**REVIEW**

**under the procedure for acquisition of the educational and scientific degree "Doctor of Science" by Dr. Borislav Radkov Draganov author of the Doctor of Science Thesis entitled: "Simultaneous approximation by the Bernstein operator"**

In the Scientific field: **4. Natural Sciences, Mathematics and Informatics**

Professional field: **4.5. Mathematics (Mathematical analysis)**

Department "Mathematical analysis"
Faculty of Mathematics and Informatics (FMI), Sofia University "St. Kliment Ohridski" (SU),

The review has been prepared by: Prof. Dr. Pencho Petrushev, University of South Carolina, as a member of the scientific jury for the defense of this Doctor of Science Thesis according to Order RD 38-627/28.11.2023 of the Rector of Sofia University.

**1. General characteristics of the dissertation thesis and the presented materials**

The Doctor of Science Thesis of Dr. Borislav Draganov entitled "Simultaneous approximation by the Bernstein operator" contains 178 pages and consists of an introduction and six chapters. The bibliography of the thesis contains 100 titles (monographs and journal articles). As indicated in the title the dissertation is concerned with the simultaneous approximation of functions and their derivatives on $[0, 1]$ by the Bernstein operator and some of its modifications.

**2. Short CV and personal impressions of the candidate**

I have been following the research of Dr. Borislav Draganov since 2004 when he defended his PhD Thesis. In reading his papers he proved to me that he is a very able analyst who is capable to grasp and develop new ideas and techniques at high intellectual level. I look forward to new excellent results from him in Approximation theory and other areas in Analysis, which I expect with confidence. I am also familiar with Dr. Draganov's style of lecturing. I have listened to his talks at several of the Sozopol Approximation theory conferences. He is an excellent communicator. He has presented his results with competence, clarity and mastery of the subject. In summary, Dr. Borislav Draganov is a talented young mathematician with promising future. I have high expectations for Dr. Draganov as his career unfolds.

**3. Content analysis of the scientific achievements of the candidate, contained in the presented Doctor of Science Thesis and in the publications to it, included in the procedure**

S.N. Bernstein introduced in 1912 his famous operator:

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},$$

which maps each continuous function $f$ on $[0,1]$ to an algebraic polynomial $B_n f$ of degree $n$. This operator allows to easily prove the famous Weierstrass theorem for approximation of continuous functions by algebraic polynomials on $[0,1]$, which can be expressed by the identity

$$\lim_{n\to\infty} \|f - B_n f\| = 0, \quad \forall f \in C[0,1],$$

where $C[0,1]$ stands for the space of all continuous functions on $[0,1]$ and $\|\cdot\|$ is the sup-norm on $[0,1]$.

As for now there is a great deal of publications on the approximation by the Bernstein operator and its properties. Dr. Draganov's dissertation is devoted to the simultaneous approximation of functions and their derivatives on $[0,1]$ with Jacobi weights by the Bernstein operator and some of its modifications, in particular, Boolean sums of Bernstein operators and Bernstein operators with integer coefficients.

In the introduction of the dissertation Dr. Draganov reviews some of the main existing results on approximation by the Bernstein operator, including the following fundamental result by Ditzian and Totik: For any $f \in C[0,1]$

$$\|f - B_n f\| \le c\omega_\varphi^2(f, n^{-1/2}), \tag{1}$$

where the modulus of smoothness $\omega_\varphi^2(f,t)$ is defined by

$$\omega_\varphi^2(f,t) = \sup_{0<h\le t} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \quad t>0.$$

Here $\varphi(x) := \sqrt{x(1-x)}$. The inverse estimate is also presented:

$$\|f - B_n f\| \ge c\omega_\varphi^2(f, n^{-1/2}), \quad n \ge n_0. \tag{2}$$

The famous Voronovskaya result is also given: For any $f \in C^2[0,1]$

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x) \tag{3}$$

uniformly on $[0,1]$. Furthermore, the following result for simultaneous approximation is presented: For any $f \in C^s[0,1]$, $s \ge 0$,

$$\lim_{n\to\infty} (B_n f)^{(s)}(x) = f^{(s)}(x) \quad \text{uniformly on} \quad [0,1]. \tag{4}$$

One of the main objectives of the dissertation is to quantify the rates of approximation in (3) and (4) in the spirit of estimates (1)–(2) with Jacobi weights. Sharp direct estimates and matching strong inverse estimates are obtained.

