R E V I E W

under the procedure for acquisition of the educational and scientific degree "Doctor"

by candidate Deyan Zhivkov Dzhundrekov

of the PhD Thesis entitled:

"Graded Algebras and Noncommutative Invariant Theory

In the Scientific field: **4. Natural Sciences, Mathematics and Informatics**

Professional field: **4.5. Mathematics**

Doctoral program "Algebra, Number Theory and Applications"- Topology,

Department „Algebra", Faculty of Mathematics and Informatics (FMI),

Sofia University "St. Kliment Ohridski" (SU)

The review has been prepared by Prof. Ph.D. Azniv Kirkor Kasparian, Section of Algebra, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", as a member of the scientific jury for the defense of this PhD thesis according to Order № 38-64/01.02.2024 of the Rector of the Sofia University.

1 General characteristics of the dissertation thesis and the presented materials

The dissertation contains 84 pages and consists of an introduction, four chapters and bibliography. One of the chapters collects preliminaries from commutative and noncommutative invariant theory and provides a detailed analysis of results of Koryukin from 1984 and of Margarete Wolf from 1936. The next two chapters reflect original contributions of the author. These include positive and negative results on finite generation of the algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of the noncommutative symmetric polynomials in d variables, endowed with an additional permutation S-action on the positions of the monomials. The aforementioned results are published in two articles from extremely prestigious specialized scientific journals, pertaining to the first and the second quartile, according to the Web of Science. One of the chapters reflects a work in progress on the algebra $K\langle X_d\rangle^{\mathrm{Alt}(d)}$ of the noncommutative polynomial invariants of the alternative group $\mathrm{Alt}(d)$, endowed with an additional S-action. The last chapter lists the main scientific contributions of the thesis and discusses their approbations. The bibliography comprises 56 titles, from which 34 articles in specialized scientific journals,

13 monographs or textbooks, as well as 7 preprints or PhD Theses. Nineteen of the references are published before 1970 and fourteen of them have appeared in the last ten years. This is a testimony that the problems, solved by the dissertation, are simultaneously classical and contemporary.

2 Short CV and personal impressions of the candidate

Deyan Zhivkov Dzundrekov has graduated from Sofia University "St. Kliment Ohridski". He has taken with me the optional courses "Introduction to Commutative Algebra" and "Introduction to Homological Algebra" with excellent grades. Deyan Dzhundrekov was prepared excellently and demonstrated a profound comprehension of the material and an original thinking. He has taught "Linear Algebra" and "Algebra 2" - Abstract Algebra in my teams. Deyan Dzhundrekov organizes perfectly his teaching by choosing appropriate problems, illustrating the theory, and explaining them accessibly to the students. In such a way, he earned the respect and the excellent feedback of his students. As a member of the committee for the entrance PhD exam of Deyan Dzhundrekov, I was impressed by his in-depth insight and broad mathematical culture. Deyan Dzhundrekov is highly respected for his honesty, qualification and the responsibility to his obligations. He performs his research and teaching with a precision and demonstrates excellent skills for work in a team.

3 Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

The dissertation under review starts with an inroductory chapter, followed by a peliminary one. The preliminaries include the Fundamental Theorem for the symmetric commutative polynomials, Emmy Noether's results on the finite generation of the algebra $K[X_d]^G$ of the invariant commutative polynomials of a finite subgroup $G \leq \mathrm{GL}_d(K)$ over a field K of arbitrary characteristic, as well as Chevalley-Shephard-Todd's Theorem on the K-transcendence of the commutative polynomial invariants $K[X_d]^G$ of a finite group $G \leq \mathrm{GL}_d(K)$, generated by pseudoreflections. Special attention is paid to the rationality of the Hilbert series of a finitely generated graded K-algebra, as well as to the explicit Molien's formula from 1897 for the Hilbert series of the algebra $K[X_d]^G$ of the commutative polynomial invariants of a finite group G over a field K of characteristic $\mathrm{char}(K) = 0$. A prevailing part of the preliminaries concerns the noncommutative invariant theory. This includes the description of the degree-lexicographic total Artinian order of the noncommutative monomials, which is compatible with left and right multiplication and introduced by Xiu in 2012. Independent results of Dick-Formanek from 1982 and of Kharchenko from 1984 establish that the algebra $K\langle X_d\rangle^G$ of the noncommutative polynomial invariants of a finite subgroup $G \leq \mathrm{GL}_d(K)$ is finitely generated if and only if G is a cyclic group of scalar matrices. The two original articles of Associate Prof. PhD Silvia Boumova, Acad. Prof. DSc Vesselin

