R E V I E W

**of the dissertation "Algebraic Methods for Studying Some Combinatorial Configurations and Their Applications"**

**by Tedis Arben Ramaj**

**for awarding the educational and scientific degree "Doctor"**

**in Scientific Field 4. Natural Sciences, Mathematics and Informatics,**

**Professional Direction 4.5 Mathematics**

**Ph.D. Program "Algebra, Topology and Applications"**

# 1 General characterisation of the dissertation and the provided materials

The dissertation consists of an introduction, three chapters and bibliography. It occupies 86 pages. The first chapter provides some preliminaries on orthogonal arrays and Krawtchouk polynomials. The second chapter presents the original results on the distance distribution of a ternary orthogonal array. The third chapter is devoted to the scientific contributions on the covering radius of a ternary orthogonal array. The bibliography consists of 44 items. More precisely, it contains 31 articles, 12 monographs and one data library. Three of the aforementioned articles reflect the scientific contributions of the dissertation. From the remaining 28 articles, seven are published after 2000, nine appeared in the period 1980-1999, three are from 1960-1979, eight from 1940-1959 and one from 1929. The twelve citations of monographs refer to eleven books. Three of these have appeared after 2000, five in the period 1990-1999, two in 1965-1975 and one in 1939. The uniform distribution in time of the appearance of the cited bibliography shows that Tedis Ramaj is acquainted, both, with the classical results and the contemporary contributions in the area of her research.

# 2 Biographical data and personal impressions

Tedis Arben Ramaj graduated as a Bachelor of Science in Mathematics in 2011 at "Aleksander Xhuvani" University of Elbasan. In 2013 she obtained a Masters Degree in Mathematics from University of Tirana. From 2013 to 2020 she has worked as a part time Assistant Professor at "Aleksander Xhuvani" University of Elbasan, University of Tirana and Polytechnic University of Tirana. From November 2020 she is a full time Assistant Professor at Section "Algebra and Number Theory" of the Faculty of Natural Sciences, University of Tirana. From 2018 Tedis Ramaj is a graduate student at PhD Program "Algebra, Topology and Applications" of the Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski".

I know Tedis Arben Ramaj from October 2018 as a member of the examination committee for her General PhD Exam in Algebra. She has prepared thoroughly and vigilantly on the curriculum and earned deservedly an excellent grade. Except from this exam, my personal impressions for Tedis Ramaj are formed by her talk on the distance distributions of ternary orthogonal arrays at the Spring Scientific Session of Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski"in 2019. Her report was very well organized, conceptual and interesting not only for the specialists in orthogonal arrays, but also for mathematicians outside this research area.

