1)^{n-i}x_1^{\lambda_1-k}x_2^{\lambda_2-s}x_3^{\lambda_3-t}p_i\circ\tau\Big). \tag{4.1}$$

where the sum runs over all the shuffles of k $z's$ into the x_1's, s z's into the x_2's and t z's into the x_3's.

Remark 4.0.2. The terms for $i = n$ and $i = n-1$ in (4.1) are equal to 0. That is obvious for $i = n$ and not hard to see for $i = n-1$.

Lemma 4.0.3. *The algebra $K\langle X\rangle^{\text{Alt}(3)}$ is generated, as a S-algebra, by the elementary symmetric polynomials e_1, e_2, e_3 and the polynomials $\sum\limits_{\text{Alt}} x_1^{\lambda_1}x_2^{\lambda_2}$.*

We denote $s_k = \sum\limits_{\text{Alt}(3)} x_1^{k-1}x_2$.

Lemma 4.0.4. *The algebra $K\langle X\rangle^{\text{Alt}(3)}$ is generated, as a S-algebra by the elementary symmetric polynomials e_1, e_2 and e_3, as well as the alternating polynomials $s_k = \sum\limits_{\text{Alt}} x_1^{k-1}x_2$, $k = 2, 3, \ldots$.*

The final step is reducing the generating set s_k, $k = 2, 3, \ldots$ to a finite set. For this, observe that for $\sigma = (n, n-1)(1, n-2)$, we have that

$$\begin{aligned}&(p_1 s_{n-1})\circ\sigma + p_{n-2}s_2 + p_{n-3}s_3 =\\ &= 2s_n + \sum_{\text{Alt}} x_1^{n-2}x_2x_3 + \sum_{\text{Alt}} x_1^{n-3}x_2^2x_3 - \sum_{\text{Alt}} x_1^{n-3}x_2x_3x_1.\end{aligned}$$

From this we obtain

$$\begin{aligned}s_n = \frac{1}{2}\Big((p_1 s_{n-1})\circ\sigma + p_{n-2}s_2 + p_{n-3}s_3 - \sum_{\text{Alt}} x_1^{n-3}x_2^2x_3\\ - \sum_{\text{Alt}} x_1^{n-2}x_2x_3\circ\big(\operatorname{id} - (n-2, n-1, n)\big)\Big).\end{aligned} \tag{4.2}$$

Since both $\sum\limits_{\text{Alt}} x_1^{n-3}x_2^2x_3$ and $\sum\limits_{\text{Alt}} x_1^{n-2}x_2x_3$ are obtained by polynomials of lower degree (see 4.1), that gives us a finite generating set.

Theorem 4.0.5. *Let* $\text{char(K)} = 0$ *or* $\text{char(K)} = p > 3$*. Then the* S*-algebra of the alternative polynomials in* 3 *noncommuting variables*$\left(K\langle X_3\rangle^{\text{Alt}(3)}, \circ\right)$ *is generated as an* S*-algebra by the elementary symmetric polynomials* $p_{1^i}, i = 1, 2, 3,$ *together with the alternating polynomials* s_2 *and* s_3*.*

Theorem 4.0.6. *The* S*-algebra* $\left(K\langle X_3\rangle^{\text{Alt}(3)}, \circ\right)$ *is not finitely generated for fields* K *of characteristic* 2 *or* 3*.*

Chapter 5

Conclusion

5.1. Main contributions

1. For a field K of arbitrary characteristic, it is proved that the S-algebra of the symmetric noncommutative polynomials in d variables has a generating set, consisting of the power sums $p_i = \sum_{k=1}^{d} x_k^i$ for $i = 1, 2, \dots$.

2. A noncommutative analogue for the Newton's identities is proved in the free associative S-algebra $\big(K\langle X_d\rangle, \circ\big)$. We relate the power sums p_i to the noncommutative elementary symmetric polynomials $e_{(1^i)} = \sum_{\sigma\in\mathrm{Sym}(d)} x_{\sigma(1)} \dots x_{\sigma(i)}$, for $i \leq d$.