Drensky, Assistant Prof. Deyan Dzhundrekov and Prof. PhD Martin Kasabov, reflected by the thesis, overcome the aforementioned obstacle to finite generation by introducing an additional permutation S-action on the positions of the letters of a monomial. This allows to "lift" results from the commutative invariant theory to the noncommutative one. The preliminaries discuss independent results of Lane from 1976 and of Kharchenko from 1978, which show that for an arbitrary field K, the algebra $K\langle X_d\rangle^G$ of the noncommutative polynomial invariants of any subgroup $G \leq \mathrm{GL}_d(K)$ is a free associative K-algebra. The dissertation explains Kharchenko's Galois correspondence, establishing that for an arbitrary finite subgroup $G \leq \mathrm{GL}_d(K)$, the free associative subalgebras of $K\langle X_d\rangle$, containing $K\langle X_d\rangle^G$, are exactly the algebras $K\langle X_d\rangle^H$ of the noncommutative polynomial invariants of the subgroups H of G. A special attention is paid to the explicit Dick-Formanek's formula for the Hilbert series of the graded K-algebra $K\langle X_d\rangle^G$, which holds for any finite subgroup $G < \mathrm{GL}_d(K)$ and is proved in 1982. An entire section of the preliminary chapter is devoted to a Koryukin's article from 1984, introducing an additional permutation S-action $\circ$ on $K\langle X_d\rangle$, which for any $n \in \mathbb{N}$ is given by the action of $\mathrm{Sym}(n)$ on the monomials $\langle X_d\rangle^{(n)}$ of degree n through permutations of the positions of the variables. The thesis provides a detailed proof of a generalization of results of Dick-Formanek from 1982 and of Kharchenko from 1984, which asserts that for an arbitrary subgroup $G \leq \mathrm{GL}_d(K)$, the algebra $K\langle X_d\rangle^G$ of the noncommutative polynomial invariants is finitely generated if and only if for the minimal K-linear subspace KY_m of the K-span KX_d of X_d with $K\langle X_d\rangle^G \leq K\langle Y_m\rangle$, the restriction of G on KY_m is a finite cyclic group of scalar matrices. A special attention is paid to Koryukin's generalization of the Hilbert Basis Theorem to S-ideals of $K\langle X_d\rangle$, as well as to the finite generation of the S-algebra $(K\langle X_d\rangle^G, \circ)$ of the noncommutative polynomial invariants of a reductive subgroup $G \leq \mathrm{GL}_d(K)$ over an arbitrary field K. The last section of the preliminaries recalls the results of Margarete Wolf from 1936 on the algebra $K\langle X_d\rangle^{\mathrm{Sym}(n)}$ of the noncomutative symmetric polynomials. She establishes that $K\langle X_d\rangle^{\mathrm{Sym}(n)}$ is a free associative algebra and any minimal homogeneous generating system of $K\langle X_d\rangle^{\mathrm{Sym}(n)}$ consists of one and a same number $\nu_n \geq 1$ of homogeneous polynomials of degree $n \in \mathbb{N}$. Except the original proof of Margarete Wolf from 1936, the dissertation provides two more proofs of the fact that any minimal homogeneous generating system of $K\langle X_2\rangle^{\mathrm{Sym}(2)}$ has exactly one homogeneous generator in any degree $n \in \mathbb{N}$.

The third chapter of the theses reflects the original results of the two published articles, while the fourth chapter discusses a special case of a work in progress on the noncommutative invariant polynomials of the alternative groups. The first section of chapter 3 establishes that the orbits of the noncommutative elementary symmetric polynomials with respect to the additional permutation S-action constitute a generating system of the K-algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of the noncommutative symmetric polynomials over a field K of characteristic $\mathrm{char}(K) = 0$ or a prime characteristic $\mathrm{char}(K) = p > d$. As a first step, the S-orbits of the power sums $p_{(n)} = x_1^n + \ldots + x_d^n$, $\forall n \in \mathbb{N}$ are shown to form a generating system of the K-algebra $K\langle X_d\rangle^{\mathrm{Sym}(n)}$. To this end, for any natural number $n \in \mathbb{N}$, the noncommutative homogeneous symmetric polynomials $p_{(\lambda)}$, associated with the partitions λ of n are proved to generate the homogeneous component $\left[K\langle X_d\rangle^{\mathrm{Sym}(d)}\right]^{(n)}$ of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ of degree n. By an

