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Variational analysis without variational principles

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ABSTRACT

of thesis
for conferring of academic and scientific degree
Doctor in professional field 4.5 Mathematics
Doctoral program "Mathematical analysis"

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Sofia, 2022

Modern variational analysis can be viewed as a further extension of the calculus of variations with focus on optimization of functions relative to various constraints and on sensitivity and stability of optimization-related problems with respect to perturbations.

One of the most characteristic features of modern variational analysis is the intrinsic presence of nonsmoothness, i.e., the necessity to deal with nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings. One reason for the growth of the subject has been, without a doubt, the recognition that nondifferentiable phenomena are more widespread, and play a more important role than smooth ones. Many fundamental objects frequently appearing in the framework of variational analysis (e.g., the distance function, value functions in optimization and control problems, maximum and minimum functions, solution maps to perturbed constraint and variational systems, etc.) are inevitably nonsmooth and also have set-valued structures requiring the development of new forms of analysis that involve generalized differentiation.

Even the simplest and historically earliest problems of optimal control are intrinsically nonsmooth, in contrast to the classical calculus of variations. Optimal control has always been a major source for applications of advanced methods of variational analysis and generalized differentiation.

Since the discovery of Pontryagin maximum principle, the dawn of optimal control theory, various versions of this result have been established, under different technical assumptions and with different proofs. As early as in 1965, Dubovickii and Miljutin realized the importance of convex approximations of closed sets for obtaining necessary optimality conditions for nonlinear problems in optimization. In a series of papers (cf., for example, the bibliography of [60]), the corresponding proofs are based on theorems for nonseparation of sets.

The classical concept of transversality has been applied successfully as a qualification condition in nonseparation results. Transversality is originally studied in the fields of mathematical analysis and differential topology. Recently, it has proven to be useful in variational analysis as well. As it is stated in [38], the transversality-oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization, as well as in proving sufficient conditions for linear convergence of the alternating projections algorithm (cf. [30]).

The classical definition of transversality at an intersection point of two smooth manifolds in a Euclidean space is that the sum of the corresponding tangent spaces at the intersection point is the whole space (cf. [32], [33]).

In order to prove the Pontryagin maximum principle (cf., for example, the bibliography of [60]), Hector Sussmann generalizes the definition of transversality for closed convex cones in $\mathbb{R}^n$: the cones C^A and C^B are transversal if and only if

$$C^A - C^B = \mathbb{R}^n$$

and strongly transversal, if they are transversal and $C^A \cap C^B \neq \{0\}$ (cf. Definitions 3.1 and 3.2 from [60]). In the finite-dimensional case, strong transversality of the approximating cones of the same type (either Clarke or Boltyanski) is a sufficient condition for local nonseparation of sets. The sets A, B containing a point x_0 are said to be locally separated at x_0, if there exists a neighborhood Ω of x_0 so that $\Omega \cap A \cap B = \{x_0\}$. In infinite-dimensional case, strong transversality of the approximating cones of the same type does not imply local nonseparation of sets, as shown by the following example.

Example 1. *Take the Hilbert cube*

$$A := \{(x_n) \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$$

and a ray $B := \{\lambda y : \ \lambda \geq 0\}$, *where* $y := (1/n^{3/4})_{n=1}^{\infty}$. *We have that the corresponding Clarke tangent cones* $\widehat{T}_A(\mathbf{0}) = l_2$ *and* $\widehat{T}_B(\mathbf{0}) = B$ *are strongly transversal, while the sets* A *and* B *are locally separated at 0.*

There are various transversality-type properties reflecting the various needs of the possible applications. In the literature there exist many notions generalizing the classical transversality as well as transversality of cones. Some of them are introduced under different names by different authors, but actually coincide. We refer to [51] for a survey of terminology and comparison of the available concepts. The central ones among them are *transversality* and *subtransversality*. They are also objects of study in the recent book [39]. One of the reasons for that is the close relation to metric regularity and metric subregularity, respectively.

The term subtransversality is recently introduced in [30] in relation to proving linear convergence of the alternating projections algorithm. However, as said earlier, it has been around for more than 20 years, but under different names – see Remark 4 in [51] and the references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cone and a sum rule for the limiting subdifferential. Here are equivalent definitions of transversality and subtransversality.

Proposition 2. *Let A and B be closed subsets of the normed space X. A and B are transversal at $\bar{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that*

$$d(x, (A - a) \cap (B - b)) \leq K(d(x, A - a) + d(x, B - b))$$

for all $x \in \bar{\mathbf{B}}_\delta(\bar{x})$ and $a, b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$.

If $a = b = 0$ in the above inequality, the sets are called subtransversal.

Another quite remarkable feature of subtransversality, was investigated in [9]. It turns out that subtransversality implies a rather general nonseparation result which is crucial for obtaining necessary optimality conditions of Pontryagin maximum principle type (including optimal control problems with infinite-dimensional state space). Moreover, subtransversality is a natural assumption for proving abstract Lagrange multiplier rule.

Another notion of tranversality - tangential transversality, was introduced recently by Bivas, Krastanov and Ribarska in [9]. The authors arrived to the study of transversality of sets when investigating Pontryagin's type maximum principle for optimal control problems with terminal constraints in infinite dimensional state space.

Definition 3. *Let A and B be closed subsets of the metric space X. We say that A and B are tangentially transversal at $\bar{x} \in A \cap B$, if there exist $M > 0$, $\delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B$, there exist sequences $t_m \searrow 0$, $\{x^A_m\}_{m\geq 1}$ in A and $\{x^B_m\}_{m\geq 1}$ in B such that for all m*

$$d(x^A_m, x^A) \leq t_m M, \quad d(x^B_m, x^B) \leq t_m M, \quad d(x^A_m, x^B_m) \leq d(x^A, x^B) - t_m \eta \,.$$

Besides the aformentioned results, the authors also established intersection rules for tangent cones in Banach spaces and some relations to masiveness of sets. Many questions about tangential transversality remained open (see [9], p. 28).

These results inspired one of the lines of research in the thesis, which is connected to the application of subtransversality and tangential transversality for obtaining necessary optimality conditions in terms of abstract Lagrange multipliers.

The intriguing thing here is to verify the subtransversality assumption in nontrivial cases. Our aim is to find some conditions which are sufficient for subtransversality of two sets. However, the approach we take is proving tangential transversality instead of subtransversality. It happens that usually tangential transversality is easier to verify than subtransversality when the information known concerns the tangential structure of the sets.

We present a general sufficient condition for tangential transversality (Theorem 26). The underlying idea is that in many cases the uniformness of the local approximation of a closed set can be used instead of some suitable compactness assumption. This is especially important in the infinite-dimensional case.

We motivate the usefulness of the obtained general results by providing some applications. One of them is finding a Lagrange multiplier when one of the sets is the epigraph of a function which is Lipschitz in one of the variables, uniformly with respect to the other.

The main application we obtained, in fact the starting point of this research, was the famous Aubin condition from [15] for the basic problem of the calculus of variations. We formulate an abstract (infinite-dimensional) version of this condition. This abstract version inspired the rest of the results in this chapter. We show that if a function (actually its epigraph) satisfies this assumption and the constraint has a specific form (tailored after the constraint in the basic problem of the calculus of variations as an infinite dimensional optimization problem), one can find a Lagrange multiplier. Sure, the proof makes use of our main result. It is worth noting that in our abstract version of Aubin condition, compactness of the operator is not necessary. For our argument, it is sufficient to assume that the image under the operator L of the correcting set is totally bounded in X. In fact, the case when L is the integration operator from $Y = L_1([a,b],\mathbb{R}^n)$ to $X = L_\infty([a,b],\mathbb{R}^n)$ could be important for future applications of our results. Clearly, this operator is bounded but not compact, and it maps weakly compact sets in Y to totally bounded sets in X, thus allowing to use weakly compact sets as "correcting sets“. This investigation has been developed in [46]. To further motivate our main result, we show that some known sufficient conditions for tangential transversality can be obtained as its particular cases. Namely, we obtain Theorem 5.2 from [9] and Proposition 3.3 from [8] as corollaries of our main result Theorem 26. Moreover, the well known notion of compactly epi-Lipschitz set is extended for a pair of closed sets (cf. Definition 30) and is shown that it could also be used as a sufficient condition for tangential transversality. This investigation has been developed in [46], where a more general necessary optimality condition, involving measures of noncompactness, is proved.

