O P I N I O N

on the PhD Thesis of Matey Boyanov Konstantinov titled

Subdifferential analysis
of convex-like functions

for obtaining the educational and scientific degree

"DOCTOR"

In Scientific field 4. Sciences, matematics and informatics
Professional field 4.5 Mathematics
Doctoral program "Operations research"
Department PORS, FMI of Sofia University

Member of the Jury: Prof. Tsvetomir Tsachev, PhD, IMI-BAS

The submitted PhD thesis consists of 79 pages, divided into introduction, three chapters, concluding remarks, appendix (containing proved propositions) and bibliography of 55 references. It is written in English.

The body of the text and the references make me believe that the author is very well acquainted with the problems, discussed in the thesis. The references to results by other authors are comprehensive and correct.

The content of the chapters is as follows:

Prox-regular sets in Hilbert spaces are studied in Chapter 1. Two characteristic properties of such sets are given (Theorem 1.1.1). These characteristic properties are known, in the present thesis they are proved in a different manner. The method used here forms the basis for obtaining the results in the next chapter.

Chapter 2 of the thesis is devoted to the study of the relationship between a class of functions, a generalization of the convex ones, and their epigraphs. This class of functions, called "epi uniformly lower regular (with parameter $\rho > 0$)" is introduced in Definition 2.1.2. In the next Definition 2.1.3 the notion "uniformly epi prox-regular (with parameter $r > 0$)" is introduced for the epigraph of a proper lower semicontinuous functions. It is proved that the two above mentioned notions are related as a convex function and its epigraph: the epigraph of an epi uniformly lower regular with parameter $\rho > 0$ function is uniformly epi prox-regular with parameter $r = \rho$ (Theorem 2.2.1) and if the epigraph of a proper lower semicontinuous function is uniformly epi prox-regular with parameter $r > 0$, then this function is epi uniformly lower regular with parameter $\rho = r/\sqrt{2}$ (Theorem 2.2.2). Two characteristic properties of the uniformly epi prox-regular sets are proved (Theorem 2.3.2). An epi uniformly lower regular function is charcterized by a geometric property of its epigraph in Theorem 2.4.1.

Chapter 3 of the thesis is devoted to a new proof of a famous result by Moreau and Rockafellar about the integrability of the subdifferential of a proper lower semicontinuous convex function, defined on a Banach space. The theorem states that if the subdifferential of g contains the subdifferential of f, then f and g differ by a constant. For the proof one uses "multistep finite increment formula" for a proper lower semicontinuous convex function which we owe to R. T. Rockafellar. The novelty of the proof presented in the thesis consists in the way "the knots" of the "multistep finite increment formula" are determined. Namely, they are found by a method for minimization of a proper lower semicontinuous convex function, which attains its minimum on the whole space. The above mentioned method is developed in the thesis.

The abstract is written in Bulgarian, consists of 28 pages, including 55 references. It describes adequately the results obtained in the thesis.

I would also like to note a few omissons in the text.

1. In a mathematical text it is not acceptable one to mention a notion ot a quantity, which is introduced later on. For instance, in the formulation of Theorem 1.1.1 on page 11 the term "uniformly prox-regular set" is used, while its definition appears on page 14 (Definition 1.1.2). Also, the quantity $c(\Delta)$ appears for the first time on page 20, but is defined on page 22.

2. In the proof of case 2 ($\lambda = 0$) of Theorem 2.2.1 the conclusion is obtained for $(x', \alpha') \in B((x, f(x)), 2\rho) \cap \mathrm{epi} f$, and it has to be obtained for
$(x', \alpha') \in B((x, \alpha), 2\rho) \cap \mathrm{epi} f$.

3. In the formulation of Lemma 3.2.1 it is stated that the function $\varphi_{x_0}(\cdot)$ is strictly monotone increasing on $(0, \infty)$ and this fact is used in the proof of Proposition A.1.11 in the Appendix. The function $\varphi_{x_0}(\cdot)$ is not strictly monotone increasing on the whole interval$(0, \infty)$, but this is unimportant, Proposition A.1.11 is still valid.

The results of the thesis are publihsed in 3 papers as follows:

– two, joint with the scientific advisor, in Journal of convex analysis
(in 2022 г. and accepted for publication in 2023, respectively), IF 0.622;
– one, joint with the scientific advisor, in Journal of applied analysis
(accepted for publication in 2023), SJR 0.212;

Based on the above I recommend the Honorable scientific jury to award Matey Boyanov Konstantinov the educational and scientific degree “Doctor” in Professional field 4.5 Mathematics, doctoral program “Operations research”.

May 31, 2023 Prof. Tsvetomir Tsachev, PhD
Sofia