Sofia University
"St. Kliment Ohridski"
Faculty of Mathematics and Informatics

A Class of Toeplitz C^*-algebras

Nikolay Petrov Buyukliev

ABSTRACT
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1 Relevance of the topic and review of the main results in the field

1.1 The class of clasical Toeplitz operators

The first class of Toeplitz operators considered were the operators associated to the unit circle in the complex plane.

Let T denote the unit circle in the complex plane, equipped with Haar measure μ. Consider the Hilbert space $L^2(T)$ of square-integrable functions on T. The functions $\{e_n(t) = e^{int} : n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2(T)$. Define $H^2(T)$ to be the closed subspace, spanned by $\{e_n : n \geq 0\}$.
Let φ be a continuous function on T. Define the multiplication operator M_φ on $L^2(T)$ by $M_\varphi(f) = \varphi.f$ Let π denote the orthogonal projection from $L^2(T)$ on $H^2(T)$. Define the Toeplitz operator T_φ on $H^2(T)$ by the formula

$$T_\varphi = \pi M_\varphi.$$

Define the Toeplitz algebra $\mathcal{T}^1$ to be the C^*-algebra generated by the operators T_φ. The algebra $\mathcal{T}^1$ is generated by the operator T_z, and T_z is an isometry. So $\mathcal{T}^1$ is just the C^*-algebra, generated by an isometry. In **[3]** Coburn determines the structure of $\mathcal{T}^1$.In Theorem 1. he proves that $\mathcal{T}^1$ contains $\mathcal{K}$– the ideal of compact operators, and in Theorem 2. – that $\mathcal{T}^1/\mathcal{K} \cong C(T)$. Thus Coburn obtains the following exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} \mathcal{T}^1 \xrightarrow{\gamma} \mathcal{T}^1/\mathcal{K} \cong C(T) \longrightarrow 0 \tag{0.1}$$

This exact sequence immediately yields the following criterion:
Theorem (Coburn, **[3]**). Operator $T \in \mathcal{T}^1$ is Fredholm if and only if $\gamma(T) \in C(T)$ is nonvanishing.

There is also the following index formula:
Theorem (Gohberg– Krein, **[1]**, **[2]**). Let $T \in \mathcal{T}$ be a Fredholm operator. Then the index of T equals the negative of the winding number of $\gamma(T)$.

These results provide index results for another class of operators. Let $\mathbb{Z}$ denote integers. Consider $l^2(\mathbb{Z})$. The functions $\{e_n(k) = \delta_{nk} : n \in \mathbb{Z}\}$ form an orthogonal basis in $l^2(\mathbb{Z})$. Let $H^2(\mathbb{Z})$ be the closed linear span of $\{e_n : n \geq 0\}$, and let π be the orthogonal projection from $l^2(\mathbb{Z})$ onto $H^2(\mathbb{Z})$. Next, given $n \in \mathbb{Z}$, define the translation operator $M_n : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ by $M_n f(k) = f(n+k)$, and define the operator

$$T_n : H^2(\mathbb{Z}) \longrightarrow H^2(\mathbb{Z}) \qquad T_n = \pi M_n.$$

The Fourier transform gives an isomorphism between $L^2(T)$ and $l^2(\mathbb{Z})$. Under this isomorphism $H^2(T)$ corresponds to $H^2(\mathbb{Z})$. Moreover, Fourier transform is a unitary equivalence splitting T_{x^n} to T_n. So, the C^*-algebra, generated by T_n is isomorphic to the Toeplitz algebra $\mathcal{T}^1$ via this unitary equivalence.

During the last fifty years, there has been an increasing interest in the problem of finding the structure of C^*-algebras, generated by multivariable Wiener-Hopf and Toeplitz operators. It is quite straightforward to generalize the above setting to several variables.