# 3 Content analysis of the dissertation

The first chapter of the dissertation recalls some properties of the orthogonal arrays and their distance distributions, which are expressed by Krawtchouk polynomials and additive characters. Section 1.1 is devoted to the history of the orthogonal arrays and some examples. Section 1.2 collects some properties of the parameters of orthogonal arrays. Section 1.3 discusses the correspondence between the orthogonal arrays and the error correcting codes. After introducing the length, the dimension and the minimum distance of a code, it considers the duality of codes, their generator and parity-check matrices. Special attention is paid to the linear orthogonal arrays and their associated linear codes. Section 1.4 recalls the definition and some properties of Krawtchouk polynomials. Most of the results are given with their proofs. The same section provides a definition of an additive character of a finite group $G$. A special attention is paid to the characters of the additive groups $(\mathbb{F}_{p^e}, +)$ of the finite fields $\mathbb{F}_{p^e}$ and the additive groups $(\mathbb{Z}_{p^e}, +)$ of the primary residue rings $\mathbb{Z}_{p^e}$. The construction of characters of $G^n$, $n \in \mathbb{N}$ from characters of $G$ is explained in detail. The exposition discusses also the statement and the proof of Delsarte's Lemma, expressing a sum of additive characters of $\mathbb{Z}_p^e$ by a value of a Krawtchouk polynomial. Section 1.4 provides also inequalities on the distance distribution of a not necessarily linear code $C \subset H(n,q)$ of cardinality $|C| = M$, which involve values of Krawtchouk polynomials. It characterises the orthogonal arrays of strength $t$ and $M$ rows in $H(n,q)$ in terms of the additive characters of the field $\mathbb{F}_q = \mathbb{F}_{p^e}$, identified with the linear space $(\mathbb{F}_p)^e$ over the prime subfield $\mathbb{F}_p$. The final, fifth section of the first chapter is devoted to the distance distributions of codes and orthogonal arrays. I would like to mention three important results on the distance distribution $(p_0, p_1, \dots, p_n)$ of an $(M, n, q, t)$-orthogonal array, which are formulated and proved in Section 1.5. All of them are expressed by linear equations on $p_0, p_1, \dots, p_n$. Delsarte's Lemma from early 1970's provides a homogeneous linear system on $p_0, p_1, \dots, p_n$, whose coefficients are values of appropriate Krawtchouk polynomials at integers from 0 to $n$. The basic result of an article of Manev from 2020 reads as a linear equation on $p_0, p_1, \dots, p_n$, whose coefficients are values of an arbitrary polynomial $f(x) \in \mathbb{R}[x]$ of degree $\deg f(x) \leq t$ at $0, 1, \dots, n$. The constant term $f_0 M$ involves the constant term $f_0$ of the expansion $f(x) = f_0 + \sum_{j=1}^{t} f_j K_j(x)$ of $f(x)$ with respect to the orthogonal system of the Krawtchouk polynomials $K_0(x), K_1(x), \dots, K_t(x)$. As an immediate consequence of Manev's result is derived a linear equation on $p_0, p_1, \dots, p_n$, whose coefficients are values of an arbitrary $f(x) \in \mathbb{R}[x]$ of $\deg f(x) \leq t$ at the points $1 - \frac{2i}{n} \in [-1, 1]$, $0 \leq i \leq n$. The constant term $a_0 M$ depends on the constant term $a_0$ of the expansion $f(x) = a_0 + \sum_{j=1}^{t} a_j Q_j(x)$ of $f(x)$ with respect to the normalized Krawtchouk polynomials $Q_0(x) = 1$, $Q_1(1), \dots, Q_t(x)$. Manev's result and its corollary are formulated and proved in Theorem 1.5.3.

