Review

according to the procedure for the defense of a dissertation on the topic:

"Class C^*-Toeplitz Algebras"

to acquire

educational and scientific degree "doctor"

candidate: Nikolay Petrov Buyukliev,
Field of higher education: 4. Natural sciences, mathematics and informatics
Professional direction: 4.5. Mathematics,
Doctoral program: Mathematical analysis , department: Mathematical analysis,
Faculty of Mathematics and Informatics (FMI),
Sofia University "St. Kliment Ohridski"(SU),

The review was prepared by: Prof. Dr. Sci. Sevdzhan Akhmedov Hakkev, IMI-BAS
in my capacity as a member of the scientific jury, according to Order No. RD-38-223 / 28.04.2023 of the Rector of Sofia University.

1. General characteristics of the dissertation work and presented materials

The presented materials have been prepared in accordance with the Law on the Development of the Academic Staff in the Republic of Bulgaria (LDASRB), the Regulations for the Application of the LDASRB, as well as the Regulations for the Terms and Procedures for Acquiring Scientific Degrees and Holding Academic Positions (RTPASDHAP) in SU"St. Cl. Ohridski". The dissertation student has submitted the complete set of documents, which includes: 1) Dissertation work; 2) Abstract in Bulgarian and English; 3) Copy of master's degree diploma; 4)Certificate of fulfillment of the minimum requirements; 5) List of scientific publications on the topic of the dissertation; 6) Autobiography; 7) Protocol for checking the originality of the dissertation work; 8) Opinion in connection with the procedure for the prevention of plagiarism; 9) Declaration of the doctoral student on originality and absence of plagiarism; 10) Records of department councils related to the opening of the procedure and the preliminary discussion of the dissertation work.

The dissertation work of the doctoral student Nikolay Buyuklev is 56 pages long, including an introduction, 5 chapters and bibliography of 52 titles.

2. Data and personal impressions about the candidate

Nikolay Buyukliev was born in 1959. He completed his higher education in 1984 with a master's degree in Mathematics at the FMM of Saint Kliment Ohridski University. In his professional career, he was successively, assistant, senior assistant and principal assistant at the Department of Mathematical Analysis of the FMI of SU. I know the dissertation student and have excellent impressions of his work

3. Content analysis of the scientific and scientific applied achievements of the candidate, contained in the presented dissertation work and the publications to it, included in the procedure

In chapter 1, the main concepts considered in the dissertation are introduced, definitions of the Toeplitz and Wiener-Hopf operators and the C^*-algebra generated by them are given as follows.

Let T be the group of complex numbers with modulus 1, μ be the Haar measure, such that $\mu(T) = 1$ and $P : L^2(T) \to H^2(T)$ is the projection operator ($H^2(T)$ is a Hardy space). For $\varphi \in C(T)$, T_φ denotes the operator $T_\varphi = P(\varphi f),\ \ f \in H^2(T)$ and $\mathcal{T}$ denotes the C^*-algebra generated by the operators T_φ

Let G be a T_2 locally compact group with unit e, P be a normal subgroup of G. For $f \in C_c(G)$, the Wiener-Hopf operator is defined

$$W_f\xi(t) = \int_G f(s)\xi(ts)1_P(ts)d\lambda(s),\ \ \xi \in L^2(P). \tag{1}$$

The C^*-algebra generated by the operators W_f, which is also called the $C*$-algebra of the Wiener-Hopf operators, is denoted by $\mathcal{B}$.

In Chapter 2 is given some preliminary information about C^*-algebras, groupoids and their algebras, examples of groupoid C^* algebras. Details of K-theory and cyclic cohomologies of C^*-algebras are also given.

In chapter 3, is considered a group C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where $\mathcal{G}$ is a Wiener-Hopf groupoid. In Section 3.1, a criteria was obtained for when an operator $T \in C^*(\mathcal{G})$ is Fredholm operator. In paragraph 3.2, a method for constructing an explicit line section is presented. It is shown that the map

$$\psi : C^*(\mathcal{G}_{|F}) \to C^*(\mathcal{G})$$
$$\psi(b)(x, n) = b(\lambda(x), n),\ \ b \in C_c(\mathcal{G}_{|F}),$$

where F is a closed and invariant subset of $\mathcal{G}^0$ and $\lambda : X \to F$ is a linear contraction is a continuous cross-section. This result is generalized to the case when F is the union of a finite

number of closed and invariant subspaces of X. It is shown that the map

$$\psi(b)(x,n) = \sum_{\sigma\subset\{1,2,\ldots,n\}} (-1)^{rank(\sigma)+1} b(\lambda_\sigma(x),n), \quad b \in C_c(\mathcal{G}_{|F}),$$

where $rank(\sigma)$ is the number of elements of σ, is a continuous cross-section.

In Chapter 4, first is presented the result of E. Park, which is related to the study of the C^*-algebra $\mathcal{T}^{\alpha,\beta}$ generated by Toeplitz operators. In particular, $\mathcal{T}^{\alpha,\beta}$ contains the $\mathcal{K}$-ideal of the compact operators and the series

$$0 \to \mathcal{K} \xrightarrow{i} \mathcal{T}^{\alpha,\beta} \xrightarrow{\gamma} \mathcal{T}^{\alpha,\beta}/\mathcal{K} \to 0$$

is exact.