Yet another notion of transversality was introduced recently by Drusvyatskiy, Ioffe and Lewis in [30]. It is intermediate between subtransversality and transversality and serves as an important sufficient condition for local linear convergence of alternating projections for solving finite dimensional nonconvex feasibility problems.

Definition 4. *The closed sets $A, B \subset \mathbb{R}^d$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B \setminus A$ it holds true that*

$$\max\left\{ d\left(\frac{x^A - x^B}{\|x^A - x^B\|},\ N_B\left(x^B\right)\right), d\left(\frac{x^B - x^A}{\|x^B - x^A\|}, N_A\left(x^A\right)\right)\right\} \geq \kappa\,,$$

where $N_D(\bar{x})$ *is the proximal or limiting normal cone to* D *at* $\bar{x}$*.*

Intrinsic transversality steadily grows in importance and number of researchers extend this transversality concept to more general settings and investigate its primal and dual characterizations. These notions (which some authors call "good arrangements of sets") and the relations between them, have been studied in details. See, e.g., [19],[20], [18], [50], and the literature therein. Still some aspects are not well understood. Indeed, one of the starting points of this investigation was a question of A.Ioffe about finding a metric characterization of intrinsic transversality. In fact, a variety of characterizations of intrinsic transversality in various settings are known (Euclidean, Hilbert, Asplund, Banach and normed linear spaces) but all of them involve the linear structure of the space. The reason is that researchers are mainly concentrated on the dual space. To the best of our knowledge, the first primal characterization of intrinsic transversality is obtained in [61] where the structure of a Hilbert space is assumed in most of the considerations.

These questions, along with the unknown relation between tangential transversality and intrinsic transversality, give rise to another line of research in the thesis.

The result of our study was somewhat surprising: it happened that intrinsic transversality and tangential transversality are "almost" equivalent. Moreover, the relation is very easy to establish, given the characterization of intrinsic transversality via the slope of coupling function due to Ioffe and Lewis. Thus a primal space characterization of intrinsic transversality has been obtained. We put a significant effort in clarifying the exact relationship of this characterization and the primal characterization of intrinsic transversality obtained by Thao et al. in [61], which they call property $(\mathcal{P})$. We proved that property $(\mathcal{P})$ implies our characterization in general Banach space setting and these properties are equivalent in Hilbert space setting. We would like to emphasize that the property we introduce is simpler (or at least it looks simpler) than the property $(\mathcal{P})$ – less variables are involved.

Establishing the exact relationship between intrinsic transversality and tangential transversality helped us to obtain primal space infinitesimal characterizations and slope characterizations of both transversality and subtransversality close in nature to tangential transversality. Thus, although the definitions and motivations for the four types of transversality properties we consider, are not similarly looking, we obtained characterizations in a unified manner for all of them. This makes obvious their close relations on the one handside, and their differences on the other handside. Indeed, it is now obvious that

$$\text{transversality} \implies \begin{array}{c}\text{tangential}\\ \text{transversality}\end{array} \implies \begin{array}{c}\text{intrinsic}\\ \text{transversality}\end{array} \implies \text{subtransversality}$$

and neither implication is invertible. This hierarchy of the properties and of their respective slope characterizations sheds new light on the topic. There have been known primal sufficient conditions and primal necessary conditions for transversality and subtransversality, but no primal characterizations (see [20] and [19]). The relationship of our characterization to these conditions is very similar to the relationship of our characterization of intrinsic transversality to property $(\mathcal{P})$ – we work with less points which makes the situation simpler. After obtaining characterizations of these transversality concepts in a unified manner, we go on to examine the regularity concepts. We obtain a characterization of metric regularity properties of a set-valued map in terms of transversality properties of sets associated with the graph of the set-valued map. We show directly that one can transfer from subtransversality to metric subregularity and from transversality to metric regularity. Similar results were already obtain in [21], [22] and [12], but there is no clear statement of such interchangeability. We moreover show proofs of some known primal space characterizations of the regularity concepts, using the already derived characterzations of their transversality counterparts. We also show how one can easily obtain from these results the characterizations of metric regularity via the *first order variation* and the *graphical derivative*.

In the last chapter of the thesis we consider continuity of the optimal value mapping for an abstract optimization problem in metric spaces, where the feasible set varies, i.e. depends on a parameter. Specifically, we deal with the function

$$S_{\mathrm{val}}(p) := \inf\{g(y) \mid y \in D(p)\}.$$

where X and Y are metric spaces, $D : X \rightrightarrows Y$ is a set-valued mapping and $g : Y \to \mathbb{R}$ is a function. The classical Maximum theorem of Berge ([7]) (in the more general setting when X and Y are merely topological spaces) considers the case when g also depends on p and says that when g is continuous (on $X \times Y$) and D is compact-valued and continuous at $\bar{p} \in X$, then S_{val} is continuous at $\bar{p}$. It is widely used in mathematical economics and optimal control.

Another version of this result is due to Berdyshev ([6]) where a so-called t-continuity (which is stronger than the well known Pompeiu-Hausdorff continuity) is required for the mapping D (see Theorem 39). The result of Berdyshev also shows that when the space is metric and g is uniformly continuous on Y, the Pompeiu-Hausdorff continuity suffices to prove continuity of S_{val}. The corresponding definitions are stated explicitly in the chapter.

Generalizations of the classical Berge theorem, which consider various well-posedness conditions of the function on the constraint set that also guarantee continuity of the value function, can be found in the book of Lucchetti ([54]). Detailed discussion on this topic could be also found in the book by Dontchev and Zolezzi ([28]).

The motivation for our investigations on this topic was Theorem 5 of Chapter IX, Section 1, in [28], which states as follows

Theorem 5. *Assume that for some point $\bar{p}$ of the topological space X, D is continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

However, in [28] it is not clearly stated what kind of continuity the authors have in mind, and this may lead to a possible confusion. We will show by a counterexample that the theorem is false if the assumed continuity of the mapping D is in the Pompeiu-Hausdorff sense in the case of metric spaces. Note that in [28] the spaces are topological (as in Theorem 1.1) so it is reasonable to assume that a topological definition of continuity is had in mind. Still this is not clearly stated. The main purpose of this chapter of the thesis is to consider this issue (in the case of metric spaces), namely when Theorem 5 holds and when it does not, and in the latter case, we examine additional assumption, under which it holds. We investigate the interplay between the continuity properties of f and D which would guarantee continuity of S_{val}. In the course of our research, we formulate a continuity assumption depending both on f and D, which we call Relaxed uniform continuity assumption, (**RUCA**). We show that it is sufficient for continuity of S_{val} but is also in some sense necessary. Moreover, we comment on how earlier results fit naturally in our approach.

Throughout the thesis, we have eschewed using variational principles, though some of our results could be obtained in this way, too. However, we prefer to lean more on geometric intuition, which, in our understanding, makes the results and their proofs more natural and well motivated.

Chapter 2 contains necessary preliminary definitions and results.