Chapter 2 is devoted to the distance distributions of the ternary orthogonal arrays. Section 2.1 formulates a result of Boyvalenkov and Kulina from 1998. Its Theorem 2.1.2 recalls the statements and the proofs of four systems of linear equations on the distance distribution of an orthogonal array, derived in Manev'c article from 2020. For one of these systems is exposed Manev's argument for obtaining an explicit basis of the solution space of the corresponding homogeneous linear system. Section 2.1 proceeds with five combinatorial identities from Riordan's book. Three of them are given with their proofs. These are used for presenting Manev's argument on a formula for inversion of a matrix of monomials and his upper bounds on the entries of the distance distribution of an orthogonal array. Section 2.2 exposes Manev's algorithm for obtaining the feasible distance distributions $p = (p_0, p_1, \dots, p_n)$ of $\mathrm{OA}(M, n, q, t)$. It starts with obtaining an upper bound $u = (u_0, u_1, \dots, u_n)$ on $p$, making use of the first part of Theorem 1.5.3 for a polynomial $f(x)$ with $f(i) \geq 0$ for all $0 \leq i \leq n$. Then one chooses $t+1$ consecutive components of $u$, having as large as possible values and denotes by $s$ the number of the components, preceding the chosen ones. Towards the description of the solution set $S$ of the linear system $Ax^T = a$ on $p$ from Theorem 2.1.2 (iv) is applied Fredholm's alternative. It claims that $S = x_o + V$ is the affine space, obtained from the linear space $V$ of the solutions of $Ax^T = \mathbb{O}_{(t+1)\times 1}$ by a translation with a partial solution $x_o$ of $Ax^T = a$. The columns of $A$, labeled by $s+1, \ldots, s+t+1$ form the binomial matrix $R_t \in M_{(t+1)\times(t+1)}(\mathbb{Z})$, whose inverse $R_t^{-1} \in M_{(t+1)\times(t+1)}(\mathbb{Z})$ is obtained explicitly in Lemma 2.1.7. That provides a row-echelon form $B = R_t^{-1}A = (B_1 I_{t+1} B_2)$ of $A$ with explicit $B_1 \in M_{(t+1)\times s}(\mathbb{Z})$, $B_2 \in M_{(t+1)\times(n-t-s)}(\mathbb{Z})$ and a matrix $B^{\perp} = \begin{pmatrix} I_s & -B_1^{\perp} & \mathbb{O}_{s\times(n-t-s)} \\ \mathbb{O}_{(n-t-s)\times s} & -B_2^{\perp} & I_{n-t-s} \end{pmatrix} \in M_{(n-t)\times(n+1)}(\mathbb{Z})$, whose rows constitute a basis of $V$. Let $x_o$ be the partial solution of $Bx^T = R_t^{-1}a = b$, obtained by annihilating the first $s$ and the last $n-s-t$ components. Denote by $V_o$ the set of the linear combinations of the rows of $B^{\perp}$ with non-negative integral coefficients, bounded above by the first $s$ components and the last $n-s-t$ components of $u$. Then any $y = (y_0, y_1, \ldots, y_n) \in x_o + V_o$ has non-negative components $y_i \geq 0$ for all $0 \leq i \leq s-1$ and $s+t+1 \leq i \leq n$. It suffices to check that $y_i \geq 0$ for all $s \leq i \leq s+t$, in order to assert that $y$ is a feasible distance distribution of an orthogonal array. Section 2.2 ends with an original application of Theorem 2.1.2 (i)-(iv) to the set of the distance distributions of the orthogonal arrays of strength 5 and index 6 in $H(13,3)$. Besides the corresponding four linear systems $Ax^T = a$, it provides the matrices $R_t^{-1}, B = R_t^{-1}A, B^{\perp}$, associated with the system $Ax^T = a$ from Theorem 1.2.1 (iv) and an upper bound $u$ on $p$. Section 2.3 is devoted to some original results of the dissertation. It provides explicit formulae for the entries of the matrix $B^{\perp}$, whose rows generate the solution space of the homogeneous linear system $Bx^T = \mathbb{O}_{(t+1)\times 1}$, as well as for the entries of a partial solution $x_o = (\mathbb{O}_{1\times s}, \xi, \mathbb{O}_{1\times(n-s-t)}) \in M_{1\times(n+1)}(\mathbb{Z})$ of $Bx^T = b$ with $\xi = (\xi_0, \ldots, \xi_t) \in M_{1\times(t+1)}(\mathbb{Z})$. In the case of an even strenght $t$, all the entries of $B_1 \in M_{(t+1)\times s}(\mathbb{Z})$, $B_2 \in M_{(t+1)\times(n-t-s)}(\mathbb{Z})$ and $\xi^T \in M_{(t+1)\times 1}(\mathbb{Z})$ from the $m$-th row are shown to have one and a same sign $(-1)^m$. As a consequence, there arise upper bounds $p_l \leq \min\left(\lfloor \frac{\xi_o}{b_{0l}} \rfloor, \lfloor \frac{\xi_1}{b_{1l}} \rfloor, \ldots, \lfloor \frac{\xi_t}{b_{tl}} \rfloor\right)$ for all $0 \leq l \leq s-1$ and all $s+t+1 \leq l \leq n$. The orthogonal arrays are objects with an abundance of symmetries, which allow to relate the parameters of $C$ to the ones for the derived arrays of $C$. That is why, the study of the distance distribution of $C$ by the distance distributions of the derived arrays of $C$ can be viewed as a realisation of Klein's ideas for studying the geometries by the means of their symmetries. More precisely, if $C$ is a $(\lambda q^t, n, q, t)$-orthogonal array with $n > t$ and $C'$ is obtained from $C$ by removing a column, then $C'$ is an $(\lambda q^t, n-1, q, t)$-orthogonal array. For any $\alpha \in \mathbb{F}_q$ or $\alpha \in \mathbb{Z}_q$, the submatrix $C_\alpha$ of $C'$, whose rows have contained $\alpha$ in the removed column is a $(\lambda q^{t-1}, n-1, q, t-1)$-orthogonal array. Corollary 2.4.1 collects some inequalities among the distance distributions of $C, C_\alpha$ and $\cup_{\beta\in\mathbb{F}_q\setminus\{\alpha\}} C_\beta$, derived in Manev's article from 2020. More precisely, the first $n$ components of an initerior distance distribution of $C$ dominate any interior distance distribution of $C_\alpha$, while the last $n$ components of an interior distance distribution of $C$ dominate any exterior distance distribution of $C_\alpha$. Any exterior distance distribution of $\cup_{\beta\in\mathbb{F}_q\setminus\{\alpha\}} C_\beta$ equals the componentwise difference of interior distance distributions of $C$ and $C_\alpha$. Theorem 2.4.2 of Boyvalenkov and Kulina from 1998 expresses the distance distribution of $C$ by the possible distance distributions of $C'$, regarded with their multiplicities. The aforementioned results are used for obtaining all the distance distributions of the $(18,7,3,2)$-orthogonal arrays by the means of their derived $(6,6,3,1)$-orthogonal arrays $C_\alpha$ with $\alpha \in \mathbb{F}_3$. The final, fifth section of Chapter 2 exposes non-existence and structural results, obtained by the means of the explicit equalities and inequalities from Section 2.3. Section 2.5.1 obtains all the possible distance distributions of the $(4.3^3, n, 3, 3)$-orthogonal arrays $C$ and their derived $(4.3^2, n-1, 3, 2)$-orthogonal arrays $C_\alpha$, $\alpha \in \mathbb{F}_3$ with $n \in \{16, 17\}$. Then it applies the inequalities from Corollary 2.4.1, in order to rule out some of them. For any posiible pair of distance distributions of $C$ and $C_\alpha$, it computes the distance distiribution of the $(8.3^2, n, 3, 2)$-orthogonal array $C_\beta \cup C_\gamma$, $\{\alpha, \beta, \gamma\} = \mathbb{F}_3$. By the means of the equalities from Theorem 2.1.2, the procedure establishes the non-existence of $(4.3^3, 17, 3, 3)$-orthogonal arrays. The non-existence of an $(4.3^3, 16, 3, 3)$-orthogonal array requires further application of

