R E V I E W

on a Thesis for awarding the degree "Doctor"

Title: Graded Algebras and Non-commutative Invariant Theory

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Scientific field: 4. Natural sciences, mathematics and informatics

Professional field: 4.5. Mathematics

Overview

The thesis is devoted to problems from non-commutative invariant theory. This subject extends a classical line of mathematical research connected with the names of D. Hilbert, P. Gordan, E. Noether, H. Weyl and many others. The goal of the thesis is to present solutions for several instances of the following problem: Given a reductive group G, find a fundamental system of generators of the algebra of the polynomials in non-commuting variables. The author solves the problem for the symmetric group and polynomials in d variables, as well as for the alternative group and polynomials in three variables. The thesis summarizes the research by the author from the last three years. My general impression is that the candidate is well acquainted with the state of the art and the new results on the problems that are in the focus of his thesis. These problems are classical and are considered important to invariant theory. The author demonstrates deep knowledge of his field of research and good capacity to apply his knowledge in the investigation of difficult problems.

Description of the results

This thesis amounts 75 pages and consists of an introduction, three chapters, an index and a list of references including 56 items. The first chapter is introductory. The author gives a historical overview of invariant theory and describes the main goals of this thesis. This chapter contains also a brief description of the main results of the thesis.

Chapter 2 is subdivided in five sections. It starts with a list of the basic notations used in the text (section 2.1). Section 2.2 is a brief introduction in commutative invariant theory. It starts with the definition of the algebra of G-invariants and then states the classical theorem of symmetric polynomials, several results by E. Noether, D.Hilbert, J.-P. Serre and T. Molien. In Section 2.3 the candidate explains some basic notions and results from non-commutative invariant theory. A paper by A. N. Koryukin which is of big importance for this thesis is discussed in Section 2.4. The candidate presents many of the results from Koryukin's paper with proofs Section 2.5 contains a summary of the results of M. Wolf on symmetric non-commutative polynomials from her PhD Thesis.

The original results are contained in the next two chapters. In Chapter 3 the author considers the S-algebra of the symmetric non-commutative polynomials in d variables $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$. The central problem here is whether or not this algebra is finitely generated. The candidate gives an afirmative answer when the characteristic is 0 or greater than the number of the variables d. In the beginning of the section the elementary symmetric polynomials in d non-commuting variables are introduced. In particular, the sums

$$p_{(n)} = \sum x_1^n = x_1^d + \ldots + x_d^n$$

and

$$p_{(1^n)} = \sum_{\sigma} x_{\sigma(1)} \ldots x_{\sigma(n)}$$

are defined explicitly since they are central for the next two sections. The next step is Lemma 3.1.2 which claims that the S-algebra of the non-commutative symmetric polynomials over any field is generated by the power sums $p_{(n)}$. The set of these polynomials is again infinite, but it is easier to grasp and more convenient to handle. Further on the candidate proves a non-commutative version of Newton's identities. These identities are formulated and proved in Lemma 3.1.4. This lemma enables the proof of the main result of this section – Theorem 3.1.5. It states that for fields K with $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ the S-algebra of the symmetric non-commutative polynomials in d variables $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is freely generated by the symmetric polynomials $p_{(1^i)} = \sum_{\sigma} x_{\sigma(1)} \ldots x_{\sigma(i)}$. A separate and independent proof for the particular case of polynomials in two non-commuting variables and a field K of charactersitic $\operatorname{char} K \neq 2$ is given in Theorem 3.1.6. These results leave open the case of the S-algebras of polynomials over finite fields K of characteristic that does not exceed the number of the variables. At the end of the section the candidate conjectures that for $\operatorname{char} K \leq d$ the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is not finitely generated (Conjecture 3.1.7).

This conjecture is tackled in the next section 3.2. The main result is contained in Theorem 3.10. It is proved there that the image of $K\langle X_d\rangle$ under a certain suitably defined homomorphism π is not finitely generated. Further in Theorem 3.2.12 the candidate constructs a minimal (with respect to inclusion) generation set for the algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$. It is proved that if $p = \operatorname{char} K \leq d$ the set of all power sums is a minimal generating set for the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$.

