Review

on a Thesis for awarding the degree "Doctor of Sciences“

Scientific field 4. Natural sciences, mathematics and informatics ,

Professional field 4.5: “Mathematics”(Mathematical analysis)

Title: „Simultaneous Approximation by the Bernstein Operator“

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1 Overview

The dissertation in English with the title **Simultaneous Approximation by the Bernstein Operator** presented by Assoc. Prof. Dr. Borislav Radkov Draganov is 178 pages long, consists of an introduction, 6 chapters, and a list of cited 100 publications. As the title indicates, the subject of the thesis is the study of the simultaneous approximation by weights of functions and their derivatives with the Bernstein operator and its modifications (iterated Boolean sums and integer variants).

2 State of current research

It is noteworthy that Assoc. Prof. Draganov has an excellent knowledge of the current state of the theory of approximations with linear operators. The literature review and the large number of cited sources demonstrate both his deep knowledge of the research field and his ability to develop it creatively.

3 Methods

In his research, the dissertation mainly uses techniques and results from classical sources in the theory of approximations, such as the monographs of Ditzian and Totik [22], of DeVore and Lorentz [18], and (for inverse inequalities) the seminal paper of Ditzian and Ivanov [23].

4 Brief description of the thesis

In the introduction, straight and inverse estimates for the weightless approximations with the Bernstein operator in terms of the clasical continuity (smoothness) moduli with uniform step and with variable step controlled by the basis weight function $\varphi(x) = \sqrt{x(1-x)}$ are given. Voronovskaya's important result justifying the bounded rate of convergence of

approximations with the Bernstein operator is cited, and its property of simultaneously approximating functions and their derivatives is stated.
Chapter 1 lists basic properties of smoothness moduli and K-functionals, defines their weighted variants, and states the existence of equivalence between smoothness moduli and (suitable) K-functionals.
In Chapter 2, Dr. Draganov proves some nesting inequalities, i.e. estimates for the norms of intermediate derivatives (a classical example is the inequality (2. 1) of Landau-Kolmogorov), but here specific variants with weights and differential operators in terms of $Df(x) = \varphi^2(x)f''(x)$, where $\varphi(x) = \sqrt{x(1-x)}$ are considered. The results presented in this chapter are published in papers (2),(3) and (9).

The first two chapters of the dissertation are of a preparatory nature, and the author refers to them in the proofs of the main results in the dissertation in order to avoid repetition of analogous reasoning. This is an important stylistic approach.
In Chapter 3, Dr. Draganov proves a straightforward estimate for the simultaneous approximation by Bernstein polynomials in the uniform norm with Jacobi weights (Theorem 3.3). This estimate has the K-function

$$K_s^D(f,t)_w = \inf_{g\in C^{s+2}[0,1]} \left\{ \|w(f-g^{(s)})\| + t\|w(Dg)^{(s)}\| \right\}.$$

The author proves that this K-function can be replaced by a sum of simpler characteristics of the approximating function (Theorem 3.5). It is shown that the resulting Jackson-type estimate is exact, proving the corresponding strong inverse inequalities in Theorem 3.8. At the end of Chapter 3, using the connection between Bernstein's and Kantorovich's operators, Draganov proves an analogous characterization for the approximation rate of the latter. The results presented in this chapter are published in papers (3) and (4).
In Chapter 4, simultaneous approximations with iterated Boolean sums of Bernstein operators are studied,

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I is the identity and $r \in \mathbb{N}_+$. For the rate of simultaneous approximations with $\mathcal{B}_{r,n}$ in the uniform norm with Jacobi weights, in Theorem 4.3 for the s-th derivative approximation, Draganov proves an estimate above with the K-functional

$$K_{r,s}^D(f,t)_w = \inf_{g\in C^{2r+s}[0,1]} \left\{ \|w(f-g^{(s)})\| + t\|w(D^r g)^{(s)}\| \right\},$$

again showing that the K-function can be replaced by equivalent simpler characteristics (Theorems 4.4-4.8). By proving a strong inverse inequality in Theorem 4.10, the author shows the accuracy of the straightforward estimate in Theorem 4.3. Finally, from the results for $\mathcal{B}_{r,n}$, Dr. Draganov derives analogous forward and backward inequalities for the simultaneous approximations with iterated Boolean sums of the Kantorovich operator. The material presented in this chapter has been published in papers (1),(2),(3),(7) and (8).
In Chapter 5, simultaneous approximations with two modifications of Bernstein polynomials, setting polynomials with integer coefficients are considered. The problem is a classical one posed by Bernstein, and the goal is to determine to what extent the requirement that the coefficients of algebraic polynomials be integers affects the order of the best approximation with algebraic polynomials in the uniform norm. In 1931, Kantorovich solved this problem using the following modification of B_n

$$\widetilde{B}_n(f)(x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k},$$

where $[\alpha]$ is the integer part of the real number α.
Following Kantorovich, B. Draganov obtains an estimate on top of the error of $\widetilde{B}_n$ for $f \in C[0,1]$ such that $f(0), f(1) \in \mathbb{Z}$. He also proves that the simultaneous approximation by $\widetilde{B}_n(f)$ satisfies a similar estimate.
Draganov also proposes a second integer modification of Bernstein polynomials via $\widehat{B}_n(f)$,

$$\widehat{B}_n(f)(x) = \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle x^k(1-x)^{n-k}$$

for $f \in C[0,1]$ and $x \in [0,1]$, where $[\alpha]$ is the closest integer to the real number α.
It is proved that the approximation processes generated by $\widetilde{B}_n$ and $\widehat{B}_n$ in the uniform norm on $[0,1]$ saturate with saturation order $1/n$ and if $\|\widetilde{B}_n(f) - f\| = o(1/n)$ or $\|\widehat{B}_n(f) - f\| = o(1/n)$, then, similar to the Bernstein operators, we have $\widetilde{B}_n(f) = \widehat{B}_n(f) = f$ and f is a polynomial of the first degree with integer coefficients. In the following theorems, it is established that the integer forms of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ possess the simultaneous approximation property, and an upper estimate for the error is obtained. It should be noted that the operators $\widetilde{B}_n$ and $\widehat{B}_n$ are not linear and continuous, and the operator $\widetilde{B}_n : C[0,1] \to C[0.1]$ is not even limited.
The requirement to approximate polynomials with integer coefficients leads to additional restrictions on the set of functions to be approximated: some of them natural (take integer values at the ends of the interval) and others unexpected (derivatives cancel at the ends of the interval, inequalities for "tangents"). Under such assumptions, Dr. Draganov proves in Theorems 5.1 and 5.4 direct estimates for the approximation with these operators, and shows that the additional restrictions imposed on the set of the functions being approximated are necessary. In Theorem 5.5 Draganov proves weak inverse relations complementary to Theorems 5.1 and 5.4. As in the previous two chapters, Draganov uses the relation between the Bernstein and Kantorovich operators to define an integer version of the latter and prove a straight bound for simultaneous approximations with them. The results in this chapter are published in papers (5) and (6).
In the Chapter 6 we study the approximation of the operator $\mathcal{D}f(x) = \frac{\varphi^2(x)}{2}f''(x)$, defined for $f \in W^2_\infty(\varphi)[0,1]$ with with the adverbial name of the Voronovskaya operator

$$D_n f(x) = n(B_n f(x) - f(x)).$$

. For these approximations in Theorem 6.1 Draganov proves Voronovskaya's regular and weak inverse inequalities, and as a consequence establishes the characterization

$$\|D_n f - \mathcal{D}f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha).$$

for functions satisfying $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$ The results presented in this chapter have been published in paper (9), written jointly with I. Gadjev.

Main results
The results obtained in this dissertation are a significant contribution to the important and topical issues in approximation theory, which shows that the work presented is suitable and sufficient for the degree of **Doctor of Science**. The concrete implementation in the setting of simultaneous approximations with weights is far from trivial and requires, besides the knowledge of the methodology, a lot of technical skill and ingenuity, qualities that Dr. Draganov definitely displays in establishing the representations of the derivatives of the Bernstein operator and in the proofs of nesting inequalities and of various inequalities of the type of Jackson, Bernstein and Voronovskaya. The consistent right and wrong approximation

theorems proved by Draganov improve and/or generalize recent results of eminent specialists in approximation theory such as Totik, Gonska, Zhou, Knoop, Mache, etc., and give the impression of a completeness of the theory of simultaneous approximations with Bernstein operators in the uniform norm with Jacobi weights.

5 Remarks and comments

No remarks can be made on the dissertation thus presented. It is very stylishly and professionally written. Exposition in each chapter includes a description of known previous results by other authors relevant to the topic, and the proofs of individual statements are well structured and detailed or with reference to another source. The dissertation has the appearance of a complete and self-contained (not an assemblage of articles) scholarly product and is recommended for publication as a monograph in a reputable publishing house. It would thus be not only a useful source for specialists in approximation theory, but also an advertisement of the Bulgarian school in this field.

6 Publications related to the thesis

The dissertation of Dr. Borislav Draganov is based on nine articles published in the last 10 years. Six of the publications are in reputable scientific journals with impact factor. Approx. Theory, (2) and (9) in Results Math, and (8) in Stud. Univ. Vabeş-Bolyai Math. The other three are: (1) e in the Annuaire of Sofia University, and (4) and (6) in proceedings of the International Conferences on Constructive Function Theory. Note that even with only publications (3) and (5), the minimum national requirements for a **Doctor of Science** in Indicator C (Pokazatel B) are satisfied.

(1) B. R. Draganov, Upper estimates of the approximation rate of combinations of iterates of the Bernstein operator, *Annuaire Univ. Sofia Fac. Math. Inform.* **101** (2013), 95–104.

(2) B. R. Draganov, On simultaneous approximation by iterated Boolean sums of Bernstein operators, *Results Math.* **66** (2014), 21–41.

(3) B. R. Draganov, Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums, J. *Approx. Theory* **200** (2015), 92–135.

(4) B. R. Draganov, An exact strong converse inequality for the weighted simultaneous approximation by the Bernstein operator, In: "Constructive Theory of Functions, Sozopol 2016" (K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 75–97, Marin Drinov Academic Publishing House, Sofia, 2018.

(5) B. R. Draganov, Simultaneous approximation by Bernstein polynomials with integer coefficients, *J. Approx. Theory* **237** (2019), 1–16.

(6) B. R. Draganov, Converse estimates for the simultaneous approximation by Bernstein polynomials with integer coefficients, In: "Constructive Theory of Functions, Sozopol 2019" (B. Draganov, K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 39–51, Marin Drinov Academic Publishing House, Sofia, 2020.

(7) B. R. Draganov, Corrigendum to "Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums" [J. Approx. Theory 200 (2015) 92–135], *J. Approx. Theory* **252** (2020), 105321.

(8) B. R. Draganov, A strong converse inequality for the iterated Boolean sums of the Bernstein operator, *Stud. Univ. Babeş-Bolyai Math.* **67** (2022), 591–598.

(9) B. R. Draganov, I. Gadjev, Direct and converse Voronovskaya estimates for the Bernstein operator, *Results Math.* **73**:11 (2018).

7 Authorship of the obtained results

Papers (1) to (8) are self-contained, and for (9) there is an attached declaration of equality of results by the authors.

8 Authors summary

The submitted abstract consists of 36 pages. The main tasks that are investigated in the dissertation are clearly and precisely formulated. It systematically presents all the obtained results and finally a bibliographical reference is given. References to publications on the dissertation and to relevant citations are included both in the dissertation itself and in the abstract.

9 Citations

Assoc. Borislav Draganov has presented a reference with 15 citations of 9 publications on the thesis submitted by him. These numerical indicators prove both the relevance of the problems on which he works on, as well as the importance of the results obtained. The reference shows that Dr. Draganov's publications have received sufficient (in terms of quantity and representativeness and despite the relatively short time since their appearance) citations, and he also meets the minimum national requirements for a **Doctor of Science** under Indicator E (Pokazatel Д).

10 Conclusion

Taking into account the high professional level of scientific research work of Assoc. Borislav Draganov and the fact that his dissertation work fully meets the set of criteria and indicators for the acquisition of the respective degree in accordance with the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria, Rules for applying of the mentioned above law, and the relevant regulations of the Sofia University concerning dissertations for degrees, I respectfully propose to the Honourable Jury **to award to Assoc. Prof. Dr. Borislav Radkov Draganov the degree of Doctor of Sciences** in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics (Mathematical Analysis).

Sofia, 21.02.2024 Member of the Scientific Panel:

(Assoc. Prof. PARVAN PARVANOV, PhD)