Chapter 1 contains the definitions and basic properties of K-functionals and moduli of smoothness on $[0,1]$. Recall the classical $K$-functional between the spaces

$L_\infty[0,1]$ and $W^m_\infty[0,1]$ defined by

$$K_m(f,t) := \inf_{g\in W^m_\infty[0,1]} \left\{\|f-g\| + t\,\|g^{(m)}\|\right\}.$$

The classical modulus of smoothness is defined by

$$\omega_m(f,t) := \sup_{0\le h\le t} \|\Delta^m_h f\|_{L_\infty[0,1-mh]}, \quad t>0,\ m\in\mathbb{N},$$

where $\Delta^m_h f$ is $m$th forward finite difference of $f\in C[0,1]$. The well known classical result of Johnen asserts that $K_m(f,t^m)\sim\omega_m(f,t)$. In Chapter 1 Dr. Draganov introduces more general K-functionals and moduli of smoothness on $[0,1]$ with Jacobi weights that are utilized later in the dissertation.

In Chapter 2 Dr. Draganov presents a number of imbedding inequalities in the spirit of the classical Landau–Kolmogorov inequality:

$$\|f^{(j)}\|_J \le c(\|f\|_J + \|f^{(m)}\|_J), \quad 0\le j\le m,\ f\in W^m_\infty(J),$$

where $J$ is an interval. The imbedding inequalities given in the dissertation involve the Jacobi weights

$$\omega(x) := x^{\gamma_0}(1-x)^{\gamma_1}, \quad \gamma_0,\gamma_1\ge 0, \tag{5}$$

and differential operators such as

$$Df(x) := \varphi(x)^2 f''(x) \quad\text{with}\quad \varphi(x) := \sqrt{x(1-x)}. \tag{6}$$

The results from Chapters 1 and 2 are ancillary and serve as a preparation for the developments in the subsequent chapters.

Chapter 3 is concerned with the simultaneous approximation of functions and their derivatives by the Bernstein operator. In the dissertation Dr. Draganov clearly describes the existing results on the subject in the literature. It should be pointed out that the very few such results are weaker and incomplete compared to the results of Dr. Dragnev.

One of the main theorems here (Theorem 3.3) asserts that if $s\in\mathbb{N}$, $0\le\gamma_0,\gamma_1<s$, $f\in C[0,1]$ and $wf^{(s)}\in L_\infty[0,1]$, then

$$\|w(B_nf-f)^{(s)}\| \le c\,K^D_s(f^{(s)},n^{-1})_w, \quad n\in\mathbb{N}, \tag{7}$$

where

$$K^D_s(f,t)_w := \inf_{g\in C^{s+2}[0,1]} \left\{\|w(f-g^{(s)})\| + t\|w(Dg)^{(s)}\|\right\}.$$

It is shown that the direct estimate from (7) is sharp. Moreover, the following strong inverse result is established (Theorem 3.8): Let $s\in\mathbb{N}$ and $0\le\gamma_0,\gamma_1<s$. Then there exists $R\in\mathbb{N}$ such that for any $f\in C[0,1]$ with $wf^{(s)}\in L_\infty[0,1]$, and all $k,n\in\mathbb{N}$ with $k\ge Rn$

$$K^D_s(f^{(s)},n^{-1})_w \le c\,\frac{k}{n}\left(\|w(B_nf-f)^{(s)}\| + \|w(B_kf-f)^{(s)}\|\right). \tag{8}$$

In particular,

$$K^D_s(f^{(s)},n^{-1})_w \le c\left(\|w(B_nf-f)^{(s)}\| + \|w(B_{Rn}f-f)^{(s)}\|\right).$$