In **Chapter 3**, *section 1* we obtain primal space characterizations of subtransversality. In the papers [19] and [20] (see Remark 3.5 in [19]) similar conditions are presented. It is proved that these conditions are characterizations (both necessary and sufficient) only in the convex case.

Our approach is to some extent motivated by the considerations in the paper [9]. In it, the notion of tangential transversality (3) is introduced as a sufficient condition for nonseparation of sets, tangential intersection properties and a Lagrange multiplier rule.

Now we introduce a weaker notion. Note that the main difference is that "there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of positive reals tending to zero such that for every t_n belonging to it ..." is replaced by "there exists a positive real θ such that ...". This is indeed a significant difference, as it will be shown later on. The other weakening in the definition, "$\bar{x} \in A \cap B$" to "$A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$", is for purely technical reasons.

Definition 6. *Let A and B be closed subsets of the metric space X and $\bar{x} \in X$. We say that A and B have property $(\mathcal{T})$ at $\bar{x}$ if there exist $\delta > 0$ and $M > 0$ such that $A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$ and* *for any* $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *and* $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *with* $x^A \neq x^B$ *there exist* $\theta > 0$, $\hat{x}^A \in A$ *and* $\hat{x}^B \in B$ *such that*

$$d(x^A, \hat{x}^A) \leq \theta M \ , \quad d(x^B, \hat{x}^B) \leq \theta M \quad \text{and} \quad d(\hat{x}^A, \hat{x}^B) \leq d(x^A, x^B) - \theta \ .$$

Equivalently, A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$ *if and only if there exist* $\delta > 0$ *and* $M > 0$ *such that* $A \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$, $B \cap \bar{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \varnothing$ *and for any* $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *and* $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *with* $x^A \neq x^B$ *there exist* $\hat{x}^A \in A$ *and* $\hat{x}^B \in B$ *such that*

$$d(\hat{x}^A, \hat{x}^B) \leq d(x^A, x^B) - \frac{1}{M} \max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\}$$

and $\max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\} > 0$.

The lemma below is the main technical result, whose direct corollaries will justify the benefits of the above definition.

Lemma 7. *Let* A *and* B *be closed subsets of the complete metric space* X *and* $\bar{x} \in X$. *Let* A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$ *with constants* δ *and* M. *Let* $x^A \in A$ *with* $d(x^A, \bar{x}) \leq \dfrac{\delta}{1+2M}$ *and* $x^B \in B$ *with* $d(x^B, \bar{x}) \leq \dfrac{\delta}{1+2M}$. *Then, there exists* $x^{AB} \in A \cap B$ *with*

$$d(x^{AB}, x^A) \leq M d(x^A, x^B) \text{ and } d(x^{AB}, x^B) \leq M d(x^A, x^B) \, .$$

Completeness is crucial in the above lemma. The following theorem is formulated in a way that enables us to use it to obtain primal space characterizations both for subtransversality and transversality.

Theorem 8. *Let* A *and* B *be closed subsets of the complete metric space* X *and* $\bar{x} \in X$. *If* A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$, *then there exist* $K > 0$ *and* $\delta > 0$ *such that*

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B)) \tag{1}$$

for all $x \in \bar{\mathbf{B}}_\delta(\bar{x})$.

If there exist $K > 0$ *and* $\delta > 0$ *such that* (1) *holds for all* $x \in \bar{\mathbf{B}}_\delta(\bar{x})$, $A \cap \bar{\mathbf{B}}_{\frac{\delta}{4K+10}}(\bar{x}) \neq \varnothing$ *and* $B \cap \bar{\mathbf{B}}_{\frac{\delta}{4K+10}}(\bar{x}) \neq \varnothing$, *then* A *and* B *have property* $(\mathcal{T})$ *at* $\bar{x}$.

As a corollary we obtain that property $(\mathcal{T})$ is an equivalent characterization of subtransversality in the presence of completeness.

Corollary 9. *If $\bar{x} \in A \cap B$, where A and B are closed subsets of the complete metric space X, then A and B have property $(\mathcal{T})$ at $\bar{x}$ if and only if A and B are subtransversal at $\bar{x}$.*

The following proposition is a reformulation of Corollary 9.

Proposition 10. *Under completeness of the space X, A and B are subtransversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ and $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$, $x \neq y$, it holds*

$$|\nabla \phi|^\diamond(x, y) = \sup_{(u,v)\neq(x,y)} \frac{\max\{\phi(x, y) - \phi(u, v), 0\}}{d((x, y), (u, v))} \geq \kappa.$$

In *Section 2*, we continue to obtain primal space characterizations of transversality. A direct consequence of the definition of transversality and Theorem 8 is a characterization of transversality in terms of "translated" subtransversality.

Proposition 11. *Let A and B be closed subsets of the Banach space X and $\bar{x} \in A \cap B$. Then A and B are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, any $x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x} + a)$ and $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x} + b)$ with $x^A - a \neq x^B - b$ there exist $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that*

$$\|x^A - \hat{x}^A\| \leq \theta M \ , \quad \|x^B - \hat{x}^B\| \leq \theta M \quad \textit{and}$$

$$\|\hat{x}^A - \hat{x}^B - (a - b)\| \leq \|x^A - x^B - (a - b)\| - \theta \ .$$

Strengthening in one of the directions of this proposition gives a characterization of transversality in terms of "translated" tangential transversality. Here

Proposition 12. *Let A and B be closed subsets of the Banach space X and $\bar{x} \in A \cap B$. Then A and B are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in \bar{\mathbf{B}}_\delta(\mathbf{0})$ and $b \in \bar{\mathbf{B}}_\delta(\mathbf{0})$, any*

$x^A \in A \cap \bar{\mathbf{B}}_\delta(\bar{x}+a)$ *and* $x^B \in B \cap \bar{\mathbf{B}}_\delta(\bar{x}+b)$ *with* $x^A - a \neq x^B - b$ *there exist* $\{x^A_m\}_{m\geq 1} \subset A$, $\{x^B_m\}_{m\geq 1} \subset B$ *and* $t_m \searrow 0$ *such that for each* m

$$\|x^A_m - x^A\| \leq t_m M \ , \quad \|x^B_m - x^B\| \leq t_m M \quad and$$

$$\|x^A_m - x^B_m - (a-b)\| \leq \|x^A - x^B - (a-b)\| - t_m \ .$$

Remark 13. *In the above proposition we can obtain the (formally) stronger statement that there exists* $\lambda > 0$ *such that the decreasing property holds for any* $t \in (0, \lambda]$ *instead of the sequence* $\{t_n\}_{n=1}^{\infty}$ *tending to zero from above.*

Analogously to Proposition 10 we can obtain similar slope type characterizations of transversality.

In *Section 3* we provide a metric characterization of intrinsic transversality. This characterization could be used as a definition of intrinsic transversality in general metric spaces. Moreover, we show that it is almost equivalent to the notion of tangential transversality, via observing a slope type characterization of the latter. Finally we show that the metric characterization we provide is equivalent in Hilbert spaces to a characterization introduced and studied in [61].

Similarly we obtained a characterization of tangential transversality in terms of the slope of the coupling function.

Proposition 14. *The subsets* A *and* B *of the metric space* X *are tangentially transversal at* $\bar{x}$ *if and only if there exist* $\delta > 0$ *and* $\kappa > 0$ *such that for any two different points* $x \in A \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *and* $y \in B \cap \bar{\mathbf{B}}_\delta(\bar{x})$ *it holds*

$$|\nabla\phi|(x,y) = \limsup_{(u,v)\to(x,y)} \frac{\max\{\phi(x,y) - \phi(u,v), 0\}}{d((x,y),(u,v))} \geq \kappa\,.$$

Drusvyatskiy, Ioffe and Lewis found a characterization of intrinsic transversality in finite dimensional spaces in terms of the slope of the coupling function (cf. Proposition 4.2 in [30]). We use this characterization as a definition of intrinsic transversality in general metric spaces.