This chapter considers a groupoid C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where $\mathcal{G}$ is a Wiener-Hopf groupoid. The goal is to construct a continuous line section ψ in a groupoid Wiener-Hopf algebra. Under the following assumptions:

- there exists a family M of operators (elementary generators $||A|| \leq 1,\ A \in M$) which gives rise to the algebra $C^*(\mathcal{G})$. If $A = A_1A_2...A_n$, where A_i are elementary expressions, then A is called a finite product

- there exists a numerical function $N(A)$, such that $A - \psi\gamma(A)$ has finite trace and $||A - \psi\gamma(A)||_1 < N(A)$

- there exist C_1, C_2 such that $N(A) > C_1$ and $N(AB) \leq C_2(N(A) + N(B))$

It is shown that the set of all operators of the type

$$S = \{A = \sum_{i=1}^{\infty} \alpha_i A_i : \sum_{i=1}^{\infty} |\alpha_i| N(A_i) < \infty\}$$

is algebra.

It is shown that the representation $\psi : \mathcal{T}^\infty \to C^*(\mathcal{G})$, where $\mathcal{T}^\infty = \gamma(S)$ is almost multiplicative, i.e. if $\gamma(A),\ \gamma(B) \in \mathcal{T}^\infty$, then $\psi\gamma(AB) - \psi\gamma(A)\gamma(B)$ is an operator with finite trace.

In the last paragraph of this chapter, the following result was obtained: If $T \in \mathcal{T}$ is a Fredholm operator and $\gamma(T),\ (\gamma(T))^{-1} \in \mathcal{T}^\infty$, then the index of T is given by the following formula

$$ind(T) = tr[\psi\gamma(T)\psi(\gamma(T)^{-1}) - \psi(\gamma(T)^{-1}\psi\gamma(T)].$$

Chapter 5 is devoted to the estimates of the K-theory of $\mathcal{B}(R^d,P)/\mathcal{K}$ and $\mathcal{B}$, where $\mathcal{B}(R^d,P) = C^*(\mathcal{G})$. In paragraph 5.1 it was shown that if $P \subset R^n$ satisfies certain geometric conditions, then $K_*(\mathcal{B}(R^n,P) = (0,0), \quad K_*(\mathcal{B}(R^d,P)/\mathcal{K}) = (0,Z)$ and the index map is an isomorphism. In Section 5.2 is shown that if P is a polyhedral cone in R^d, then in $\mathcal{B}(R^d,P)$ there exists a Fredholm operator with index one. It is also shown that if

$K_*(\mathcal{B}(R^d, P)/\mathcal{K}) = (0, Z)$, then $K_*(\mathcal{B}) = (0, 0)$ and the index map

$$ind : K_1(\mathcal{B}(R^n, P)/\mathcal{K}) \to K_0(\mathcal{K}$$

is an isomorphism.

Chapter 6 deals with Toeplitz operators $\mathcal{T}(H_3(Z))$ associated with the three-dimensional discrete Heisenberg group $H_3(Z)$ and its semigroup P. It is shown that for Toeplitz operators, there exists a set of two-sided closed ideals

$$\{0\} \subset I_0 \subset I_{1d} \subset I_2 \subset I_3 = \mathcal{T}(H_3(Z)),$$

where $I_0 \cong \mathcal{K}$, $I_3/I_2 \cong C^*(H_2(Z))$.

4. Approbation of the results

The results of the dissertation were published in 4 articles, all in English and all as sole author. A brief analysis of these publications shows the following: one of the articles was accepted for publication in a journal with an impact factor, and the other 3 articles were published in the Annuaire Univ. Sofia Fac. Math. Inform. The results of the dissertation work were reported at three international conferences and at the FMI scientific session.

The obtained 86 points exceed the minimum requirements for obtaining the educational and scientific degree "doctor" and the additional requirements of SU "St. Kliment Ohridski" for the acquisition of an educational and scientific degree "doctor" in the scientific field and professional direction of the procedure.

Based on the submitted materials, the reviewer assumes that there is no proven plagiarism in the submitted dissertation and research papers under this procedure.

5. Qualities of the Abtsract

The abstract in a volume of 14 pages adequately reflects the main ideas and essential final results that are described in the dissertation work.

6. Critical notes and recommendations

I have no substantive criticisms. There is also a thorough knowledge of the literature on the issues discussed in the dissertation, evident from the introduction, which makes the obtained results even more convincing.

7. Conclusion

Having familiarized myself with the dissertation work presented in the procedure and the scientific works accompanying it and based on the analysis of their significance and the scientific and applied scientific contributions contained in them, I confirm that the presented dissertation work and the scientific publications to it, as well as the quality and the originality of the results and achievements presented in them, they meet the requirements of LDASRB, the Regulations for its application and the relevant Regulations of SU "St. Kliment Ohridski" for the candidate's acquisition of the educational and scientific degree "doctor" in scientific field 4. Natural sciences, mathematics and informatics and professional field 4.5 Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional field and no plagiarism has been found in the submitted the competition of scientific papers.

Based on the above, I recommend the scientific jury to award Nikolay Petrov Buyukliev an educational and scientific degree "Doctor" in scientific field 4. Natural sciences, mathematics and informatics, professional direction 4.5 Mathematics, doctoral program "Mathematical analysis".

29.05.2023 year

Prepared the review:
(Prof. Dr.Sci. Sevdzhan Hakkeav)