1.2 Toeplitz C^*-algebras investigated in the thesis

In this thesis I discuss more general types of Toeplitz operators:
Let G denote a second countable, locally compact group with identity e and left Haar measure λ.
Fix a closed, normal subsemigroup P of G, which generates G and contains e.
For $f \in C_c(G)$ we define the Wiener-Hopf operator W_f on $L^2(P)$ by the formula

$$W_f\xi(t) = \int_G f(s)\xi(ts)1_P(ts)d\lambda(s), \qquad \xi \in L^2(P)$$

Observe, that W_f is an immediate generalisation of the operators of $\mathcal{T}^1$. The C^*-algebra, generated by $\{W_f : f \in C_c(G)\}$ will be denoted by $\mathcal{B}(G, P)$ or $\mathcal{T}(G, P)$, or simply by $\mathcal{B}$ or $\mathcal{T}$. It shall be referred as the C^*-algebra of Wiener-Hopf operators, associated with G and P. Whenever G is a discrete group, $\mathcal{T}$ will be called a C^*-algebra of Toeplitz operators (associated with G and P) [1]

1.3 The programme we propose

The programme we propose to study these algebras is the following:

• Construct a groupoid $\mathcal{G}$, such that the algebra $\mathcal{B}$ is isomorphic to the groupoid C^*-algebra $C^*(\mathcal{G})$.

• Determine the lattice of two-sided ideals of $\mathcal{B}$. Determine a composition series of $\mathcal{B}$ and compute its subquotients. Determine the type of $\mathcal{B}$. Whenever $\mathcal{B}$ is type I algebra, obtain a parametrisation of the spectrum of $\mathcal{B}$ and exhibit a topology on it.

• Compute the K-theory of ideals of $\mathcal{B}$, corresponding quotients and the whole $\mathcal{B}$.

• Find Fredholm criteria for operators in $\mathcal{B}$

• Obtain a formula that calculates the index of the Fredholm operators.

• Give a formula which expresses the Fredholm index in terms of topological data.

[1] Traditionally, whenever the group G is discrete, the term Toeplitz operator is used and whenever the group G is continuous, the term Wiener-Hopf operator is used.

Here both type of operators and algebras are treated the same way and we will not make any distinction between these two terms.

1.4 A review the development

Now we begin with reviewing the development of the study of C^*-algebras of the Toeplitz operators.

Pionering work in the field was done in a series of papers by Coburn and Douglas **[3]**, **[5]**,**[6]**, **[17]**, and **[18]**.

More advances were made by Upmeier **[19]**,**[20]** and **[21]** who determined a composition series of $\mathcal{B}$ for Hardy-Toeplitz algebras of all bounded symmetric domains. Moreover he developed an index theory, proving index formulae for the all Wiener-Hopf operators, associated to symmetric cones.

Another approach was taken by Dynin **[7]**, who used a procedure, based on the local decomposition of the cone P into a product relative to the fixed exposed face for the construction of the composition series.This presumes a certain tameness of the cone P, which he calls "complete tangibilyty".Due to the weakness of this assumptions, he received results about a ladge class of cones, including polyhedral, almost smooth and homogeneous cones.

The approach, which I follow in this thesis is due to Muhly and Renault. Over the last twenty years the groupoid algebra techniques have been used with spectacular success to study Toeplitz and Wiener-Hopf C^*-algebras $\mathcal{B}$.

Muhly and Renault describe in **[11]** a general procedure to produce a locally compact groupoid, whose groupoid C^*- algebra is just the Wiener-Hopf algebra and obtain composition series for the C^*- algebra $\mathcal{B}$ of Wiener-Hopf operators in the case when the cone P is polyhedral or symmetric.Their construction is based on the convinient compactification of the cone P.
Nica in **[8]** has given a uniform construction of this Wiener-Hopf compactification for all pointed and solid cones.
Recently A. Aldridge and T. Johansen in **[12]** and **[13]** studied an multivariable generalisation of the clasical Wiener-Hopf algebra, associated with convex cones in $\mathbb{R}^n$. Using groupoid methods they constructed composition series for the Wiener-Hopf C^*-algebra $\mathcal{B}$.
Aldridge and Johansen computed the spectrum of $\mathcal{B}$ and in the framework of Kasparov KK-theory give a topological expression of the index maps.

2 Author's contribution and content of the thesis

The thesis contains 56 pages, 3 of them are the list of used references. The list of references is composed by 52 items. Four of them are autor's.

2.1 In section 1.

In section 1 we give the Definition of the multivariable Toeplitz and Wiener-Hopf operators and the C^*-algebras which are generated by these operators. We explaine problems in the investigations of these algebras and the approaches we can use to solve them. Also we present a programme how to study these algebras.