Definition 15. *Let X be a metric space. The closed sets $A, B \subset X$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x^A \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\delta(\bar{x}) \cap B \setminus A$ it holds true that*

$$|\nabla \phi|(x^A, x^B) \geq \kappa\,.$$

It is obvious there is "almost" equivalence between intrinsic transversality and tangential transversality. Here Due to Proposition 14 we have that the only difference between tangential transversality and intrinsic transversality is that in the original definition of tangential transversality the required condition should hold for all points of A and B (respectively) near the reference point, whereas in intrinsic transversality – only for points in $A \setminus B$ and $B \setminus A$ (respectively). We introduce the following property.

Definition 16 (Property $(\mathcal{LT})$)**.** *We say that the closed sets A and B satisfy property $(\mathcal{LT})$ at $\bar{x} \in A \cap B$, if there exist $\varepsilon > 0$ and $\theta > 0$ such that for any two different points $x^A \in \bar{\mathbf{B}}_\varepsilon(\bar{x}) \cap A \setminus B$ and $x^B \in \bar{\mathbf{B}}_\varepsilon(\bar{x}) \cap B \setminus A$, there exist sequences $t_m \searrow 0$, $\{x_m^A\}_{m\geq 1} \subset A$ and $\{x_m^B\}_{m \geq 1} \subset B$ such that for all m*

$$d(x_m^A, x^A) \leq t_m, \quad d(x_m^B, x^B) \leq t_m, \quad d(x_m^A, x_m^B) \leq d(x^A, x^B) - t_m\theta\,.$$

The comments above yield the following

Corollary 17. *The sets A and B are intrinsically transversal at $\bar{x} \in A \cap B$ if and only if they satisfy property $(\mathcal{LT})$ at $\bar{x}$.*

In this way we answer a question of Prof. A. Ioffe about finding a metric characterization of intrinsic transversality, as well as some of the questions posed in [9].

We provide an example which shows that although the difference is slight, the notion of tangential tranvsersality is stronger than the one of intrinsic transversality.

It is known that intrinsic transversality and subtransverslity coincide for convex sets in finite-dimensional spaces (cf. Proposition 6.1 in [38] or Corollary 3.4 in [50] for an alternative proof). Both proofs exploit the dual characterizations of intrinsic transversality and substransversality. Now we can easily obtain the slightly stronger result

Corollary 18. *Let X be a Banach space. The closed convex sets $A, B \subset X$ are tangentially transversal at the point $\bar{x} \in A \cap B$, if and only if they are subtransversal at $\bar{x}$.*

In the papers [50] and [61] a generalization of intrinsic transversality to Hilbert spaces is derived. It is based on the normal structure - Definition 2(ii) in [50] and Definition 3 in [61]. Moreover, in paper [61] a so called property $(\mathcal{P})$ is introduced. It is in primal space terms and is shown to be equivalent to the afformentioned extension of intrinsic transversality in Hilbert spaces based on the normal structure (Definition 2(ii) in [50] and Definition 3 in [61]).

In order to state it we need the following notation - for a normed space X,

$$\mathrm{d}(A, B, \Omega) := \inf_{x \in \Omega, a \in A, b \in B} \max\{\|x - a\|, \|x - b\|\}, \quad \text{for } A, B, \Omega \subset X$$

with the convention that the infimum over the empty set equals infinity.

Here is the corresponding definition.

Definition 19 (Property $(\mathcal{P})$). *A pair of closed sets $\{A, B\}$ is said to satisfy property $(\mathcal{P})$ at a point $\bar{x} \in A \cap B$ if there are numbers $\alpha \in (0, 1)$ and $\varepsilon > 0$ such that for any $a \in (A \setminus B) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$, $b \in (B \setminus A) \cap \bar{\mathbf{B}}_\varepsilon(\bar{x})$ and $x \in \bar{\mathbf{B}}_\varepsilon(\bar{x})$ with $\|x - a\| = \|x - b\|$ and number $\delta > 0$, there exists $\rho \in (0, \delta)$ satisfying*

$$\mathrm{d}\left(A \cap \bar{\mathbf{B}}_\lambda(a), B \cap \bar{\mathbf{B}}_\lambda(b), \bar{\mathbf{B}}_\rho(x)\right) + \alpha\rho \le \|x - a\|, \ \textit{where } \lambda := (\alpha + 1/\sqrt{\varepsilon})\rho$$

The following two theorem show that in general normed spaces property $(\mathcal{P})$ implies property $(\mathcal{LT})$, while in Hilbert spaces they are equivalent.

Theorem 20. *Let X be a normed space, A and B be closed subsets of X and $\bar{x} \in A \cap B$. Assume that A and B satisfy property $(\mathcal{P})$ at $\bar{x}$. Then they satisfy property $(\mathcal{LT})$ at $\bar{x}$. If X is moreover a Hilbert space, then the reverse is also true - if the sets satisfy property $(\mathcal{LT})$ at $\bar{x}$, then they satisfy property $(\mathcal{P})$ at $\bar{x}$.*

In *section 4* we show that regularity and subregularity could be characterized in terms of transversality and subtransversality. The same sets as in the formulations below appear in the papers [21](Theorem 5.2), [22] (Theorem 4.2) and [12] (Theorem 4), but the equivalence with (sub)regularity is not explicitly stated.

Theorem 21. *Let $F : X \rightrightarrows Y$ be a set-valued mapping between the metric spaces X and Y, and $(\bar{x}, \bar{y}) \in Gr\ F$. Define the sets $A := Gr\ F$ and $B := X \times \{\bar{y}\}$. Then F is subregular at $(\bar{x}, \bar{y})$ if and only if A and B are subtransversal at $(\bar{x}, \bar{y})$.*

Corollary 22. *Let $F : X \rightrightarrows Y$, X and Y be metric spaces, and $(\bar{x}, \bar{y}) \in Gr\ F$ as above. Define the sets $A := Gr\ F$ and $B_y := X \times \{y\}$. Then F is regular at $(\bar{x}, \bar{y})$ if and only if there are constants $\delta > 0$ and $K > 0$ such that for any $(x, y) \in \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and any $\hat{y} \in \bar{\mathbf{B}}_\delta(\bar{y})$*

$$d((x, y), A \cap B_{\hat{y}}) \leq K(d((x, y), A) + d((x, y), B_{\hat{y}}))\,. \tag{2}$$

If in addition X and Y are normed spaces, then this is also equivalent to A and $B := B_{\bar{y}}$ being transversal at $(\bar{x}, \bar{y})$.

In *sections 5 and 6* we prove characterizations of subregularity and regularity using our previously established characterizations for subtransversality and transversality. As corollaries we obtain the classical "rate of descent" characterizations of subregularity and regularity.

Using the above theorem, we establish a characterization of metric regularity of a map $F : X \rightrightarrows Y$, X – complete metric space and Y – Banach space, using its *first order (contingent) variation* $F^{(1)}(x, y)$. This is first done in [31] (see also Theorem 4.13 and Remark 4.14(c) in [3] for a proof in Banach spaces or [41] for an alternative proof). Given $(x, y) \in \mathrm{Gr}F$, define $F^{(1)} : X \times Y \rightrightarrows Y$ by

$$F^{(1)}(x, y) := \limsup_{t \to 0_+} \frac{F(\bar{\mathbf{B}}_t(x)) - y}{t}\,,$$

where lim sup stands for the Kuratowski limit superior of sets. Our proof is done via a sequential characterization of metric regularity, which we have not seen stated anywhere in the literature.

