БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
Институт по Математика и Информатика (ИМИ)
Ул.”Акад. Г. Бончев”, бл. 8, 1113 София
Академик Петър Ст. Кендеров, асоцииран член на ИМИ
Тел.0887 499 044, ел.поща: vorednek@gmail.com

Review

according to the procedure for the defense of a dissertation on the topic:

"Variational analysis without variational principles"

(dissertation presented for obtaining the educational

and scientific degree "doctor")

Candidate: **Stoyan Raychev Apostolov**

Field of graduate studies: **4. Natural sciences, mathematics, and informatics**

Professional direction**: 4.5. Mathematics**

Doctoral program: **„Mathematical Analysis“,** department: **„ Mathematical Analysis “,**

Faculty of Mathematics and Informatics (FMI) of Sofia University “Kliment Ohridski” (SU)

The review was prepared by: **DMS Petar Stoyanov Kenderov**,
in my capacity as a member of the scientific jury, according to Order № РД-38-309/01.07.2022 г. of the Rector of Sofia University.

1. General characteristics of the doctoral thesis and the materials provided

The dissertation is written in English and consists of 72 pages. It is structured as follows: Introduction, Preliminaries, three chapters ("On transversality-type property", "Sufficient conditions for tangential transversality", "On Continuity of Optimum value map"), Conclusion and Bibliography. The latter contains 62 titles.

In addition to the dissertation work, I also got access to an impressive number of other documents (about 30 pieces), accompanying the conduct of the doctoral studies and showing remarkable neatness in compliance with a number of administrative requirements and measures to prevent plagiarism. This shows that the scientific supervisor (Professor Nadezhda Ribarska), the Department of Mathematical Analysis and the Faculty of Mathematics and Informatics have done their duties well - they have built a neat system for the smooth progress of the preparation of the doctoral students. The thorough and high-quality "internal review" provided by Prof. Nadia Zlateva should be especially noted. It presents in detail the main achievements of the candidate and makes relevant critical remarks.

Data and personal impressions about the candidate

From the scarce "Autobiography", written according to the "Europass" model, it can be seen that Stoyan Apostolov was born on 29.12.1992 in Panagyurishte. His diploma for "Bachelor of Applied Mathematics" was issued by FMI on 04.12.2017. He completed a master's degree in Applied Mathematics - Optimization at the same faculty in 2019. He then entered the doctoral program. While he was a student and doctoral student, he was used in the educational process at FMI as a part-time assistant (in Linear Algebra - from 2013 to 2016 and in Differential and Integral Calculus - from 2016 to 2021). From the certificates accompanying the bachelor's and master's diplomas, in which the courses and the marks of the examinations passed, are entered, it appears that 55 examinations were passed. There are only 7 non-excellent grades.

I have direct impressions of Mr. Apostolov's participation in seminars and projects financed by the National Science Fund and Sofia University. I also follow his publications.

2. Content analysis of the candidate's scientific and scientific-applied achievements, contained in the presented dissertation work and the publications to it, included in the procedure