2.2 In section 2.

In section 2 we collect some necessary preliminary definitions and results.
This section contains facts concerning C^*-algebras; groupoids and their C^*-algebras, some basic examples of groupoids, K-theory of C^*-algebras, and Cyclic cohomology.

2.3 In section 3.

In Section 3 we consider the groupoid C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where the groupoid $\mathcal{G}$ is a Wiener-Hopf groupoid, i.e., $\mathcal{G}$ is a reduction of a transformation group $\mathcal{G} = (Y \times G)|X$, where Y and X are suitable topological spaces.
We give a criterion for an operator $T \in C^*(\mathcal{G})$ to be Fredholm. Also we give a method to construct continious linear cross-sections using contractions in $\mathcal{G}^0$–the unit space of $\mathcal{G}$.
The results will be published in **[24]**.

In § 3.1 we establish a criterion for an operator $T \in C^*(\mathcal{G})$ to be Fredholm.

Let X be a regular compactification of P. Then $U = i(P)$ is an open and invariant subset of $X = \mathcal{G}^0$, and therefore we have an exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C^*(\mathcal{G}) \xrightarrow{\gamma} C^*(\mathcal{G})/\mathcal{K} = C^*(\mathcal{G}_{|F}) \longrightarrow 0$$

This short exact sequence gives a criterion for an operator $T \in \mathcal{T}$ to be Fredholm:
Theorem 3.1. An operator $T \in \mathcal{B}$ is Fredholm if and only if $\gamma(T)$ is invertible in $C^*(\mathcal{G}_{|F})$.

In § 3.2 we give a method how to construct a continious linear cross-section in Wiener-Hopf groupoid algebras using contractions in the unit space of $\mathcal{G}$

Let F be a closed and invariant subset of $X = \mathcal{G}^0$, and let $\lambda : X \longrightarrow F$ be a continuous contraction (i.e. $\lambda(x) = x$, $\forall x \in F$).

Theorem 3.1. In the above notations, the map

$$\psi(b)(x,n) = b(\lambda(x),n) \quad b \in C_c(\mathcal{G}_{|F})$$

is a continuous cross-section.

There is an analogue of this formula,which defines continuous linear cross- section, in the case when F is a union of finite number of closed and invariant subsets of X. Suppose that $F_1, F_2, \ldots, F_n$ are closed and invariant subsets of X and $F = \bigcup_{i=1}^{n} F_i$. For $\sigma \subset \{1,2,\ldots,n\}$, define $rank(\sigma)$ to be the number of the elements of σ and denote $F_\sigma = \bigcap_{i\in\sigma} F_i$. Let $\lambda_\sigma : X \longrightarrow F_\sigma$ be continuous contractions, such that $\lambda_{\sigma\cup\tau} = \lambda_\sigma \circ \lambda_\tau$ for all $\sigma, \tau \subset \{1,2,\ldots,n\}$.

Theorem 3.2. In the above notations, the map ψ given by the formula

$$\psi(b)(x,n) = \sum_{\emptyset\neq\sigma\subset.\{1,2,\ldots,n\}} (-1)^{rank(\sigma)+1} b(\lambda_\sigma(x),n) \quad b \in C_c(\mathcal{G}_{|F})$$

is a continuous cross-section.

2.4 In section 4

B Section 4 we impose additional constraints on a cross-section ψ, which give us the opportunity to define cyclic 1-cocycle and to obtain a formula that calculates the index of the Fredholm operators. The results will be published in **[27]**.

In **[23]** A.Connes gives a connection between $H^*_\lambda(A)$ and almost commutative maps ϱ (i.e. , maps $\varrho : A \longrightarrow L(H)$ such that $\varrho(x.y) - \varrho(.y)\varrho(y)$ are a trace class operators for all $x, y \in A$). Whenever ϱ is an almost commutative map, he constructs a cyclic 1-cocycle $\tau \in H^1_\lambda$ and proves that the index map $K_1(A) \longrightarrow \mathbb{Z}$ is given by the formula:

$$index(\varrho(U)) = \langle U, \tau \rangle \qquad \forall U \in GL(A).$$

In **[10]**, E. Park considers the C^*- algebra $\mathcal{T}^{\alpha,\beta}$, generated by the Toeplitz operators in the quarter plane. He proves in **[10]** , Prop. 2.3 that $\mathcal{T}^{\alpha,\beta}$ contains $\mathcal{K}$ – the ideal of the compact operators, and therefore he obtains the following exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} T^{\alpha,\beta} \xrightarrow{\gamma} T^{\alpha,\beta}/\mathcal{K} \longrightarrow 0$$

He constructs a continuous cross-section $\rho : T^{\alpha,\beta}/\mathcal{K} \longrightarrow T^{\alpha,\beta}$. The map ρ has a property that for all x and y in $T^{\alpha,\beta}/\mathcal{K}$, the operator $\rho(xy) - \rho(x)\rho(y)$ is compact.