Corollary 23. *Let us consider $F : X \rightrightarrows Y$ with closed graph, where X is a complete metric space and Y is a Banach space. Then, the following are equivalent*

(i) *F is regular at $(\bar{x}, \bar{y}) \in Gr\, F$*

(ii) *there exist $\delta > 0$ and $r > 0$ such that*

$$\mathbf{B}_r(\mathbf{0}) \subset F^{(1)}(x, y) \text{ for all } (x, y) \in \bar{\mathbf{B}}_\delta(\bar{x}, \bar{y}) \cap Gr\, F$$

(iii) *there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\, F \cap \bar{\mathbf{B}}_\delta((\bar{x}, \bar{y}))$ and all $\hat{y} \in \bar{\mathbf{B}}_\delta(\bar{y})$, there is a sequence $\{(x_n, y_n)\}_{n\geq 1} \subset Gr\, F \setminus \{(x, y)\}$ converging to (x, y) such that for all n it holds*

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y))\,.$$

We also obtain the as a corollary the classical result (cf. Theorem 1.2 in [25] and Theorem 4.13 and Remark 4.14(b) in [3]) establishing the relation between the metric regularity of a map $F : X \rightrightarrows Y$, X and Y – Banach spaces, and its *graphical (contingent) derivative.*

In **Chapter 5**, *Section 1 and 2*, we state some necessary preliminary definitions and results. The following definition is from [44]:

Definition 24. *Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is nonempty for each $t \in [0, \lambda]$.*

Next we remind the classical concept of compactly epi-Lipschitz sets in Banach spaces. It was introduced by J.M. Borwein and H.M. Strojwas in 1985 in [11] and it includes all finite-dimensional and all epi-Lipschitsian sets in Banach spaces. Since then, it has been an important notion in nonsmooth analysis and has been frequently used in qualification conditions for obtaining normal intersection properties and calculus rules concerning limiting Fréchet cones and subdifferentials (in Asplund spaces, cf. [55] and [56]) and G-cones and G-subdifferentials (in general Banach spaces, cf. [43] and [38]). Compactly epi-Lipschitz sets are called *massive* in [39]. Here is the corresponding

Definition 25. *Let A be a closed subset of the Banach space X and $x_0 \in A$. We say that A is compactly epi-Lipschitz (massive) at x_0, if there exist $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset X$, such that for all $t \in [0, \delta]$ the following inclusion holds true*

$$A \cap (x_0 + \delta \overline{\mathbf{B}}) + \varepsilon \overline{\mathbf{B}} \subset A + tK \ .$$

In *section 3*, we state and prove the main result of the chapter.

For its statement we will need the notion of ε-density: we say that a set A is ε-dense in the set B, if for all $v \in B$ there is $u \in A$ such that $\|v - u\| < \varepsilon$. It is in part motivated by the notion of massive sets, which is now "split" between the sets.

Theorem 26. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$, $q_1 > 0$, $q_2 > 0$, such that $q_1 + q_2 < 1$ and:*

(i) there exist bounded "ball covering" sets M_A and M_B such that $M_A - M_B$ is εq_1-dense in $\varepsilon \overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that

$$A \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;

(ii) there exist two bounded sets D_A and D_B such that $D_A - D_B$ is εq_2-dense in $U_A - U_B$ and they are "η-uniform" with $\eta := (1 - q_1 - q_2)/3$, i.e. for each $t \in [0, \delta]$

$$A \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tD_A \subset A + t\eta \overline{\boldsymbol{B}} \text{ and } B \cap (x_0 + \delta \overline{\boldsymbol{B}}) + tD_B \subset B + t\eta \overline{\boldsymbol{B}}.$$

Then A and B are tangentially transversal at x_0.

In *Section 4* we provide applications of the main result. Two sufficient conditions for tangential transversality, namely Theorem 5.2 from [9] and Proposition 3.3 from [8] are obtained from the main result of this chapter in a unified manner.

Theorem 27. *Let X and Y be Banach space and let $f : X \times Y \to \mathbb{R}$ be proper lower-semicontinuous function. Let $L : Y \to X$ be continuous linear operator and*

$$S = \{(Ly, y) \mid y \in Y\}.$$

Let $(\bar{x}, \bar{y}) \in S$ be such that there exists $\bar{\delta} > 0$ and $K > 0$, such that for all $y \in \bar{y} + \bar{\delta}\bar{\boldsymbol{B}}_Y$ and for all $x' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$, $x'' \in \bar{x} + \bar{\delta}\bar{\boldsymbol{B}}_X$ holds

$$|f(x', y) - f(x'', y)| \leq K\|x' - x''\|.$$

Moreover, let $\widehat{T}_{epi\ f}((\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))) - \widehat{T}_{S\times(-\infty, f(\bar{x},\bar{y})]}((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})))$ be dense in $X \times Y \times \mathbb{R}$ and X and Y are separable Banach spaces. Then epi f and $S \times (-\infty, f(\bar{x}, \bar{y})]$ are tangentially transversal.

Below we formulate an abstract (infinite-dimensional) version of the well-known Aubin condition from [15] for the basic problem of the calculus of variations:

Definition 28. *Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which has finite value at $(\overline{x}, \overline{y}) \in X \times Y$. It is said that f satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ iff there exist positive reals $\overline{\delta} > 0$ and $K > 0$ such that for every $t \in [0, \overline{\delta}]$ the following inclusion holds true:*

$$\operatorname{epi} f \cap \left((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X\times Y\times R}\right) + t\left(\overline{\mathbf{B}}_X, \mathbf{0}, 0\right) \subset$$

$$\subset \operatorname{epi} f + t\left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, K[-1, 1]\right).$$

The next theorem is the main motivation of our research:

Theorem 29. *Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$. Let $L : Y \longrightarrow X$ be a compact linear operator and $S := \{(Ly, y) : y \in Y\}$. We assume that*

$$\widehat{T}_{\operatorname{epi} f}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) - S \times (-\infty, 0]$$

is dense in $X \times Y \times R$. Then epi f and $S \times (-\infty, f(\overline{x}, \overline{y})]$ are tangentially transversal at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.

The next definition is an extension of the notion of a massive set to "massiveness" of two sets as a pair.

Definition 30. *Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. We say that A and B are jointly massive at x_0 if there exist $\varepsilon > 0$, $\bar\delta > 0$, bounded sets $M_A \subset X$, $M_B \subset X$ and a compact set $K \subset X$ such that:*
(i) $\varepsilon \overline{\boldsymbol{B}}_X \subset \overline{M_A - M_B}$;
(ii) $A \cap (x_0 + \bar\delta \overline{\boldsymbol{B}}) + tM_A \subset A + tK$ and $B \cap (x_0 + \bar\delta \overline{\boldsymbol{B}}) + tM_B \subset B + tK$ whenever $t \in [0, \bar\delta]$.

We easily observe that if the sets A and B are closed, $x_0 \in A \cap B$ and A is massive at x_0, then A and B are jointly massive at x_0.

The next assertion is a direct generalization of Theorem 4.3 of [9].