The realization of the practical significance of the Maximum Principle of L.S. Pontryagin and the discovery of various proofs of it led to the rise of a number of new studies and the emergence of new mathematical fields. One such example is the Nonsmooth Analysis, where the tangential approximation of a set A at a given point x_0 of it is not a linear space, but a cone C^A, which depends on x_0. In many optimization problems, finding necessary conditions for extremum often rests on considerations of separation, at least locally, of two sets A and B with a common point x_0. Local separation of two sets at point x_0 here is understood as the existence of an open neighbourhood U of the point x_0, such that $A \cap B \cap U = \{x_0\}$, i.e. in a neighbourhood U of the point x_0 the two sets have no other common point except the point x_0. In a considerable number of investigations, especially in the finite-dimensional case, the proof of a necessary condition for an extremum rests on the intuitively clear fact that if two sets are locally separated at a given point x_0, then their tangential approximations C^A and C^B also „do not have much in common", i.e. in some sense „they do not intersect". The formal term for this is that „they are not strongly transversal". Strong transversality means that the set $C^B - C^A$ (understood as a Minkowski difference) is the whole space and that the cones C^A and C^B have no common point except the null element of the space. To what extent this necessary condition is "strong" and can be used to obtain meaningful information about the optimal object in the corresponding practical problem depends on the choice of the definition of tangential cones and on the meaning of the notion of strong transversality. This led to the emergence of an astonishing variety of definitions of tangential cones (Clarke's, Boulligand's, and many other authors), as well as to the emergence of various transversalities. The dissertation also follows this trend, but the emphasis, except in chapter 5, is on the different types of transversality, in infinite-dimensional spaces, where the strong transversality of cones C^A and C^{B} in the sense indicated above, no longer implies local nonseparation of the sets A and B. In my opinion, the candidate's most advanced contributions are in this direction. He examines a whole palette of different transversalities: transversality, tangential transversality, intrinsic transversality, and subtransversality, each one implying the next. The doctoral student correctly indicates (pages 64-65 of the dissertation) other contributions of his as the most significant. I briefly mention some of them here:

1. Finding a sufficient condition for tangential transversality (Theorem 4.3.2.), as well as establishing that this condition includes as special cases other known sufficient conditions for tangential transversality.

2. Applying this sufficient condition for deriving tangential transversality of the admissible set of a minimization problem and the epigraph of the objective function in three important cases (4.4.3., 4.4.6. and 4.4.8.), as well as derivation, in all three cases, of a "Lagrange multipliers – type" rule (4.4.9.).

3. A suitable definition of intrinsic transversality in the case of infinite-dimensional spaces is given. It is shown (3.3.9. and 3.3.10.) that this definition is equivalent, in the case of Hilbert spaces, to a definition introduced in 2020 in a paper by Thao, N. H.; Bui, H. T.; Cuong, N.D.; Verhaegen, M. (citation 60 of the bibliography at the end of the thesis).

4. It is proved (3.4.1. and 3.4.2.) that transversality and subtransversality can be used to characterize metric regularity and subregularity.

In my view, there are many more "inconspicuous" contributions that are part of the proofs and the considerations, and show remarkable ingenuity on the part of the candidate. Even this part of the dissertation alone fully meets the requirements for the award of the educational and scientific degree "Doctor".

The last (fifth in a row) chapter of the dissertation contains discussions of a completely different type. It is concerned with a series of minimization problems with the same objective function $g(y)$, defined in the topological space Y, where the admissible sets $D(p)$ of each particular problems lie in Y and depend on a parameter p. When the parameter set X is endowed with a topology, the following question is reasonable:

Under what conditions is the optimal value $s(p) = inf\{g(y): y \in D(p)\}$ a continuous function of the parameter p?

As a motivation for examining this question, the candidate points the removal of an inaccuracy in a statement present in a book and formulated in the thesis as Theorem 1.0.4:

Assume that for some point $p°$ of the topological space X, D is continuous at $p°$ and g is continuous on D($p°$). Then s is continuous at $p°$.

This formulation allows for an interpretation that makes the statement obviously false. When said „*g is continuous on D($p°$)*“, this usually is understood as a „continuity of the restriction of $g(y)$ on the set *D($p°$)*“. It would be better to use the wording of the form: „*g is continuous at the points of D($p°$)*”. This is what the candidate actually had in mind, as is observed from the proofs of the statements in this chapter, where exactly this form of continuity is used.