Unfortunately, in this generality, this is the most one can say: $\rho(xy) - \rho(x)\rho(y)$ is not always a trace class operator. E. Park gets around this problem by restricting his choices of x and y to lie in a dense subalgebra $T_{\infty}^{\alpha,\beta}$ of $T^{\alpha,\beta}/\mathcal{K}$.

In § 3.2, we give a method how to construct continuous linear cross-sections ψ in Wiener-Hopf groupoid algebras, using contractions in the unit space of $\mathcal{G}$.
We have the same troubles as E. Park in **[10]** : the operator $\psi(xy) - \psi(x)\psi(y)$ is compact, but not always a trace class operator. The main purpose of this section is to give sufficient conditions for ψ, such that we are able to define a subalgebra $\mathcal{T}^{\infty}$, dense in $\mathcal{T}/\mathcal{K}$, with the property that $\psi(x.y) - \psi(x).\psi(y)$ is a trace class operator for all $x, y \in \mathcal{T}^{\infty}$.

In the § 4.1 we impose some additional constraints on ψ.

In the § 4.2 we define the algebras S and $\mathcal{T}^{\infty}$.

In the § 4.3 we prove that $\rho = \psi \circ \gamma$ is almost multiplicative on $\mathcal{T}^{\infty}$.

And in the final § 4.4 we prove a formula for the Fredholm operators, which calculates their index:

Theorem 4.4 Let $T \in \mathcal{T}$ be Fredholm operator. Let $\gamma(T)$ and $\big(\gamma(T)\big)^{-1}$ are in $\mathcal{T}^{\infty}$. Then the Fredholm index $ind(T)$ of T is given by the following formula:

$$ind(T) = tr\left[\psi\gamma(A)\psi(\gamma(A)^{-1}) - \psi(\gamma(A)^{-1})\psi\gamma(A)\right]$$

2.5 In section 5.

Whenever the lattice of ideals and corresponding quotients of the algebra $\mathcal{B}$ are known, the next problem is to determine the K-theory of $\mathcal{B}(\mathbb{R}^d, P)/\mathcal{K}$ and $\mathcal{B}$.
One possible approach is to use the exact sequence of Mayer-Viotoris and the standart six term exact sequence. This is done in **[25]** and in § 5 under additional assumption that the cone P is an "exhaustible cone".
In section 5 we prove that if the cone P satisfies some suitable geometric conditions (P to be "exhaustible"), then $K_*(\mathcal{B}(R^n, P) = (0,0)$, $K_*(\mathcal{B}(R^n, P)/\mathcal{K}) = (0, \mathbb{Z})$, and the index map is an isomorphism. The proof uses the Mayer-Vietoris exact sequence and the standart six term exact sequence in K-theory.
This results are published in **[25]**.
The main results in this section are:

• Consruction of a Fredholm operator with index 1.
This is done in § 5.2 This result is cited and used in **[14]** and **[15]**.

If we know an ideal J of the C^*-algebra $\mathcal{B}$ and the corresponding quotient then we can use the standard six term exact sequence (see § 2.4.1.). But even when K-theories of J and $\mathcal{B}/J$ are known this is not enough to obtain K-theory of $\mathcal{B}$. We need additional information for the maps in the diagram.

This note explains the importance of the following:
Theorem (§ 5.2. Theorem 5.2.) There exists a Fredholm operator $S \in \mathcal{B}(\mathbb{R}^d, P)$ such that $ind(S) = 1$

Corollary (**[25]**. Corollary 2.3) If $K_*(\mathcal{B}(\mathbb{R}^d, P)/\mathcal{K}) = (0, \mathbb{Z})$, then
(i)$K_*(\mathcal{B}(\mathbb{R}^d, P)) = (0, 0)$ and
(ii)The index map $ind : K_1(\mathcal{B}(\mathbb{R}^d, P)/\mathcal{K}) \longrightarrow K_0(\mathcal{K})$ is an isomorphism.