Proposition 31. *Let A and B be jointly massive at x_0 and $\widehat{T}_A(x_0) - \widehat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.*

The obtained sufficient conditions could be applied with Theorem 3.3 from [9], to derive Lagrange multipliers in different situations, as summarized in the following

Theorem 32. *Let X and Y be Banach spaces. We consider the optimization problem*

$$f(x,y) \to \min \quad \textit{subject to} \quad (x,y) \in S \ ,$$

where $f : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper and S is a closed subset of $X \times Y$. Let one of the following three conditions be satisfied:

1. *X and Y are separable, $S = \{(Ly, y) \mid y \in Y\}$, where L is a continuous linear operator and there exist $\bar\delta > 0$ and $K > 0$, such that for all $y \in \bar y + \bar\delta \bar{\boldsymbol{B}}_Y$ and all $x' \in \bar x + \bar\delta \bar{\boldsymbol{B}}_X$, $x'' \in \bar x + \bar\delta \bar{\boldsymbol{B}}_X$ holds $|f(x', y) - f(x'', y)| \le K\|x' - x''\|$*

2. *epi f and $S \times (-\infty, f(\bar x, \bar y)]$ are jointly massive at $(\bar x, \bar y, f(\bar x, \bar y))$.*

3. *$S = \{(Ly, y) \mid y \in Y\}$, where L is a compact linear operator and f satisfies the Aubin condition at $(\bar{x}, \bar{y})$.*

Then there exists a triple $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$ such that

(i) $(\xi, \eta, \zeta) \neq (\mathbf{0}, \mathbf{0}, 0)$;

(ii) $\zeta \in \{0, 1\}$;

(iii) $\langle \xi, Ly \rangle + \langle \eta, y \rangle \leq 0$ *for every* $y \in Y$;

(iv) $\langle \xi, u \rangle + \langle \eta, v \rangle + \zeta s \geq 0$ *for every* $(u, v, s) \in \widehat{T}_{epif}(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$.

Remark 33. *In cases 1. and 3. equality holds in (iii) since Y is a vector space. In case 2. we might just have $f : X \to \mathbb{R}$.*

In **Chapter 5**, we study the optimal value map associated to an optimization problem. We will assume that S_{val} assumes only finite values. Throughout the chapter, all the topological spaces involved will be metric spaces with the property that every open ball is connected (clearly this is the case for normed vector spaces).

For a subset A of X and $\varepsilon > 0$ we define

$$A_\varepsilon = \bigcup_{x \in A} \mathbf{B}_\varepsilon(x) = \{z \in X \mid \exists x \in A,\ \rho(z, x) < \varepsilon\}.$$

We will consider set-valued maps with closed values only.

We introduce the two continuity (semi-continuity) notions considered in this chapter.

Definition 34. *Two notions of upper semicontinuity.*

- *Topological upper semicontinuity (t-usc). $F : X \rightrightarrows Y$ is t-usc at $\bar{x} \in X$ if for any open U containing $F(\bar{x})$, there exists an open neighbourhood V of $\bar{x}$ such that $F(x) \subseteq U$ for all $x \in V$.*

- *Pompeiu-Hausdorff upper semicontinuity (h-usc). $F : X \rightrightarrows Y$ is h-usc at $\bar{x} \in X$ if for any $\varepsilon > 0$, there exists an open neighbourhood V of $\bar{x}$ such that $F(x) \subseteq F(\bar{x})_\varepsilon$ for all $x \in V$.*

Clearly t-usc implies h-usc. However, the reverse implication might not hold, since in general there are open sets U containing $F(\bar{x})$ which do not contain a set of the form $F(\bar{x})_\varepsilon$. However, both notions coincide when $F(\bar{x})$ is compact as observed in [2], [6], [27]. There are a number of concepts of continuity of set-valued mappings that are usually tied with corresponding concepts of convergence of sequences of sets; among them the popular Kuratowski-Painleve continuity ([27]), based on the notion of set convergence introduced by Painleve and elaborated by Kuratowski. A good reference for convergence of sets is the survey by Sonntag and Zalinescu [59].

In *Section 1* we provide a counterexample and a remedy. We begin with a counterexample to Theorem 5 if continuity is in the Pompeiu-Hausdorff sense.

Counterexample 35. *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$. Consider*

$$D(p) = \{(x, y) \in \mathbb{R}^2 \mid y \geq -|p|\}$$

and

$$g(x,y) = \begin{cases} 0, & y \leq -\frac{1}{1+x^2} \\ (1+x^2)y + 1, & -\frac{1}{1+x^2} < y < 0 \\ 1, & y \geq 0 \end{cases}$$

Then D is continuous at $\bar{p} = 0$, g is continuous on all of $\mathbb{R}^2$ (in the sense of Pompeiu-Hausdorff), $S_{val}(0) = 1$, while for $p \neq 0$, $S_{val}(p) = 0$.

It is evident, that the conclusion fails, because the function g is arbitrary steep around $\partial D(0)$. To circumvent this possibility, we introduce a "relaxed" uniform continuity around $\partial D(\bar{p})$.

According to the authors knowledge the following definition is new.

Definition 36. *Let $F : X \rightrightarrows Y$ be a set-valued map and $f : Y \to \mathbb{R}$. We say that the couple (F, f) satisfies the relaxed uniform continuity assumption (**RUCA**) at $\bar{x}$ if*

Theorem 37. *Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Assume moreover that the couple (D, g) satisfies (**RUCA**) at $\bar{p}$. Then S_{val} is continuous at $\bar{p}$.*

It is easy to observe that the pair (D, g) from Counterexample 35 does not satisfy (**RUCA**). From this theorem we obtain the following corollary, which could also be derived as a special case of the theorem of Berge, since, as noted, when $D(\bar{p})$ is compact, h-continuity is equivalent to t-continuity.

Corollary 38. *Let X be a complete metric space. Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$, g is continuous on $D(\bar{p})$ and $D(\bar{p})$ is totally bounded (bounded if X is finite dimensional normed vector space). Then S_{val} is continuous at $\bar{p}$.*

In *section* 2 we turn our attention to the case of t-continuity of the feasibility mapping. As noted earlier, the following theorem follows from a result of Berdyshev [6] (and is essentially equivalent to it in the case of metric spaces).

Theorem 39. *Assume that for some $\bar{p} \in X$, D is t-continuous at $\bar{p}$ and g is continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

The results developed at the end of the section generalize the preceding theorem.

Another result following from Berdyshev's work is the following

Theorem 40. *Assume that for some $\bar{p} \in X$, D is h-continuous at $\bar{p}$ and g is uniformly continuous on $D(\bar{p})$. Then S_{val} is continuous at $\bar{p}$.*

It could be proved along the lines of our results so far. Next we involve the measure of noncompcatness as an intermediary to obtain characterization of t-usc via (**RUCA**).

Clearly (**RUCA**) for (F, f) at $\bar{x}$ is a property depending on both the set-valued map F and the real-valued function f. However, in some cases, strong properties of only one of the objects ensures (**RUCA**) independently of the other object. For example, if the function f is uniformly continuous on the whole of Y, (**RUCA**) is satisfied independently of the properties of the set-valued map F - i.e. for any map F. On the other hand, as in Corollary 38, if F is h-usc at $\bar{x}$ and $F(\bar{x})$ is totally bounded, then (**RUCA**) is satisfied for any function f which is continuous on $F(\bar{x})$.

The following Proposition clarifies when such a situation is present. It shows that if F is h-usc at $\bar{x}$, then (**RUCA**) for (F, f) at $\bar{x}$ holds for any function f continuous on $F(\bar{x})$ if and only if F is t-usc.

Proposition 41. *Let $F : X \rightrightarrows Y$ and $\bar{x} \in X$. The following are equivalent*

(i) F is t-usc at $\bar{x}$;

(ii) F is h-usc at $\bar{x}$ and for every $\varepsilon > 0$ there exists an open neighbourhood V of $\bar{x}$ such that

$$\alpha\left(\bigcup_{x\in V} F(x) \setminus F(\bar{x})\right) < \varepsilon;$$

*(iii) F is h-usc at $\bar{x} \in X$ and for any function $f : Y \to \mathbb{R}$ which is continuous on $F(\bar{x})$, the couple (F, f) satisfies (**RUCA**) at $\bar{x}$.*

1 Author's reference

These are the main accomplishments in the thesis due to the author:

1. A general sufficient condition for tangential transversality is obtained. It is shown that it has as special cases some known sufficient conditions for tangential transversality

2. The general condition for tangential transversality is applied to derive tangential transversality of the feasible set of a minimization problem and the epigraph of the function in interest, at a given reference point. More specifically, three different scenarios are considered: the function satisfies Lipschitz condition with respect to the first variable, uniformly in the second, the feasible set is the graph of a continuous linear operator, and there exists a uniform tangent set generating the Clarke tangent cone to the epigraph at the reference point; The function satisfies the Aubin condition at the reference point and the feasible set is the graph of a compact linear operator; The graph and the feasible set are jointly massive at the referrence point. In each of the three cases, we used the obtained tangential transversality to derive a Lagrange multiplier rule if the reference point is a solution to the minimization problem.

