

# REPORT

of the procedure for defending the PhD Thesis:

”A Class of Toeplitz  $C^*$  - algebras”

by Nikolay Petrov Buyukliev

a self-study PhD student in the field of Higher Education 4 Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics, doctoral program ”Mathematical Analysis” , section ” Mathematical Analysis” , Faculty of Mathematics and Informatics, Sofia University ”St. Kliment Ohridski” .

The report is prepared by Assoc. Prof. DSc. Ognyan Borisov Christov, FMI, SU, as a member of the scientific juri, according to order No RD - 38-223 from 28.04.2023 of the Rector of Sofia University.

## 1. General characteristics of the dissertation and the presented documents.

The presented dissertation consists of 56 pages, organized in six chapters and a bibliography containing 52 titles.

In addition to the dissertation, the procedure documentation also includes CV, an abstract in Bulgarian and English, list of publications, reference contributions, statement of authorship, protocol of the preliminary defense of the dissertation, scientific papers, related to the dissertation etc.

## 2. Personal data about the candidate.

Nikolay Buyukliev graduated from a mathematics high school in Gabrovo in 1977 with a golden medal. The same year, he participated in the nineteenth IMO in Belgrade, Yugoslavia, where he won a silver medal. He has completed his higher education at FMM, Sofia University with Master’s degree in Mathematics under the supervision of Prof. Roni Levi with the diploma thesis ”Wiener-Hopf operators” .

He worked as an assistant, senior assistant and chief assistant in the section of Mathematical Analysis at FMI, SU since 1985. He led exercises in Differential and Integral Calculus 1 and 2 and Mathematical Analysis, lectures in Mathematical Analysis and Differential Geometry, a special course ”Introduction to Functional Analysis” for full-time students at FMI, SU, as well as lectures and exercises in Mathematical Analysis for part-time students at FMI and FzF of SU.

In co-authorship with Prof. R. Levi and Prof. R. Maleev, he has prepared freely distributable materials on Differential and Integral Calculus 1 and 2 under the EOS-2014 and EOS-2015 programs.

He was a self-study PhD student in the doctoral program ”Mathematical Analysis” , section Mathematical Analysis, FMI, SU from 2018 till 2022.

The scientific interests of N. Buyukliev are in the field of the theory of operators and K-theory. He has submitted 4 publications, related with his dissertation. One paper is

published in the ANNUAIRE DE L'UNIVERSITE DE SOFIA, two others accepted for publication *ibid.*, authenticated by the relevant note and one article submitted for publication, all of them not co-authored.

Notice that the first paper [42] is cited in [20] and [21].

I know Nikolay from our student years at FMM. As mentioned above, he was a scholar and a student with outstanding mathematical capabilities. Ever since, he has enjoyed exceptional authority and respect among colleagues and students.

### **3. Content analysis of the scientific achievement of the candidate, contained in the presented dissertation and the publications to it, included in the procedure.**

The dissertation is devoted to the development of the theory of Wiener-Hopf and Toeplitz  $C^*$  - algebras. Essentially, this is an area that requires non-trivial knowledge of functional analysis, cohomology algebra, Lie groups and algebras, etc.

**Chapter 1** is more of an Introduction in which the author explains the philosophy of research related to the study of  $C^*$  - algebras and gives a description of the results in the dissertation. **Chapter 2** recalls some necessary definitions and results concerning algebras, Fredholm operators and their indices, groupoids and their algebras, K-theory of algebras, etc. The original results of the dissertation are presented in Chapters 3 - 6.

To make clear what was done in the dissertation, very briefly, without going into the complex details, I will recall the philosophy of such studies, as described by the author in Chapter 1.

Let  $G$  be a group and  $P$  be a subgroup  $P \subset G$ . Wiener-Hopf operators are defined in the corresponding spaces (Toeplitz operators in the case of discrete groups). The following problems can be set for the  $C^*$  - algebras of these operators:

Problem I: to describe the chain of ideals in the  $C^*$  - algebra and their factors, respectively.

Problem II: to calculate the K-theory of the  $C^*$  - algebra and more importantly - to write an index formula for the corresponding Fredholm operators.