Theorem 2.4.2. The original results from Section 2.3 and the aforementioned procedures, which rule out the presence of ternary orthogonal arrays of strength 3, length $n \in \{16, 17\}$ and index 4, reflect the results of an article of Boumova, Ramaj and Stoyanova in "Comptes rendus de l'Académie bulgare des Sciences" from 2021. The non-existence of a $(4.3^3, 17, 3, 2)$-orthogonal array is established also by a different algorithm in an article of Boumova, Marinova, Ramaj and Stoyanova in Annual of Sofia University "St. Kliment Ohridski" from 2019. This algorithm expresses the distance distributions of $C, C', C_0$ and $C_1 \cup C_2$ by the number $y_i$, respectively, $\overline{y_i}$ of the vanishing, respectively, of the non-zero elements in the removed column of the $i$-th block of $C$. By its very definition, the $i$-th block of $C$ consists of the rows of $C$ at distance $i$ from the reference point. Besides the aforementioned non-existence results, the article of Boumova, Ramaj and Stoyanova from "Comptes rendus de l'Académie bulgare des Sciences" - 2021 provides two structural results, which are exposed in Section 2.5.2. One of these results reduces the possible distance distributions of ternary orthogonal arrays of strength 3, length 15 and index 4. The second one establishes the existence of at most one explicitly given distance distribution of a ternary orthogonal array of strength 5, length 16 and index 6.