3. Characterization of subtransversality, transversality and intrinsic transversality are obtained in the spirit of the original definition of tangential transversality, i.e. primal space characterizations. The question of the relation between all these notions is fully answered. A characterization of transversality in terms of "translated" tangential transversality is derived.

4. Extension of intrinsic transversality to infinite-dimensional spaces is proposed. It is shown to be implied by a previously proposed extension ([61]), and is proved that both coincide in the case of Hilbert spaces.

5. It is clearly stated and proved that transversality and subtransversality could be used as a characterization of metric regularity and

metric subregularity. This is later used to obtain new proofs of the well-known primal space characterizations of the regularity concepts. We use sequential primal space characterization of metric regularity to provide new proof of the characterization of regularity via the first order variation and via the graphical derivative.

6. The optimal value map associated with a minimization problem whose feasible set depends on a parameter is considered. It is provided a counterexample to a probable interpretation of a result concerning the continuity of such map. We propose an additional assumption (**RUCA**) under which we could prove continuity. We go on to show that (**RUCA**) is in some sense necessary to obtain continuity of the map: (**RUCA**) is satisfied for all functions in interest if and only if the multivalued map which defines the feasible set is topologically continuous.

2 Publications related to the thesis

1. Apostolov, S.; Krastanov, M.; Ribarska, N. (2020) *"Sufficient Condition for Tangential Transversality"*, Journal of Convex Analysis 27, 19-30
2. Apostolov, S. (2021) *"On continuity of optimal value map"*, Comptes rendus de l'Academie bulgare des Sciences, Vol 74, No4, pp 506-513
3. Apostolov, S.; Bivas, M.; Ribarska, N. (2022) *"Characterizations of Some Transversality-Type Properties"*. Set-Valued and Variational Analysis. https://doi.org/10.1007/s11228-022-00633-4
4. Apostolov, S.; Bivas, M. *Characterizations of metric (sub)regularity via (sub)transversality*, submitted.

3 Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. *"Sufficient conditions for tangential transversality"*, 47th Winter School in Abstract Analysis, Svratka, Czech Republic, 2019, https://www2.karlin.mff.cuni.cz/ lhota/ (based on a joint work with Mikhail Krastanov and Nadezhda Ribarska)

2. *"Intrinsic transversality and tangential transversality"*, 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, June 28 - July 2, 2021, Borovets, Bulgaria, http://www.math.bas.bg/ bio/WP21/ (based on a joint work with Mira Bivas and Nadezhda Ribarska)

3. *"Intrinsic transversality and tangential transversality"*, The 13th International Conference on Large-Scale Scientific Computations LSSC 2021, June 7 - 11, 2021, Sozopol, Bulgaria (based on a joint work with Mira Bivas and Nadezhda Ribarska)

4. *"Intrinsic transversality and tangential transversality"*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 27 March 2021 (based on a joint work with Mira Bivas and Nadezhda Ribarska)

5. *"On continuity of optimal value map"*, Spring Scientific Session, Faculty of Mathematics and Informatics, Sofia University, 26 March 2022

4 Declaration of originality

The author declares that the thesis contains original results obtained by him or in cooperation with his coauthors. The usage of results of other scientists is accompanied by suitable citations.

5 Acknowledgements

I would like to most sincerely express my gratitude to Professor Nadezhda Ribarska for her valuable help, motivation and support. I also express my most heartfelt gratefulness for the guidance to Professor Asen Dontchev, who, unfortunately, passed away last autumn.

References

[1] Apostolov, S.; Bivas, M.; Ribarska, N. (2022) Characterizations of Some Transversality-Type Properties Set-Valued and Variational Analysis. https://doi.org/10.1007/s11228-022-00633-4

[2] Aubin, J.-P.; Frankowska, H. (2009) Set-valued analysis, Boston. Reprint of the 1990 edition [MR1048347]. Modern Birkhauser Classics. *Birkhauser Boston, Inc.*. xx+461 pp.

[3] Azé, D. (2006) A unified theory for metric regularity of multifunctions. J. Convex Analysis 13, 225-252

[4] Azé, D.; Corvellec J.-N. (2004) Characterizations of error bounds for lower semicontinuous func- tions on metric spaces, ESAIM Control Optim. Calc. Var. 10 , 409–425

[5] Azé, D.; Corvellec, J.-N.; Lucchetti R. E. (2002) Variational pairs and applications to stability in nonsmooth analysis, Nonlinear Anal., Theory Methods Appl. 49A(5) 49 ,643–670

[6] Berdysev, V. I. (1977) Stability of the minimization problem in the case of the perturbation of the set of admissible elements. (Russian) *Mat. Sb. (N.S.)* 103(145), no. 4, 467-479, 630.

[7] Berge, C. (1997) Topological spaces. Including a treatment of set-valued functions, vector spaces and convexity. Translated from the French original by E. M. Patterson. Reprint of the 1963 translation. *Dover Publications, Inc., Mineola, NY,* xiv+270 pp.

[8] Bivas, M.; Krastanov, M.; Ribarska, N. (2018) On strong tangential transversality, preprint, https://arxiv.org/abs/1810.01814

[9] Bivas, M.; Krastanov, M.; Ribarska, N. (2020) On tangential transversality, Journal of Mathematical Analysis and Applications, Volume 481, Issue 1, 123445, ISSN 0022-247X, https://doi.org/10.1016/j.jmaa.2019.123445

[10] Bivas, M.; Ribarska, N.; Valkov, M. (2018) Properties of uniform tangent sets and Lagrange multiplier rule, Comptes rendus de lвЂ™Acade'mie bulgare des Sciences, 71 (7), 875-884

[11] Borwein, J.M.; Strojwas, (1985) H.M. Tangential approximations, Nonlinear Analysis: Theory, Methods & Applications, 9 (12), 1347-1366

[12] Bui, H.T.; Cuong, N.D.; Kruger, A.Y. (2020) Geometric and Metric Characterizations of Transversality Properties, Vietnam Journal of Mathematics 48:277вЂ“297 https://doi.org/10.1007/s10013-020-00388-1

[13] Bui, H. T.; Kruger, A. Y. (2019) Extremality, Stationarity and Generalized Separation of Collections of Sets, J Optim Theory Appl, 182, 211вЂ“-264

[14] Cibulka, R.; Fabian, M. (2016) On primal regularity estimates for set-valued mappings, Journal of Mathematical Analysis and Applications, 438(1), 444–464

[15] Clarke, F. (2005) Necessary Conditions in Dynamic Optimization, Memoirs of the American Mathematical Society

[16] Clarke, F. (1990) Optimization and Nonsmooth Analysis, Canadian Mathematical Society series of monographs and advanced texts, Canadian Mathematical Society

[17] Cuong, N. D.; Kruger, A.Y. (2019) Nonlinear transversality of collections of sets: Primal space characterizations, Preprint, arXiv: 1902.06186

[18] Cuong, N. D.; Kruger, A.Y. (2020) Dual sufficient characterizations of transversality properties, Positivity, 1–47.