Later in Chapter 4 it is shown in Theorem 4.4 that the above $K$-functional $K^D_s(f,t)_w$ can be characterized in terms the following simpler $K$-functionals:

$$K_m(f,t)_w = \inf_{g\in AC^{m-1}_{loc}(0,1)} \left\{\|w(f-g)\| + t\|wg^{(m)}\|\right\} \tag{9}$$

and

$$K_{m,\varphi}(f,t)_w = \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{ \|w(f-g)\| + t\|w\varphi^m g^{(m)}\| \right\}. \tag{10}$$

Using in addition that $K_1(f,t)_w \sim \omega_1(f,t)_w$ and $K_{2,\varphi}(f,t^2)_w \sim \omega_\varphi^2(f,t)_w$ (the weighted Ditzian-Totik modulus of smoothness) Dr. Draganov derives the following direct result (Theorem 3.5): If $f \in C[0,1]$ and $wf^{(s)} \in L_\infty[0,1]$, $s \in \mathbb{N}$, then

$$\begin{aligned}
&\|w(B_n f - f)^{(s)}\| \\
&\quad \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2})_w + \omega_1(f', n^{-1})_w, & s = 1,\ 0 \le \gamma_0, \gamma_1 < 1, \\[2ex] \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\, \|f^{(s)}\|, & s \ge 2,\ \gamma_0 = \gamma_1 = 0, \\[2ex] \omega_\varphi^2(f^{(s)}, n^{-1/2})_w + \dfrac{1}{n}\, \|wf^{(s)}\|, & s \ge 2,\ 0 < \gamma_0, \gamma_1 < s. \end{cases}
\end{aligned}$$

For the proof of the direct estimate (7) Dr. Draganov applies a standard method, used for instance in the monograph of Ditzian and Totik “Moduli of smoothness”. For the proof of the inverse estimate (8) he uses the general scheme developed by Ditzian and Ivanov. The proofs of the above results in the dissertation are nontrivial and highly technical. To prove these results Dr. Draganov establishes a number of auxiliary results that are interesting in their own right. For example, in order to prove the direct estimate (7) Dr. Draganov establishes the following important estimates (see Proposition 3.14 and Corollary 3.18): For any $f \in C[0,1]$ such that $wf^{(s)} \in L_\infty[0,1]$

$$\|w(B_n f)^{(s)}\| \le c\|wf^{(s)}\|,$$

and for any $f \in AC^{s+1}[0,1]$ such that $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\|w(Df)^{(s)}\|.$$

For the proof of the inverse estimate (8) Dr. Draganov establishes the following interesting Bernstein-type inequalities (see Corollary 3.24 and Corollary 3.25): For any $f \in C[0,1]$ such that $wf^{(s)} \in L_\infty[0,1]$

$$\|w(DB_n f)^{(s)}\| \le cn\|wf^{(s)}\|,$$

and for any $f \in C[0,1]$ such that $w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$

$$\|w(D^2 B_n f)^{(s)}\| \le cn\|w(Df)^{(s)}\|.$$

In addition, in the following somewhat more restricted case Dr. Draganov manages to straighten the inverse estimate (8) (Theorem 3.26): Let $1 \le s \le 6$ and $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}$ such that for any $f \in C[0,1]$ with $wf^{(s)} \in L_\infty[0,1]$

$$K_s^D(f^{(s)}, n^{-1})_w \le c\,\|w(B_n f - f)^{(s)}\|, \quad n \ge n_0.$$

This estimate along with (7) implies that under the above assumptions

$$\|w(B_n f - f)^{(s)}\| \sim K_s^D(f^{(s)}, n^{-1})_w.$$

Finally, in Section 3 Dr. Draganov considers the simultaneous approximation of functions and their derivatives by the Kantorovich operator, defined by

$$K_n f(x) = \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt\, p_{n,k}(x),\ \ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Using the relationship between the Kantorovich and Bernstein operators he obtains direct and inverse estimates similar to estimates (7) and (8) for the simultaneous approximation by the Kantorovich operator.