3. A noncommutative analogue for the fundamental theorem of symmetric polynomials is proven. We prove that the elementary noncommutative polynomials e_i, $i = 1, \dots, d$, generate the S-algebra $\big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)$ for fields of characteristic 0 or greater than the number of variables d.

4. The question about infinite generation of $\big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)$ when the field K has a positive characteristic p, less or equal to the number of variables, is reduced to the case when the characteristic is equal to the number of variables.

5. It is proven that $M_d := \big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)^+ / \circ\left(\left(\big(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\big)^+\right)^2\right)$, obtained by the factoring the augmentation ideal of the symmetric noncommutative S-algebra by its square, is spanned both as a $\circ$-module and as a vector

space, by the power sums p_i for $i = 1, 2, \ldots$.

6. We prove that the abealization map $\pi : K\langle X_d\rangle \to K[X_d]$ sends a generating set of the S-algebra of the noncommutative symmetric polynomials $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ to generating set of its image, the commutative algebra $\pi\left(\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)\right) \subset K[X_d]^{\mathrm{Sym}(d)}$.

7. For $\mathrm{char}(K) = p = d$ we prove that $\pi\left(\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)\right)$ is spanned by all the products $e_1^{m_1} \ldots e_d^{m_d}$ of the elementary symmetric polynomials, except the powers e_p^m of the p-th power sum e_p.

8. We prove that for $d \geq \mathrm{char}(K) = p > 0$, the S-algebra of the noncommutative symmetric polynomials $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ is not finitely generated.

9. In the same setting for $\mathrm{char}(K) = p = d$ we prove that the power sums $\{p_i \mid i = 1, 2, \ldots\}$ are a minimal generating set for the S-algebra $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$. This is done by proving that the power sum p_n does not belong to the S-subalgebra of $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$, generated by the power sums $p_1, \ldots, p_{n-1}$.

5.2. Publications, related to the thesis

1. Boumova, S.; Drensky, V.; Dzhundrekov, D.; and Kassabov, M. (2022) "*Symmetric polynomials in free associative algebras*", Turkish Journal of Mathematics: Vol. 46: No. 5, Article 4. https://doi.org/10.55730/1300-0098.3225

2. Boumova, S.; Drensky, V.; Dzhundrekov, D.; Kassabov, M. (2023) "*Symmetric Polynomials in Free Associative Algebras—II*". Mathematics 2023, 11, 4817. https://doi.org/10.3390/math11234817

The results from the above publications, have been presented in the following talks:

1. *"Symmetric polynomials in noncommuting variables"*, Spring Science Session of Faculty of Mathematics and Informatics, Sofia, March 27, 2021.

2. *"On the symmetric polynomials in noncommuting variables"*, National Seminar on Coding Theory "Acad. Stefan Dodunekov", November 7-11, 2021.

3. *"Symmetric polynomials in d noncommuting variables"*, Annual Seminar on Algebra and Geometry, November 14-17, 2021.

4. *" Symmetric polynomials in free associative algebras"*, Spring Science Session of Faculty of Mathematics and Informatics, Sofia, March 26, 2022.

5. *"Symmetric polynomials in free associative algebras"*, Annual Seminar on Algebra and Geometry, August 28-September 2, 2021.

6. *"Symmetric polynomials in free associative algebras (Part 2)"*, National Seminar on Coding Theory "Acad. Stefan Dodunekov", Arbanasi, November 10-13, 2022.

7. *"Alternative polynomials in free associative algebras"*, Spring Science Session of Faculty of Mathematics and Informatics, Sofia, March 25, 2023.

5.3. Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his coauthors. The usage of results of other scientists is accompanied by suitable citations.

This thesis is not used for conferring a scientific or academic degree in any other university or institute.

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> *"The mediocre teacher tells. The good teacher explains. The superior teacher demonstrates. The great teacher inspires."* - William Arthur Ward

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