induction on the number of the components of λ are counted the S-orbits of $p_{(\lambda)}$. Towards a construction of a finite generating system of the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$, the dissertation introduces special subsets $\mathrm{Sh}_i \subset \mathrm{Sym}(k)$ of permutations, called shuffles. For arbitrary $k \in \mathbb{N}$ and $\min(d - \min(k,d), 1) \leq i \leq \min(k,d)$, the set Sh_i consists of those $\rho \in \mathrm{Sym}(k)$, for which ρ^{-1} preserves the mutual positions of the members of the sets $\{1, 2, \ldots, \min(k,d) - i\}$ and $\{\min(k,d) - i + 1, \min(k,d) - i + 2, \ldots, \min(k,d)\}$. For arbitrary $k \in \mathbb{N}$ are proved noncommutative Newton formulae, relating the power sums $p_{(k)}$, with the noncommutative elementary symmetric polynomials $p_{(1^i)} = \sum_{\sigma \in \mathrm{Sym}(d)} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(i)}$, $1 \leq i \leq d$. This is an extremely complicated result, expressed as vanishing of appropriate alternating sums over the orbits of $p_{(1^{k-i})} p_{(\max(k-d,0),i)}$ under Sh_i, for all $0 \leq i \leq \min(k,d)$. As a result, the S-orbits of the noncommutative elementary symmetric polynomials $p_{(1^i)}$, $1 \leq i \leq d$ turn to be a free generating system of the K-algebra $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ over a field K of characteristic $\mathrm{char}(K) = 0$ or a prime characteristic $\mathrm{char}(K) = p > d$.

The second section of the third chapter establishes that if the basic field is of prime characteristic $\mathrm{char}(K) = p \leq d$, then the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is not finitely generated. After reducing the considerations to the case of $\mathrm{char}(K) = p = d$, the natural epimorphism $\pi : K\langle X_d\rangle \to K[X_d]$ of the free associative algebra $K\langle X_d\rangle$ onto the corresponding free comutative and associative algebra $K[X_d]$ is proved to map $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ onto the K-span of 1 and the monomials $e_1^{m_1} \ldots e_d^{m_d}$ of the commutative elementary symmetric polynomials $e_k = \sum x_1 \ldots x_k$, $1 \leq k \leq d$ with $m_k \geq 1$ for at least one $1 \leq k \leq d-1$. Let B^+ be the ideal of the commutative polynomials from $\pi(K\langle X_d\rangle^{\mathrm{Sym}(d)})$, which vanish at $x_1 = \ldots = x_d = 0$. In a geometric context and with an extraordinary finesse is constructed an infinite K-linearly independent subset $\{e_1 e_d^m + (B^+)^2, \ldots, e_{d-1} e_d^m + (B^+)^2 \,|\, m \in \mathbb{N}\}$ of the quotient space $B^+/(B^+)^2$. The existence of the aforementioned infinite linearly independent system implies that the subalgebra $\pi\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}\right)$ of $K[X_d]$ and its pre-image $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ are not finitely generated. The same section establishes that over a field K of $\mathrm{char}(K) = p \leq d$ the S-orbits of the noncommutative power sums $p_{(n)}$, $\forall n \in \mathbb{N}$ constitute a minimal generating system of $K\langle X_d\rangle^{\mathrm{Sym}(d)}$.