From above it becomes clear that Dr. Draganov has obtained impressive results on simultaneous approximation of functions and their derivatives by the Bernstein operator. The gist of Dr. Draganov's results is that they utilize the right characteristics: K-functionals and moduli of smoothness of Ditzian and Totik which account for the improvement of the approximation near the ends of the interval. In addition to this Dr. Draganov study simultaneous approximation in the more general setting with Jacobi weights. At the same time all existing results (except for one) are in terms of classical modili of smoothness with a fixed step. With his results in Chapter 3 Dr. Draganov has practically exhausted the research theme "simultaneous approximation of functions and their derivatives by the Bernstein operator".

Chapter 4 of the dissertation deals with weighted simultaneous approximation of functions and their derivatives by iterated Boolean sums of Bernstein operators $\mathcal{B}_{r,n}$, defined by

$$\mathcal{B}_{r,n} := I - (I - B_n)^r,$$

where $I$ stands for the identity.

The rates of $\|\mathcal{B}_{r,n} f - f\|$ are sufficiently well characterized in the literature. Dr. Draganov focuses on the simultaneous approximation by the operators $\mathcal{B}_{r,n}$ on $[0,1]$ with Jacobi weights. The following direct estimate is established in the dissertation (Theorem 4.3): Let $r, s \in \mathbb{N}$ and $0 \le \gamma_0, \gamma_1 < s$. Then for any $f \in C[0,1]$ such that $wf^{(s)} \in L_\infty[0,1]$

$$\|w(\mathcal{B}_{r,n} f - f)^{(s)}\| \le c\, K^D_{r,s}(f^{(s)}, n^{-r})_w, \tag{11}$$

where the K-functional $K^D_{r,s}(f,t)_w$ is defined by

$$K^D_{r,s}(f,t)_w := \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\| + t\|w(D^r g)^{(s)}\| \right\}.$$

Also, a matching inverse estimate is established (Theorem 4.10): Let $r, s \in \mathbb{N}$ and $0 \le \gamma_0, \gamma_1 < s$. Then there exists $R \in \mathbb{N}$ such that for any $f \in C[0,1]$ with $wf^{(s)} \in L_\infty[0,1]$, and all $k, n \in \mathbb{N}$ with $k \ge Rn$

$$K_{r,s}(f^{(s)}, n^{-r})_w \le c \left(\frac{k}{n}\right)^r \left( \|w(\mathcal{B}_{r,n} f - f)^{(s)}\| + \|w(\mathcal{B}_{r,k} f - f)^{(s)}\| \right). \tag{12}$$

In particular,

$$K_{r,s}(f^{(s)}, n^{-r})_w \le c \left( \|w(\mathcal{B}_{r,n} f - f)^{(s)}\| + \|w(\mathcal{B}_{r,Rn} f - f)^{(s)}\| \right).$$

It is shown (Theorems 4.4, 4.5) that the $K$-functional $K^D_{r,s}(f,t)_w$ can be characterized in terms of the simpler $K$-functionals from (9)–(10). Furthermore, using the relations $K_r(f, t^r)_w \sim \omega_r(f,t)_w$ and $K_{2r,\varphi}(f, t^{2r})_w \sim \omega^{2r}_\varphi(f,t)_w$ Dr. Draganov derives the following direct estimates: Given $r, s \in \mathbb{N}$ and $0 < \gamma_0, \gamma_1 < s$, for any $f \in C[0,1]$ such that $wf^{(s)} \in L_\infty[0,1]$

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2})_w + \omega_1(f', n^{-r})_w, & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2})_w + \dfrac{1}{n^r}\,\|wf^{(s)}\|, & s \ge 2, \end{cases}$$

and for any $f \in C^s[0,1]$

$$\|(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2}) + \omega_r(f', n^{-1}) + \omega_1(f', n^{-r}), & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2}) + \omega_r(f^{(s)}, n^{-1}) + \dfrac{1}{n^r}\,\|f^{(s)}\|, & s \ge 2. \end{cases}$$

In addition, Dr. Draganov establishes a direct estimate in terms of the operator $D$ (Theorem 4.9): Given $r, s \in \mathbb{N}$ for any $f \in C^{2s}[0,1]$

$$\|D^s(\mathcal{B}_{r,n}f - f)\| \le c\,\widehat{K}_{r,s}(D^s f, n^{-r}),$$

where

$$\widehat{K}_{r,s}(F,t) = \inf_{g \in C^{2(r+s)}[0,1]} \{\|F - D^s g\| + t\|D^{r+s}g\|\}.$$

Finally, in Section 4 Dr. Draganov obtains a direct estimate for simultaneous approximation by iterated Boolean sums of the Kantorivich operator as a consequence of the respective estimate for approximation by $\mathcal{B}_{r,n}$.