The final, Chapter 3 derives analytic upper bounds on the covering radius $\rho(C)$ of an orthogonal array $C$, making use of the linear system on the distance distribution of $C$, given by Theorem 2.1.2 (iv). The corresponding results are published in a joint article with Boumova and Stoyanova from the Proceedings of the International Workshop on Algebraic and Combinatorial Coding Theory 2020. For the distance distribution $p(x) = (p_0(x), p_1(x), \ldots, p_n(x)) \in (\mathbb{Z}^{\geq 0})^{n+1}$ of a fixed orthogonal array $C$ with respect to an exterior point $x \in H(n,q) \setminus C$, note that $p_0(x) = 0$ and denote by $j(x) \in \mathbb{Z}^{\geq 0}$ the non-negative integer with $p_0(x) = \ldots = p_{j(x)}(x) = 0$ and $p_{j(x)+1}(x) \neq 0$. Section 3.1 recalls that if $J$ is the maximum of $j(x)$ over all $x \in H(n,q) \setminus C$ then the covering radius of $C$ is $\rho(C) = J + 1$. Namely, for any $x \in H(n,q) \setminus C$ one has $p_{j(x)+1}(x) \neq 0$ for some $j(x) + 1 \leq J + 1$ and there is a point $y(x) \in C$ with $d(x, y(x)) = j(x) + 1 \leq J + 1$. Moreover, if $J = \max\{j(x) \,|\, x \in H(n,q) \setminus C\} = j(x_o)$ is attained at a point $x_o \in H(n,q) \setminus C$ then $p_0(x_o) = \ldots = p_J(x_o) = 0$ and $p_{J+1}(x_o) \neq 0$, so that the distance $d(x_o, C) := \min\{d(x,y) \,|\, y \in C\}$ from $x_o$ to $C$ is $d(x_o, C) = J+1$. Bearing in mind that $\rho(C) = \max\{d(x,C) \,|\, x \in H(n,q) \setminus C\}$, one concludes that $\rho(C) = J+1$. Concerning the set of all the orthogonal arrays $C$ with fixed strength $t$, length $n$, index $\lambda$ and $q$ levels, one has $\rho(C) \leq J+1$, where the maximum $J$ is taken over the set $Q(\lambda q^t, n, q, t)$ of all the feasible exterior distance distributions of $(\lambda q^t, n, q, t)$-orthogonal arrays. Section 3.2 makes use of the original results from Section 2.3 towards the analytic upper bound $\rho(C) \leq n - t$ on the covering radius $\rho(C)$ of an $(\lambda q^t, n, q, t)$-orthogonal array $C$. More precisely, if one restricts to the distance distributions $p \in M_{1 \times (n+1)}(\mathbb{Z})$ with $p_0 = \ldots = p_{n-t-1} = 0$ then the corresponding linear system $Bx^T = b$ on $p$ has coefficient matrix $B = (B_1 I_{t+1})$ with $B_1 \in M_{(t+1) \times (n-t)}(\mathbb{Z})$ and unique solution $p = (\mathbb{O}_{1 \times (n-t)} b_0, \ldots, b_{t+1})$. Corollary 3.2.1 establishes that $b_0 = \lambda\binom{n}{t}$ and $b_1 = -\lambda\binom{n}{t-1}(n-t-q-1)$, whereas $p_{n-t} = b_0 = \lambda\binom{n}{t} \neq 0$ and the covering radius satisfies the inequality $\rho(C) \leq J + 1 = n - t$. Moreover, making use of $b_1 < 0$, Section 3.2 proves that $\rho(C) \leq n-t-1$ whenever $n-t > q-1$. By explicit examples it shows that the equality $\rho(C_1) = n - t$ is attained by an orthogonal array $C_1$ with $n - t = q - 1$, while $\rho(C_2) = n - t - 1$ is attained by an orthogonal array $C_2$ with $n - t > q - 1$. The final Section 3.3 makes use of the inequalities on the distance distributions of $C$ and $C_\alpha$, $\alpha \in \mathbb{F}_q$, given by Corollary 2.4.1, as well as of the formula for $b_0$, applied to the orthogonal arrays $C_0$ and $\cup_{\beta \in \mathbb{F}_q^*} C_\beta$, towards the upper bound $\rho(C) \leq n - t - 2$ in the case of $n > 2(t + q - 1)$. The dissertation illustrates the upper bound $\rho(C) \leq n - t - 2$ on three examples of ternary orthogonal arrays with known distance distributions.