Chapter 4 deals with non-commutative alternating polynomials. The goal here is to generalize the results from the previous chapter about the algebra $(K\langle X_d\rangle^{\mathrm{Alt}(d)}, \circ)$ of the non-commutative polynomials in d non-commuting variables, invariant under the action of the alternative group Alt(d). The main result is contained in Theorem 4.0.5. It states that if $\operatorname{char} K = 0$ or $\operatorname{char} K = p > 3$, the algebra of the alternative polynomials in three non-commuting variables is finitely generated. This is done by presenting a finite system of generators, namely, the symmetric polynomilas p_{1^i}, together with the alternating polynomials s_2 and s_3 defined as

$$s_2 = x_1x_2 + x_2x_3 + x_3x_1 - x_2x_1 - x_3x_2 - x_1x_3,$$

and

$$x_3 = x_1^2x_+x_2^2x_3 + x_3^2x_1 - x_2^2x_1 - x_3^2x_2 - x_1^2x_2.$$

Finally, the candidate proves that for fields K of characteristic 2 or 3, the S-algebra $(K\langle X_d\rangle^{\mathrm{Alt}(3)}, \circ)$ is not finitely generated.

The material in chapter 3 is based on the two publications in *Mathematics* and the *Turkish Journal of Mathematics*. Section 4 contains results by the author that are not published yet and are part of an ongoing research.

Publications related to the thesis

The thesis is based on two papers that are published in good mathematical journals with an impact factor, as follows:

- Mathematics (Q1, IF 2.4) – 1 paper

- Turkish Journal of Mathematics (Q2, 1.0) - 1 paper

In both publications Deyan Dzhundrekov has three co-authors. All co-authors declare that the contribution of Deyan Dzhundrekov is significant. I accept that his results are well-known and highly valued in the professional community.

The presented publications meet the minimal national requirements as given in the corresponding documents.

The results in this thesis have been presented in seven talks during the last three years.

I could not detect any plagiarism in this thesis.

Remarks and comments

I have the following remarks, questions and comments related to this thesis:

(1) The thesis is written in English. The exposition is clear and easy to follow. I find it very convenient that the author has included an index which makes the work with the text a lot easier.

(2) It is odd to number the results in chapter 4 as Lemma 4.0.1, Remark 4.0.2 and so on. I think it is better to have Lemma 4.1, Remark 4.2 and so on.

(3) The detailed discussion in sections 2.4 and 2.5 of the work of Korykin and Margarete Wolf gives this thesis a certain tutorial value.

(4) In the definition of s_3 on page 67 (somewhere in the middle) there is a misprint. It should probably read $s_3 = \sum_{\mathrm{Alt}} x_1^2 x_2 = x_1^2 x_2 + \ldots$ and so on.

(5) A couple of minor misprints do not spoil the generally favorable impression from this text.

(6) I would recommend the presentation of the results at prestigious international conferences abroad.

All the above remarks are insignificant mathematically and do not change my very good impression of the deep research carried out by the author.

Author's summary

The author's summary is made according to the regulations and reflects properly the main results and contributions of this thesis.

Conclusion

This thesis is focused on problems from invariant theory. The obtained results mark progress in the field of invariant theory.

I am deeply convinced that the presented thesis "Graded Algebras and Noncommutative Invariant Theory" by Deyan Dzhundrekov contains results that are an original contribution to the field of invariant theory. The candidate demonstrates deep knowledge of his field and the capacity to develop it in a new and important way. With this, he meets the national requirements prescribed by the law and the specific regulations of Sofia University and the Faculty of Mathematics and Informatics for the professional field 4.5 Mathematics. I assess **positively** the presented thesis and recommend to this panel to award **Deyan Dzhundrekov** the scientific degree "Doctor" in the scientific field 4. Natural sciences, mathematics and informatics, professional field 4.5 "Mathematics", doctoral program "Algebra, Number Theory and Applications – Topology".

Sofia, 28.03.2024 Member of the Scientific Panel:

(Prof. Ivan Landjev)