The main contribution of Dr. Draganov in this section is to the *simultaneous approximation* by the operators $\mathcal{B}_{r,n}$ on $[0,1]$ with Jacobi weights. To prove the above estimates Dr. Draganov applies the method of proof of the results for simultaneous approximation by the Bernsten operator $B_n$. To achieve this he proves a number of intermediate estimates; most of them are interesting in their own right. The proofs are involved and require a lot of work, ingenuity and persistence.

In Chapter 5 Dr. Draganov concentrates on the simultaneous approximation by Bernstein polynomials with integer coefficients. The problem (posed by S.N. Bernstein) is to determine to what extent the rate of approximation from algebraic polynomial in the uniform norm on $[0,1]$ is affected by the requirement that the coefficients of the approximating algebraic polynomials are integer. To solve this problem L.V. Kantorovich introduced the operator

$$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[ f\left(\frac{k}{n}\right)\binom{n}{k} \right] x^k(1-x)^{n-k},$$

where $[\alpha]$ denotes the largest integer $\le \alpha$. Kantorovich showed that if $f \in C[0,1]$ is such that $f(0), f(1) \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0.$$

Dr. Draganov introduces another integer modification of the Bernstein polynomial, defined by

$$\widehat{B}_n(f)(x) := \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right)\binom{n}{k} \right\rangle x^k(1-x)^{n-k},$$

where $\langle\alpha\rangle$ stands for the nearest integer to the real $\alpha$.

Observe that the operator $\widetilde{B}_n$ is not bounded, while $\widehat{B}_n$ is bounded, but not continuous. Both $\widetilde{B}_n$ and $\widehat{B}_n$ are not linear.

In this chapter Dr. Draganov concentrates on the simultaneous approximation by the operators $\widetilde{B}_n$ and $\widehat{B}_n$. He establishes direct estimates for $\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\|$ and $\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\|$ in terms of the Ditzian-Totik modulus of smoothness $\omega_\varphi^2(\cdot,\cdot)$ under appropriate boundary conditions on $f$ as well as weak inverse estimates. As a consequence he derives the following characterizations (Corollaries 5.6, 5.7): Under the assumptions of Theorem 5.1 for $0 < \alpha < 1$

$$\begin{aligned}&\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha})\\ &\qquad\qquad \iff \quad \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha),\end{aligned}$$

and under the assumptions of Theorem 5.4 for $0 < \alpha < 1$

$$\begin{aligned}&\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha})\\ &\qquad\qquad \iff \quad \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha).\end{aligned}$$

A saturation result is also obtained (Theorem 5.14): Let $f \in C^s[0,1]$, $s \in \mathbb{N}_0$, be such that $f(0), f(1) \in \mathbb{Z}$. If

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = o(1/n) \quad \text{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = o(1/n),$$

then $f(x) = px + q$ for some $p, q \in \mathbb{Z}$ and thus $\widetilde{B}_n(f) = \widehat{B}_n(f) = f$ for all $n$.

The results from Chapter 5 are also nontrivial and obtaining them requires a thorough analysis and a lot of work. The idea for the introduction and study of the operator $\widehat{B}_n$ is novel. Although not in the core of the dissertation the results from this chapter are a nice addition to it as a whole.