The fourth chapter is devoted to an unpublished result on the noncommutative polynomial invariants $K\langle X_d\rangle^{\mathrm{Alt}(d)}$ of the alternative group $\mathrm{Alt}(d)$. It shows that an arbitrary $f \in K\langle X_d\rangle^{\mathrm{Alt}(d)}$ decomposes into a sum $f = f_1 + f_2$ of a symmetric noncomutative polynomial $f_1 \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ and an alternating noncommutative polynomial $f_2 \in K\langle X_d\rangle^{\mathrm{Alt}(d)}$, i.e. $\tau(f_2) = -f_2$ for any transposition $\tau = (i,j) \in \mathrm{Sym}(d)$, $1 \leq i < j \leq d$. Once, we have a minimal generating set of the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$, the construction of a minimal generating system of $(K\langle X_d\rangle^{\mathrm{Alt}(d)}, \circ)$ is reduced to obtaining a generating set of S-orbits of the alternating noncommutative polynomials $\sum_{\sigma \in \mathrm{Sym}(d)} (-1)^\sigma u^\sigma$, generated by monomials $u \in \langle X_d\rangle$. For $d = 3$ and a field K of arbitrary characteristic, the S-orbits of the elementary symmetric polynomials $p_{(1)}, p_{(1^2)}, p_{(1^3)}$ and the S-orbits of the alternating polynomials $s_k := \sum_{\sigma \in \mathrm{Sym}(3)} (-1)^\sigma (x_1^{k-1} x_2)^\sigma$, $k \in \mathbb{N} \setminus \{1\}$ are proved to generate the K-algebra $K\langle X_3\rangle^{\mathrm{Alt}(3)}$. If K is of characteristic $\mathrm{char}(K) = 0$ or a prime characteristic $\mathrm{char}(K) = p > 3$, the K-algebra $K\langle X_3\rangle^{\mathrm{Alt}(3)}$ is generated by the S-orbits of $p_{(1)}, p_{(1^2)}, p_{(1^3)}$, $s_2 = \sum_{\sigma \in \mathrm{Sym}(3)} (-1)^\sigma (x_1 x_2)^\sigma$ and $s_3 = \sum_{\sigma \in \mathrm{Sym}(3)} (-1)^\sigma (x_1^2 x_2)^\sigma$. The results of the second section of chapter 3 imply that

over a field K of characteristic $\mathrm{char}(K) = 2$ or 3, the K-algebra $K\langle X_3\rangle^{\mathrm{Alt}(3)}$ is not generated by finitely many S-orbits.

The last, fifth chapter formulates the main contributions of the dissertation and describes the approbation of the obtained results, i.e., the publications and the talks, reflected by the original results.

4 Approbation of the results

The original results of the dissertation are published in two articles. One of them appeared in Turkish Journal of Mathematics with Impact Factor 1 from 2022, which is classified in the second quartile by Web of Science. The other one is published in MDPI Mathematics with Impact Factor 2,4 from 2022, pertaining to the first quartile. These earn the total of 135 points, instead of the required 30 ones. In such a way, Deyan Zhivkov Dzhundrekov not only meets, but exceeds considerably the minimal national requirements under Article 2b, paragraphs 2 and 3 of the Act on Development of the Academic Staff in the Republic of Bulgaria and, respectively, to the additional requirements of Sofia University "St. Kliment Ohridski" for acquiring the educational and scientific degree "Doctor" in the scientific field and professional field of the procedure.The publications, reflected by the thesis are joint with Associate Prof. PhD Silvia Boumova, Acad. Prof. DSc Vesselin Drensky and Prof. PhD Martin Kasabov. To the best of my knowledge, the contributions of all co-authors to the aforementioned publications are equipollent.

The results, presented by the candidate in the dissertation work and the scientific works to it do not repeat such from previous procedures for acquiring a scientific title or academic position. There is no plagiarism, proven in the legally established order, in the submitted dissertation work and the scientific papers under this procedure.

5 Qualities of the abstract

The abstracts in Bulgarian and English reflect truthfully the results and the content of the dissertation. They contain the formulations of all the statements from the thesis, as well as a list of the main contributions, a description of the approbation of the obtained results, a declaration on originality and the entire bibliography.

6 Critical notes and recommendations

I have no critical remarks and suggestions.

7 Conclusion

Having become acquainted with the PhD thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific

and applied contributions contained therein, **I confirm** that the presented PhD thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by by the candidate of educational and scientific degree "Doctor" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics. In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, I strongly recommend the scientific jury to award

Deyan Zhivkov Dzhundrekov

the educational and scientific degree "Doctor" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics, doctoral program "Algebra, Number Theory and Applications"- Topology.

Date: March 25, 2024

Reviewer:
Azniv Kirkor Kasparian, Prof., Ph.D.