Both problems are actual and complex, and there are their implementations in small dimensions and mainly for abelian groups.

One of the approaches in the study of these problems is called groupoid: the  $C^*$  - algebra of the Wiener-Hopf operators (respectively, the Toeplitz operators) is identified with a  $C^*$  - algebra of an explicitly constructed groupoid  $\mathcal{G}$  and then problems I and II are studied for the resulting groupoid  $C^*$  - algebra  $C^*(\mathcal{G})$ . This approach is adopted in the dissertation.

In **Chapter 3** the groupoid  $C^*$  - algebras  $\mathcal{T} = C^*(\mathcal{G})$  are considered, where  $\mathcal{G}$  is a Wiener-Hopf groupoid, that is,  $\mathcal{G}$  is a reduction of a group of transformations  $\mathcal{G} = (Y \times G)|X$ , where  $X, Y$  are suitable topological spaces and  $G$  is a locally compact group. The continuous linear sections are explicitly constructed and with their help the indexes of certain Fredholm operators are calculated. The results of this chapter are published in [43].

In **Chapter 4** the results of the previous chapter are applied. The indexes of certain Fredholm operators are calculated in dense subalgebra of  $C^*$  - algebra of the Wiener - Hopf

groupoid, naturally imposing some restrictions on the linear section. The results of this chapter are published in [45].

In **Chapter 5** the author deals with the  $C^*$  - algebra  $\mathcal{B}(\mathbb{R}^n, P)$  of Wiener-Hopf operators, associated with the group  $G = \mathbb{R}^n$  and a polyhedral cone  $P \subset \mathbb{R}^n$ . The chain of ideals was found earlier, so here the K-theory is clarified. The first main result is the construction of a Fredholm operator with index 1. Next, the combination of the Mayer - Vietoris exact sequence and the standard six term exact sequence gives the needed K - theory.

Notice that the last result is generalized by Aldridge, who essentially uses the construction of a Fredholm operator with index 1. The results of this chapter are published in [42] and this work cited twice.

In **Chapter 6** the  $C^*$  - algebra of the Toeplitz operators in discrete three-dimensional Heisenberg group  $G = H_3(\mathbb{Z})$  is studied. According to the general scheme, a groupoid is constructed whose  $C^*$  - algebra is isomorphic to  $\mathcal{T}(H_3(\mathbb{Z}))$ .

The K-theory was considered earlier, so here the effort is focused on finding the chain of ideals. This chain of ideals, along with the corresponding factors, is written explicitly. The results of this chapter are published in [44].

In summary, the dissertation contains original results that enrich the theory of  $C^*$  - algebras and are subject to development.

#### **4. Approbation of the results.**

The results of the dissertation are presented in 4 publications, all of them not co-authored. This fully satisfies the minimum national requirements (under Art. 2b, para. 2 and 3 of ZRASRB) for obtaining a PhD degree in the scientific field and professional direction of the procedure.

There is no legally proven plagiarism in the submitted dissertation and the scientific papers on this procedure.

Nikolay has given talks at the following international conferences: Shumen 2008, Berlin 2010, Baku 2019, as well as at several FMI Spring Science Sessions.

#### **5. Quality of the Abstract.**

The abstract correctly reflects the results and the contributions of the dissertation.

#### **6. Critical notes and recommendations.**

I have no critical remarks.

## 7. Conclusion.

After getting acquainted with the dissertation presented in the procedure and the accompanying scientific papers and based on the analysis of their significance and the scientific contributions contained in them, **I confirm** that the presented dissertation meets the requirements of the Bulgarian legislation and the respective Regulations of Sofia University "St. Kliment Ohridski" for obtaining a PhD degree in the scientific field 4 Natural sciences, mathematics and informatics and professional field 4.5. Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional field and no plagiarism has been established in the scientific papers submitted in the procedure.

**I strongly recommend that the scientific juri to suggest to the Faculty Council of FMI, SU "St. Kliment Ohridski" to award Mr. Nikolay Petrov Buyukliev the educational and scientific degree "Doctor" in the scientific field 4 Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics.**

May 14, 2023

Prepared the report :  
Assoc. Prof. DSc Ognyan Christov