REVIEW

of the PhD Thesis entitled:
"**Subdifferential analysis of convex-like funcitons**"

under the procedure for acquisition
of the educational and scientific degree "Doctor"
in Professional Field 4.5.Mathematics,Doctoral
program:"Operations research",
At Sofia University "St.Kliment Ohridski",Faculty of
Mathematics and Informatics
Candidate:Matey Boyanov Konstantinov
Reviewer: Prof. Nadezhda Kostadinova Ribarska,
DSci May 31, 2023

I am writing this review in my capacity as a member of the scientific jury, according to Order No. RD-38-113/6.03.2023 of the Rector of Sofia University. The presented dissertation "Subdifferential analysis of convex-like functions" (Субдиференциален анализ на функции, подобни на изпъкналите) consists of 79 pages and is written in English. It contains an introduction, three chapters, a conclusion, an appendix and a bibliography including 55 titles. An abstract in Bulgarian and an abstract in English (both 28 pages long), as well as all other documents required by the procedure (including a report from the anti-plagiarism system) are submitted. I accept for review all submitted materials.

Matey Boyanov Konstantinov was born in 1994. He graduated from the Sofia Mathematical High School in 2013 and immediately began his studies at the Faculty of mathematics and informatics of the Sofia University "St. Kliment Ohridski", majoring in mathematics. He graduated as a Bachelor of Mathematics in July 2017. During his studies, he received awards from the National Student Olympiad in Mathematics and the National Student Olympiad in Computer Mathematics. He did his master's degree in the master's program "Optimization" and graduated with honors in 2019 with a thesis on "Research of advertising models" under the supervision of Prof. DSci Mikhail Krastanov. From February 2020 Matey Konstantinov is a full-time doctoral student in the doctoral program "Operations Research" at FMI, SU with scientific supervisor Prof. DSci Nadia Zlateva. Since April 2023 he occupies a position of a researcher (first level) at FMI, Sofia University.

I know Matey Konstantinov since 2018, when he enrolled in the optional course on Introduction to functional analysis lectured by me. I think he is very fair and diligent. I especially want to emphasise his enthusiastic attitude towards teaching. During his studies, as well as as a master's and doctoral stu-

dent, he taught classes in Probability and Statistics, Differential Equations and Applications, Differential and Integral Calculus – 1, Differential and Integral Calculus – 2, Operations Research, Linear Optimization, Mathematical Optimization. I have excellent reports about him as an assistant from most of the lecturers of the respective disciplines. I also want to acknowledge the tremendous work he did in ensuring the smooth running of the remote part of the conference 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics, Borovets 2021.

Matey Konstantinov's dissertation is in the field of non-smooth analysis. It is a modern mathematical discipline, part of the so-called variational analysis. The rapid development of this part of mathematics is largely due to the realization that non-smooth phenomena are much more widespread than previously thought, as well as to the needs of optimization and optimal control. Specifically, the thesis contains results on prox-regular sets, primal lower nice functions, and integrability of the subdifferential of lower semicontinuous convex functions.

The first and the second chapter of the dissertation are devoted to the study of primal lower-nice functions and prox-regular sets. The corresponding class of functions and the corresponding class of sets are in a natural relationship among themselves. These classes were introduced by Poliquin in 1991 (in a finite-dimensional space) and by Poliquin and Rockafellar in 1996, respectively, and represent a natural generalization of convex, on the one hand, and C^2-smooth, on the other hand, functions and sets. A rich theory has been built around the properties of these classes, including regularization, integrability, second-order properties, and more. Characterizations of these objects that do not use the normal structure are obtained in the dissertation. Of course, this could be extremely useful.

The results in the first and second chapters are obtained for subsets of a Hilbert space and for functions defined on subsets of a Hilbert space. The rich structure allows for many considerations that are essentially geometric. As an example, roughly speaking, a closed set is uniformly prox-regular with constant r when any point on its boundary outside can be touched by a ball of radius r. The exact definition includes the term "proximal normal". An internal characterization of these sets (which does not use normals) is obtained in the first chapter of the dissertation. Unfortunately, the characterization is already known, but it is obtained in a new way – essentially via the convexity modulus of a Hilbert space.

Chapter two is central to this dissertation. A new class of functions – epi uniformly lower regular – is introduced, which is somewhat broader than the class of primal lower-nice functions (a well-established and widely studied concept). A characterization of the subsets of the Cartesian product of the Hilbert space and the real line, which are epigraphs of functions of the specified

class, has been established. One remark: in the proof of Theorem 2.2.1 the conclusion is obtained for points in the set near $(x, f(x))$, and not ones that are close to (x, α). I guess this could be fixed easyly. Then, using a method similar to that used in the first chapter, a characterization (up to the multiplication of a constant by $\sqrt{2}$) of such epigraphs that does not use normals is found. As a consequence, the main result – Theorem 2.4.1 – characterization of epi uniformly lower regular functions is obtained. I believe that the difference in the constant is not significant. Overall, the proofs are highly technical, and the result is interesting.

The third chapter of the thesis is devoted to a new proof of the classical Moreau-Rockafellar result on the integrability of the subdifferential of lower semicontinuous convex functions defined on a Banach space (by this we mean that the condition that the subdifferential of g contains the subdifferential of f implies that f and g differ by a constant). It is based on a proof proposed by Milen Ivanov and Nadia Zlateva, but with a different construction of the approximating sequences.

The results contained in the dissertation are published in three articles – two in the Journal of Convex Analysis (the journal has an impact factor) and one in the Journal of Applied Analysis (the journal has an impact rank). The three articles are joint with the supervisor.

Part of the results of the dissertation were reported at prestigious scientific forums: 10-th International Conference on Numerical Methods and Applications August 22 – 26 August, 2022, Borovets (Bulgaria) and 15-th International Workshop on Well-Posedness of Optimization Problems and Related Topics June 28 – July 2, 2021, Borovets (Bulgaria). Matei Konstantinov also took part in the 14th International Workshop on Well-Posedness of Optimization Problems and Related Topics August 20-24, 2018, Borovets (Bulgaria) and at the 46th Winter School of Abstract Analysis, January 2018, Svratka (Czech Republic).

The results obtained in Matey Konstantinov's dissertation and the publications based on these results exceed a lot the minimum national requirements (according to Art. 2b, Para. 2 and 3 of the ZRASRB) and, respectively, the additional requirements of SU "St. Kliment Ohridski" for the acquisition of the educational and scientific degree "doctor" in professional field 4.5 Mathematics (Operations Research).

An abstract in Bulgarian and an abstract in English are presented, consisting of 28 pages and 55 titles of cited literature. The abstracts accurately and comprehensively reflect the results described in the dissertation.

My remark is that there are a lot of language errors and minor inaccu-

racies that could be easily rectified. It would also be good to submit the full text of the article published in the Journal of Applied Analysis.

The dissertation contains original results. Referencing other people's results is comprehensive and correct. The presentation is good. The dissertation contains scientific results that are an original contribution to the scientific field and which can be developed in future.

In conclusion, Matey Konstantinov's dissertation is an original research in the field of subdifferential calculus. Received results are published in peer-reviewed journals. I confirm that the presented dissertation and the scientific publications related to it meet the requirements of ZRASRB, the Regulations for its application and the relevant Regulations of SU "St. