REVIEW

under the procedure for acquisition
of the the scientific degree "Doctor of Science"

by Borislav Radkov Draganov
Faculty of Mathematics and Informatics (FMI),
Sofia University "St. Kliment Ohridski" (SU)

of the Doctor of Science Thesis entitled:

Simultaneous approximation by the Bernstein operator

in the Scientific field: 4. Natural Sciences, Mathematics and Informatics,
Professional field: 4.5. Mathematics

The review has been prepared by professor DrSci Kamen Ganchev Ivanov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, as a member of the scientific jury for the defense of this DrSci thesis according to Order № РД 38-627 / 28.11.2023 of the Rector of the Sofia University.

The review has been prepared according to the requirements of:

- the Act on Development of the Academic Staff in the Republic of Bulgaria (ADASRB);
- the Rules for the Implementation of ADASRB;
- the The regulations on the terms and conditions for acquiring scientific degrees and occupying academic positions at the Sofia University "St. Kliment Ohridski" (SU)

1. General characteristics of the dissertation thesis and the presented materials

The presented dissertation is written in English, contains 178 pages and consists of an Introduction and 6 chapters. The presented bibliography of 100 titles occupies 9 pages.

Two abstracts of the dissertation are also presented: in Bulgarian (36 pages) and in English (35 pages). Each of the abstracts is accompanied by identical bibliographies of 55 titles. The order of bibliography entries differs in the two abstracts, because three of the titles in the abstract in Bulgarian are written in Cyrillic and are sorted at the beginning of the list.

2. Short CV and personal impressions of the candidate

Borislav Radkov Draganov graduated from secondary education in 1993 at the First English High School – Sofia. In 1998, he received a master's degree from the Faculty of Mathematics and Informatics Sofia University "St. Kliment Ohridski", and in 2004 he obtained the educational and scientific degree "Doctor" after successfully defending a dissertation on "A new method for the characterization of K-functionals and application in approximation theory".

Since 2002, Borislav Draganov has worked as an assistant and chief assistant, and from 2011 until now – as an associate professor in the Department of "Mathematical Analysis", Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski". At the same time, since 2004, he has been working in the Mathematical Modeling section, later converted to the Mathematical Modeling and Numerical Analysis section, of the Institute of Mathematics and Informatics at the BAS – Sofia.

I know Borislav Radkov Draganov from the time of his study at the FMI of SU. I was the supervisor of his master's thesis, as well as the dissertation for obtaining the educational and scientific degree "Doctor". Since then, we have jointly carried out significant in volume scientific research in the field of Approximation Theory, reflected in a number of joint publications. Throughout the period, Borislav Draganov appeared as a scientist devoted to the creative process, striving for scientific discoveries, and at the same time a very capable and extremely fair colleague.

3. Content analysis of the scientific achievements of the candidate, contained in the presented DrSci thesis and the associated publications, included in the procedure

The natural formulation of the problems related to the simultaneous approximation of functions f with the Bernstein operators $B_n f$, leads to upper estimates with norms of the form $\|w\mathcal{D}f\|_{L^\infty}$, where w is an appropriate Jacobi weight and $\mathcal{D}$ is a polynomial differential operator. Of course, such an

expression can be evaluated from above with a sum of several monomial differential operators, but this in the most prevalent number of cases coarsens the estimate and prevents proving accompanying inverse statements. One of the main values of the results obtained by Borislav Draganov is finding the exact differential operators $\mathcal{D}$, which allows for obtaining consistent direct and strong inverse theorems. Let us clarify that "simultaneous approximation" means approximation of several successive derivatives of a given function with the corresponding derivatives of the approximating operator.

Chapter 1 is of an auxiliary nature. It presents the definitions and properties of classical K-functionals and their characterization with appropriate moduli of smoothness. Here, by "classical" we mean K-functionals, which are defined using monomial differential operators and are extensively studied in the monograph by Ditzian and Totik. Combinations of these characteristics of functions are used in the statements in the following chapters.

Chapter 2 is also auxiliary and contains various embedding theorems between weight spaces of differentiable functions. Some of the results are well known, while others are new, their value arising from questions raised in the dissertation. These results are published in papers 26, 27 and 33 of the dissertation bibliography.

Consistent direct and strong inverse estimates are proved in Chapter 3 for the simultaneous approximation with Bernstein operator in the L^∞ weighted norm.

Theorem 3.3 contains the direct estimate

$$\|w(B_n f - f)^{(s)}\| \leq c\, K_s^D(f^{(s)}, n^{-1})_w.$$

Here, the order of derivatives s is an arbitrary natural number, the weight w is of Jacobi type, i.e. $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$, the constraint being $0 \leq \gamma_0, \gamma_1 \leq s$. The K-functional $K_s^D(g, \delta)_w$ contains the term $\|w(Dg)^{(s)}\|_{L^\infty}$, i.e. L^∞ norm with weight w of the s-th derivative of the operator Dg, $Dg(x) = x(1-x)g''(x)$.

Under the conditions of Theorem 3.3, the strong two-terms inverse estimate is proved in Theorem 3.8

$$K_s^D(f^{(s)}, n^{-1})_w \leq c\, \left(\|w(B_n f - f)^{(s)}\| + \|w(B_{Rn} f - f)^{(s)}\|\right),$$

where R is an appropriate integer constant.

Theorem 3.26 contains the strong one-term inverse estimate

$$K_s^D(f^{(s)}, n^{-1})_w \le c\,\|w(B_nf - f)^{(s)}\|.$$

Here, the derivative order s is up to 6, not arbitrary as in Theorem 3.8. The proof of Theorem 3.26 is much more complicated than the proof of Theorem 3.8, which also leads to the constraint $s \le 6$.

In the case $s = 0$, these theorems were obtained in the 1990s in the works of Ditzian and the reviewer, Totik, and Knopp and Zhu. The results for $s \ge 1$ are an original contribution of Borislav Draganov.

Chapter 3 also contains a characterization of the K-functional $K_s^D(g, \delta)_w$ with combinations of simpler functionals and of suitable moduli, see e.g. Theorem 3.31. Results similar to the ones described so far have also been obtained for generalized Kantorovich operators, see e.g. in Theorem 3.41.

The results of Chapter 3 are published in articles 27, 28 of the dissertation bibliography.

Chapter 4 deals with the iterated Boolean sums $\mathcal{B}_{r,n}$ of the Bernstein operator, $\mathcal{B}_{r,n} = I - (I - B_n)^r$. Such sums are designed to improve the saturation order n^{-1} of the Bernstein operator – up to n^{-r} for $\mathcal{B}_{r,n}$. Results established before the ones discussed here are those of Gonska and Zhu and Ding and Cao.

The results in Chapter 4 generalize those of Chapter 3 (except for the strong one-term inverse estimation). In Theorem 4.3, the direct estimate with the appropriate K-functional is obtained, and in Theorem 4.10 – the strong two-term inverse estimate. Chapter 4 also contains a characterization of the K-functionals used in the above estimates, as well as the corresponding results for the iterated Boolean sums of generalized Kantorovich operators, see e.g. Theorem 4.25.

The results of Chapter 4 are published in articles 25, 26, 27, 31, 32 of the dissertation bibliography.

In Chapter 5, direct and weak inverse estimates for the error of simultaneous approximation are proved for two modifications (of Kantorovich and of Draganov) of Bernstein polynomials, which are algebraic polynomials with *integer coefficients*. Both modifications define *nonlinear operators*, Kantorovich's one being unbounded and discontinuous, and Draganov's – bounded but also discontinuous. These properties do not allow obtaining complete analogues of the results of Chapters 3 and 4. Direct estimates are

obtained in Theorems 5.1 and 5.4, and weak inverse estimates are obtained in Theorem 5.5. These estimates hold under a number of natural restrictions at the edges of the interval on the function being approximated, e.g. the obvious requirement that $f(0)$ and $f(1)$ be integers.

The results of Chapter 5 are published in articles 29, 30 of the dissertation bibliography.

Chapter 6 examines the rate of convergence to 0 of the expression

$$B_n f(x) - f(x) - \frac{1}{2n} Df(x),$$

which represents the remainder in Voronovskaya's theorem. Direct and weak inverse estimates are proved (Theorem 6.1 and Corollary 6.3).

Continuing the above development of $B_n f(x) - f(x)$ in the power of n^{-1}, Bernstein obtains new operators. In Theorem 6.4 and Corollary 6.5, a direct and weak inverse estimate for the next operator in this sequence are obtained.

The results of Chapter 6 are published in article 33 (co-authored with I. Gadjev) of the dissertation bibliography.

Summarizing the above, the dissertation contains a comprehensive statement of the current results for simultaneous approximation with Bernstein operators and their modifications. The author has completely reproduced other scientists results and very significantly improved them. In my view, the proof of the strong one-term inverse estimate in Theorem 3.26 presents the greatest difficulty overcome. Another very important contribution is finding the appropriate K-functionals (accompanied by the proof of consistent direct and strong inverses inequalities), as well as their characterization by combinations of several simple moduli.

4. Approbation of the results

The new results in the presented dissertation are published in nine papers (numbered 25 to 33 in the bibliography). Six of the articles are in impact factor journals:

- Three papers (numbered 27, 29, 31) are in *Journal of Approximation Theory* (WoS Q1/Q2), as 31 is a corrigendum to 27;
- Two papers (numbered 26, 33) are in *Results in Mathematics* (WoS Q2);

- One paper (numbered 32) is in *Studia Univ. Babes-Bolyai Math.* (WoS Q4).

One of the remaining three articles is in the Annual of Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics, and the other two are in the proceedings of the International Conference "Constructive theory of functions".

Eight of the articles are single-authored, and article number 33 is co-authored with Ivan Gadjev. For the latter, there is a declaration of equal contribution of the two co-authors.

Borislav Draganov has attached a list of 15 citations (without self-citations) of the 9 mentioned articles.

My opinion is that:

- the scientific papers meet the minimum national requirements (according to Art. 2b, paras. 2 and 3 of ADASRB) and accordingly to the additional requirements of SU "St. Kliment Ohridski" for obtaining the scientific degree "Doctor of Sciences" in the field of higher education: 4. Natural sciences, mathematics and informatics; professional direction 4.5 Mathematics. Only 2 publications, for example numbers 27 and 29 in the Journal of Approximation Theory, are enough for Borislav Draganov to exceed the minimal national requirements for indicator Г. The list of citations indicates that they are sufficient to meet the minimal national requirements for indicator Д.

- the results presented by the candidate in the dissertation work and the related scientific publications do not repeat those from previous procedures for acquiring a scientific title or an academic position;

- there is no plagiarism in the submitted dissertation and scientific publications under this procedure, with all results of other authors properly and fully cited.

5. Qualities of the abstract

The abstract (in the Bulgarian and English versions) meets all the requirements for its preparation.

It correctly and fully presents the obtained results and the content of the dissertation work.

6. Critical notes and recommendations

The dissertation is very well written. The theorems are well formulated, their proofs are complete. Individual chapters begin with the known results of other authors, simultaneously comparing the author's results with others' results. The author posses very good knowledge of the subject literature and extensively cites all currently known results.

I have no critical remarks about the dissertation. The number of typos is minimal and does not hinder the smooth exposition of this complex material.

7. Conclusion

Having become acquainted with the DrSci thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific contributions contained therein, I confirm that the presented DrSci thesis and the accompanying scientific publications, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the ADASRB, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" for acquisition by the candidate of the scientific degree "Doctor of Science" in the Scientific field "4. Natural Sciences, Mathematics and Informatics", Professional field "4.5. Mathematics". In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for this procedure.

Based on the above, I **strongly recommend** the scientific jury to award to

associate professor Borislav Radkov Draganov, PhD

the scientific degree "Doctor of Science" in the Scientific field "4. Natural Sciences, Mathematics and Informatics", Professional field "4.5. Mathematics".

23.02.2024 Reviewer:

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