[19] Cuong, N. D.; Kruger, A.Y. (2021) Primal necessary characterizations of transversality properties, Positivity 25, 531–558

[20] Cuong, N. D.; Kruger, A.Y. (2021) Transversality properties: primal sufficient conditions, Set-Valued Var. Anal 29, 221–256

[21] Cuong, N. D.; Kruger, A.Y. (2021) Transversality properties: Primal sufficient conditions, Set-Valued and Variational Analysis 29, no. 2, 221-256

[22] Cuong, N. D.; Kruger, A.Y. (2021) Primal Space Necessary Characterizations of Transversality Properties, Positivity, 25:531вЂ“558 https://doi.org/10.1007/s11117-020-00775-5

[23] De Giorgi, E.; Marino, A.; Tosques, M. (1980) Problemi di evoluzione in spazi metricie curve di massima pendenza, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 68, 180–187

[24] Dolecki, S.; Rolewicz, S. (1978) A characterization of semicontinuity-preserving set-functions. *J. Math. Anal. Appl.* 65, no. 1, 26-31.

[25] Dontchev, A. L.; Quincampoix M.; Zlateva, N. (2006) Aubin criterion for metric regularity, Journal of Convex Analysis, 13, No 2, 281-297

[26] Dontchev, A. L.; Rockafellar, R. T. (2014) Implicit Functions and Solution Mappings: A View from Variational Analysis, Springer-Verlag New York

[27] Dontchev, A. L.; Rockafellar, R. T. (2014) Implicit functions and solution mappings. A view from variational analysis. Second edition. New York, Springer Series in Operations Research and Financial Engineering. *Springer,* xxviii+466 pp.

[28] Dontchev, A. L.; Zolezzi, T. (1993) Well-posed optimization problems, Berlin, Lecture Notes in Mathematics, 1543. *Springer-Verlag* xii+421 pp.

[29] Drusvyatskiy, D.; Ioffe, A. D.; Lewis, A.S. (2014) Alternating projections and coupling slope, `http://www.optimization-online.org/DB_HTML/2014/01/4217.html`

[30] Drusvyatskiy, D.; Ioffe, A. D.; Lewis, A.S. (2015) Transversality and alternating projections for nonconvex sets, Found. Comput. Math., 15, 1637вЂ“1651

[31] Frankowska, H. (1990) Some inverse mapping theorems, Ann. Inst. H. Poincaré, Anal. Non Linéaire, 7, 183вЂ“234

[32] Guillemin, V.; Pollack, A. (1974) Differential Topology. Prentice-Hall Inc, Englewood Cliffs, N.J.

[33] Hirsch, M. (1976) Differential Topology. Springer, New York

[34] Ioffe, A. D. (1984) Approximate subdifferentials and applications I: The finite dimensional theory, Trans. Amer. Math. Soc., 281, 389-416

[35] Ioffe, A. D. (1989) Approximate subdifferentials and applications III: The metric theory, Mathematica, 36 (1), 1-38

[36] Ioffe, A. D. (1999) Variational methods in local and global nonsmooth analysis, In Clarke F.H., Stern, R.J. (eds.) Nonlinear Analysis, Differential Equations and Control, NATO Science Series C: Mathematical and Physical Sciences, vol. 258, pp. 447-502. Kluwer, Dordrecht, Boston, London

[37] Ioffe, A. D. (2000) Metric regularity and subdifferential calculus, Russian Math.Surveys 55(3), 501–558

[38] Ioffe, A. D. (2017) Transversality in Variational Analysis, J Optim Theory Appl 174(2), 343–366

[39] Ioffe, A. D. (2017) Variational Analysis of Regular Mappings: Theory and Applications, Springer Monographs in Mathematics, Springer

[40] Ivanov, M.; Zlateva, N. (2016) Long Orbit or Empty Value Principle, Fixed Point and Surjectivity Theorems, Compt. rend. Acad. bulg. Sci., 69, No 5, 553-562

[41] Ivanov, M.; Zlateva, N. (2020) On Characterizations of Metric Regularity of Multi-Valued Maps, Journal of Convex Analysis 27, No. 1, 381–388

[42] Jech, T. (1978) Set theory, Academic press

[43] Jourani, A.; Thibault, L. (1996) Extensions of subdifferential calculus rules in Banach spaces, Canadian Journal of Mathematics, 48 (4), 834-848

[44] Krastanov, M. I.; Ribarska, N. K. (2017) Nonseparation of Sets and Optimality Conditions, SIAM J. Control Optim., 55 (3), 1598-1618

[45] Krastanov, M. I.; Ribarska, N. K. (2018) A Functional Analytic Approach to a Bolza Problem, Lecture Notes in Economics and Mathematical Systems, Control Systems and Mathematical Methods in Economics, 687, Gustav Feichtinger, Raimund Kovacevic, Gernot Tragler, 97–117

[46] Krastanov, M. I.; Ribarska, N. K. (2021) A necessary optimality condition involving measures of noncompactness, Pure and applied functional analysis, Volume 6, Number 6, 1361-1381

[47] Krastanov, M. I.; Ribarska, N. K.; Tsachev, Ts. Y. (2011) A Pontryagin maximum principle for infinite-dimensional problems, SIAM Journal on Control and Optimization, 49 (5), 2155–2182

[48] Kruger, A. Y. (2006) About regularity of collections of sets, Set-Valued Anal. 14, 187–206

[49] Kruger, A. Y. (2015) Error bounds and metric subregularity, Optimization, 64:1, 49-79, DOI: 10.1080/02331934.2014.938074

[50] Kruger, A. Y. (2018) About Intrinsic Transversality of Pairs of Sets, Set-Valued Var. Analysis 26, 111-142, https://doi.org/10.1007/s11228-017-0446-3

[51] Kruger, A. Y.; Luke, D. R.; Thao, N. H. (2018) Set regularities and feasibility problems, Mathematical Programming B 168, 279–311

[52] Kruger, A. Y.; Thao, N. H. (2015) Quantitative Characterizations of Regularity Properties of Collections of Sets, Journal of Optimization Theory and Applications, 164 (1), 41вЂ“67

[53] Lucchetti, R. (1989) Stability in optimization. *Serdica* 15, no. 1, 34-48.

[54] Lucchetti, R. (2006) Convexity and well posed problems, New York, CMS Books in Mathematics/Ouvrages de MathΓ©matiques de la SMC, 22, *Springer*, xiv+305 pp

[55] Mordukhovich, B. S. (2006) Variational Analysis and Generalized Differentiation, I: Basic Theory, Grundlehren der mathematischen Wissenschaften, Springer, New York

[56] Penot, J. P. (2013) Calculus Without Derivatives, Graduate Texts in Mathematics, Springer, New York

[57] Rockafellar, R. T. (1979) Clarke's tangent cones and the boundaries of closed sets in R^n, Nonlinear Analysis: Theory, Methods & Applications, 3 (1), 145 –154.

[58] Rockafellar, R. T. (1980) Generalized directional derivatives and subgradients of nonconvex functions, Canad. J. Math., 32, 257-280

[59] Sonntag, Y.; Zalinescu, C. (1993) Set convergences. An attempt of classification. *Trans. Amer. Math. Soc.* 340 , no. 1, 199-226.

[60] Sussmann, H. (2006) On the validity of the transversality condition for different concepts of tangent cone to a set, Proceedings of the 45-th IEEE CDC, San Diego, CA, December, 3-15, 241–246.

[61] Thao, N. H.; Bui, H. T.; Cuong, N.D.; Verhaegen, M. (2020) Some new characterizations of intrinsic transversality in Hilbert spaces, Set-Valued and Variational Analysis volume 28, pages 5-39

[62] Van Ngai, H.; Théra, M. (2008) Error bounds in metric spaces and application to the perturbation stability of metric regularity. SIAM Journal on Optimization 19(1), 1–20