In Chapter 6 Dr. Draganov deals with the rate of approximation in the Voronovskaya result (3) for the Bernstein operator. The following two differential operators play an important role here

$$D_n f(x) := n(B_n f(x) - f(x)) \quad \text{and} \quad \mathcal{D} f(x) := \frac{\varphi^2(x)}{2} f''(x).$$

The main result in this chapter involves the K-functional

$$\widetilde{K}(F, t) := \inf_{g \in W_\infty^4(\varphi)[0,1]} \left\{ \|F - \mathcal{D} g\| + t \left( \|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\| \right) \right\}.$$

The main Theorem 6.1 here reads as follows: If $f \in C[0,1]$ and $\varphi^2 f'' \in L_\infty[0,1]$, then

$$\|D_n f - \mathcal{D} f\| \le c\, \widetilde{K}(\mathcal{D} f, n^{-1}) \le c \left( K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{1}{n} \|\varphi^2 f''\| \right).$$

Conversely, if $f \in C[0,1]$ and $\varphi^2 f'' \in L_\infty[0,1]$, then for all $k, n \in \mathbb{N}$

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\, \|D_k f - \mathcal{D} f\| + c \frac{k}{n}\, K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{n} \|\varphi^2 f''\|.$$

As a consequence of the above Dr. Draganov derives the following characterization: If $f \in C[0,1]$ and $\varphi^2 f'' \in L_\infty[0,1]$ and $0 < \alpha < 1$, then

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \iff \quad K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha).$$

To proof the above results Dr. Draganov obtains several nontrivial properties of the Voronovskaya operator and related estimates.

From all of the above it is clear that Dr. Draganov has developed an impressive body of work on simultaneous approximation by the Bernastein operator and some of its modifications. In his dissertation he shows a deep understanding of the approximation by linear operators, in particular by the Bernstein operator, and an excellent familiarity with the existing results in the literature on the subject. In each chapter of the dissertation Dr. Draganov includes and discusses all relevant existing results in the literature. Although standard the proofs of his results are nontrivial, original and highly technical. Dr. Draganov has displayed considerable energy, independence, and ingenuity in his work. The dissertation is very well written, the results and their proofs are carefully structured and presented in sufficient detail, which makes it easy to reed. He does a great job in explaining the basic concepts and the ideas of his proofs. Overall Dr. Draganov's dissertations has the character of a comprehensive treatise on the simultaneous approximation of functions and their derivatives by the Bernstein operator, and some of its modifications, in particular, Boolean sums of Bernstein operators and Bernstein operators with integer coefficients, which leaves little room for further development.

## 4. Approbation of the results

Dr. Draganov's dissertation is based on nine of his articles published in good journals: Journal of Approximation Theory, Results in Mathematics and Studia Universitatis Babeş-Bolyai Mathematica. Two of his papers are published in conference proceedings. Dr. Draganov's results from the dissertation have been cited in 15 publications.

The scientific works meet the minimum national requirements (under Art. 2b, para. 2 and 3 of ADASRB*) and the additional requirements of Sofia University "St. Kliment Ohridski" for acquiring the scientific degree "Doctor of Science" in the scientific field and professional field of the procedure.

The results presented by Dr. Draganov in the dissertation do not overlap the results from his PhD Thesis.

Dr. Draganov's results are original and impressive. Plagiarism is out of question. In the dissertation the existing relevant results are presented and cited very carefully and in detail.

## 5. Qualities of the abstract

The abstract is well written and presents correctly the results and the content of the dissertation. It meets all requirements for preparation of abstracts.

## 6. Critical notes and recommendations

This reviewer has no critical remarks concerning Dr. Draganov's dissertation. He looks forward to new excellent results from Dr. Draganov.

## 7. Conclusion

Having read the Doctor of Science Thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific contributions contained therein, I **confirm** that the presented Doctor of Science thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the ADAS in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by the candidate of the scientific degree "Doctor of Science" in the

Scientific field: Natural Sciences, Mathematics and Informatics, Professional field: Mathematics. In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, I **strongly recommend** the scientific jury to award Dr. Borislav Draganov the scientific degree "Doctor of Science" in the Scientific field: Natural Sciences, Mathematics and Informatics and Professional field: Mathematics.

Date: February 20, 2024 Reviewer:

(Prof. Dr. Pencho Petrushev)