Actually, the research of the candidate in this chapter, really removes possible confusion, but in a different direction. The result for the continuity of $s(p)$ is true (and easy to prove), if the function $g(y)$ is continuous and the set-valued map $D: X => Y$ is compact valued and continuous with respect to the Vietoris topology in the space of closed subsets of Y. When Y is a metric space, the continuity of D could be considered with respect to the Hausdorff distance, in which case it is said that D is metrically continuous. The two types of continuity coincide when the values of $D(p)$ are compact. The mechanical transferring of the validity of the considered statement from the usual continuity to the case of metrically continuous mapping D with noncompact values is wrong, as shown by Example 5.2.1. The situation could be corrected if one requires uniform continuity of the function $g(y)$. A very good contribution of the candidate in this chapter is the identification of a notion, weaker than uniform continuity, Definition 5.2.2., which plays the same role. The proofs become significantly harder, but the results are more satisfying.

3. Approbation of the results

Part of the results of the dissertation have been published in three journals with an impact factor (Journal of Convex Analysis, Set-Valued and Variational Analysis, Comptes rendus de l'Academie bulgare des Sciences). One paper is submitted for publication.

Results of the dissertation have been reported at three international scientific forums (one in the Czech Republic and two in Bulgaria), as well as at two spring scientific sessions organized by the Faculty of Mathematics and Informatics (in 2021 and 2022). The article published in Set-Valued and Variational Analysis in 2022 has already been referenced in at least one of the two authoritative mathematics databases - zbMATH (https://zbmath.org/?q=an%3A7563239). The same applies to the article published in в Comptes rendus de l'Academie bulgare des Sciences (https://zbmath.org/?q=an%3A1488.49037) .

Although recently published, there are already favorable reviews of Apostolov's results. This is what was written in NGUYEN DUY CUONG's thesis submitted for defense at the Australian Federal University (https://www.researchgate.net/profile/Cuong-Nguyen-Duy-4/publication/353605783_Thesis-_NguyenDuyCuong/links/6105c1581e95fe241a9e63b3/Thesis-NguyenDuyCuong.pdf):

Very recently, another important property called "tangential transversality" has come to life. The property is used to obtain necessary optimality conditions for optimization problems in terms of abstract Lagrange multipliers and to formulate intersection rules for tangent cones in Banach spaces. For more discussions about the property as well as connections with other transversality properties, we refer the reader to these recent papers [3, 4, 30, 31].

In the papers mentioned under numbers 3 and 4 Apostolov is a coauthor or author.

The minimal national requirements according to the Law of promoting academic staff of Republic of Bulgaria are exceeded many times (for group Γ more than three times, and for the total - more than twice).

4. Qualities of the abstract

The abstract reflects well the results and considerations presented in the dissertation. I find it too bulky - almost half the dissertation.

5. Critical remarks and recommendations

I have no other critical remarks except the above-mentioned. My recommendation to the author is to continue to deal with the same topic, which is currently very relevant and has attracted strong and productive researchers.

It is reasonable, apart from the continuity of the optimal value function $\boldsymbol{s}(\boldsymbol{p})$, to consider the continuity of the set of points, where $\boldsymbol{g}(\boldsymbol{y})$ attains its minimum on $\boldsymbol{D}(\boldsymbol{p})$. There are a lot of results in this field, and it is not clear whether and how much of them will remain true in the weaker setting regarding the function $\boldsymbol{g}(\boldsymbol{y})$, considered in the thesis.

6. Conclusion

Having familiarized myself with the dissertation work presented in the procedure and the scientific works accompanying it and based on the analysis of their significance and the scientific and scientific-applied contributions contained in them, I confirm that the presented dissertation work and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Law for promoting of academic staff, the Rules for its application and the relevant Rules of the SU "St. Kliment Ohridski" for the candidate's acquisition of the educational and scientific degree "doctor" in the scientific field 4. Natural sciences, mathematics and informatics, and professional direction 4.5. Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional direction and no plagiarism has been found in the scientific works submitted for the competition.

Based on the above, I recommend the scientific jury to award to Stoyan Raichev Apostolov the educational and scientific degree "doctor" in the scientific field 4. Natural sciences, mathematics and informatics, professional direction. 4.5. Mathematics.

01.10.2022 г. Review prepared by:

DMS Petar Stoyanov Kenderov