• Using an induction on the dimension in **[25]** is proved the following:
Theorem 5.2.(**[25]**. Theorem 3.5) Let P be an exhaustible cone in $\mathbb{R}^d$. Then:
(i) $K_*(\mathcal{B}(\mathbb{R}^d, P) = (0, 0)$.
(ii) $K_*(\mathcal{B}(\mathbb{R}^d, P)/\mathcal{K}) = (0, \mathbb{Z})$
(iii)The index map $ind : K_1(\mathcal{B}(\mathbb{R}^d, P)/\mathcal{K}) \longrightarrow K_0(\mathcal{K})$ is an isomorphism.

The definition of exhaustible cones is clumsy and unconfortable.A. Aldridge in **[14]** proved a result that is stronger than Theorem 5.2 in two directions: he does not need any assumptions about P and he prove a result of KK-theory, such that Theorem 5.2 is a corrolary. So this assumption is necessary only for the proof of above theorem and is not essential.
Here I cite the result of Aldridge:

Теорема 5.3 (Aldridge, **[14]** Thm 0.3) Let P be a polyhedral cone. Then $\mathcal{B}$ is KK-contractible.

Finally I note that in the works of Aldridge **[14]** and **[15]** is used the existence of Fredholm operator with index 1.

2.6 In section 6.

In section 6 we consider an interesting non-euclidian example given by the discrete Heisenberg group $H_3(\mathbb{Z})$ and its positive semigroup P.
The discrete three-dimensional Heisenberg group $H_3 = H_3(\mathbb{Z})$ can be realised as the multiplicative group of upper-triangular matrices:

$$H_3 = \left\{ s = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}, \quad P = \{s \in H_3 : \quad a, b, c \geq 0\}$$

We note that P is a normal subsemigroup, generating H_3, and $P \cap P^{-1} = \{e\}$.

In section 6 we represent $\mathcal{T}(H_3(\mathbb{Z}))$ as a groupoid C^*-algebra. We use this representation to show that $\mathcal{T}$ is not postliminal and to find a composition series with explicit ideals and subquotiens.
The results of this section are reported in IECMSA-2019-Baku and are submitted in Proceedings of the Bulgarian Academy of Sciences – see **[26]**.
The main rezults of this section is:

Theorem 6.3. The closed two-sided ideals of $\mathcal{T} \cong C^*(\mathcal{G})$ are

$$\{0\} \subset I_0 \subset I_1 \subset I_{1d} \subset I_2 \subset I_3 = \mathcal{T},$$

where $I_0 \cong \mathcal{K}$ and $I_3/I_2 \cong C^*(H_3(\mathbb{Z})$. Also we have $I_2/I_{1d} \cong (C(T^2) \times \mathcal{K})^2$, $I_{1d}/I_1 \cong C(T) \times \mathcal{K}$ and $I_1/I_0 \cong (C(T) \times \mathcal{K})^2$.

Corolary.6.4 The ideal I_2 is a type I C^*-algebra, but $\mathcal{T}$ is not a type I C^*-algebra.

2.7 References

The list of used references covers 3 pages and contains 52 items. Four of them are autor's.

2.8 Declaration of originality

The author declares that the thesis contains original results obtained by him. The usage of results of other scientists is accompanied by suitable citations.

Nikolay Petrov Buyukliev

2.9 Acknowledgements

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2.10 Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. **FMI Spring Science Sessions** 2019, 2020, 2022.

2. INTERNATIONAL SCIENTIFIC CONFERENCE Dedicated to the 105 Anniversary of John Atanasoff and John von Neumann, **Shoumen—2008**

3. 21^{st} International Workshop on Operator Theory and its Applications (**IWOTA 2010**)–Berlin, 2010.

4. International Eurasian Conference on Mathematical Sciences and Applications (**IECMSA-2019**), August 27-30, 2019, Baku, Azerbaijan, 2019

2.11 List of publications related to the thesis

1 Buyukliev, N., K-Theory of the C^*-Algebra of Multivariable Wiener–Hopf Operators Associated with some Polyhedral Cones in $\mathbb{R}^n$, Annuaire Univ. Sofia Fac. Math. Inform. 91(1997), no. 1–2, 115–125.

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