The contributions of the dissertation under review include explicit formulae for a partial solution $x_o = (\mathbb{O}_{1 \times s}, \xi, \mathbb{O}_{1 \times (n-s-t)}) \in M_{1 \times (n+1)}(\mathbb{Z})$ of the system $Bx^T = b$ on the distance distribution $p \in M_{1 \times (n+1)}(\mathbb{Z})$ of an orthogonal array, as well as explicit formulae for the entries of the matrix $B^\perp$, whose rows generate the solution space of $Bx^T = \mathbb{O}_{(n+1) \times 1}$. These are used towards the non-existence of ternary orthogonal arrays of strength 3, index 4 and length

$n \in \{16, 17\}$. The aforementioned explicit formulae reduce considerably the set of the possible distance distributions of the $(4.3^3, 15, 3, 3)$-orthogonal arrays and the $(6.3^5, 16, 3, 5)$-orthogonal arrays. Among the author's contributions is an explicit upper bound on the covering radius of an orthogonal array and two improvements of this bound for special values of the parameters. Appropriate examples illustrate that all the three bounds can be attained by orthogonal arrays with specific parameters.

# 4 Approbation of the results

The dissertation of Tedis Ramaj reflects the results of three article. One of them is published in 2021 in a journal with IF from the fourth quartile, another appeared in 2019 in a refereed and indexed journal and the last one is in the proceedings of an international conference from 2020. The publications of Tedis Arben Ramaj earn her 72 points according to Decree 26/13.02.2019 on the amendments of the Rules of implementation of the Law on Development of Academic Staff of Republic Bulgaria. This exceeds more that two times the required 30 points for acquisition of the educational and scientific degree "Doctor". More precisely, the article in a journal with IF from the fourth quartile provides 36 points, while the remaining two articles are assessed by 18 points each. All the publications of Tedis Ramaj are joint with her scientific advisors Associate Professor Maya Stoyanova and Associate Professor Silvia Boumova. The article from 2019 is also joint with Tanya Marinova. According to the submitted declarations, the contributions of all co-authors are commensurable. A computer test has proved that the results of the dissertation are original and there is no plagiarism. The scientific contributions of the dissertation are reported six times at international and national conferences and workshops.

The aforementioned scientometric criteria and content analysis convinced me that the dissertation "Algebraic Methods for Studying Some Combinatorial Configurations and Their Applications" of Tedis Arben Ramaj complies with the minimal national requirements of Decree 26/13.02.2019 on the amendments of the Rules of implementation of the Law on Development of Academic Staff of Republic Bulgaria, as well as with the requirements of the Rules on the terms and conditions for acquisition of academic degrees and occupation of academic positions at Sofia University "St. Kliment Ohridski".

# 5 Qualities of the abstract

The English and the Bulgarian abstracts reflect appropriately the content and the scientific contributions of the dissertation. The works of other authors are cited appropriately by accurate formulation of the results and exact reference to the corresponding source.

# 6 Critical remarks and suggestions

Few misprints in the dissertation do not decrease its high quality. I recommend Tedis Ramaj to keep on working in the area of orthogonal arrays with the same thoroughness, as she did in her dissertation.

# 7 Conclusion

According to my impressions for the dissertation and its corresponding scientific works, and based on the above analysis of their scientific significance and applicability, I confirm that the presented dissertation and its corresponding scientific publications, together with the quality and the originality of their results and achievements, comply with the requirements of the Law on Development of Academic Staff of Republic Bulgaria, the Rules on its implementation and the Rules on the terms and conditions for acquisition of academic degrees and occupation of academic positions at Sofia University "St. Kliment Ohridski" for acquisition of the educational and scientific degree "Doctor" in Scientific Field 4. Natural Sciences, Mathematics and Informatics, Professional Directions 4.5 Mathematics. In particular, Tedis Arben Ramaj complies with the minimal national requirements in the professional direction and no plagiarism was found in the presented scientific works.

**Based on the aforementioned, I assess positively and strongly recommend the Scientific Juri to award Tedis Arben Ramaj the educational and scientific degree "Doctor" in Scientific Field 4. Natural Sciences, Mathematics and Informatics, Professional Directions 4.5 Mathematics (Algebra, Topology and Applications).**

April 12, 2021

Prof. Azniv Kirkor Kasparian
Section of Algebra
Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"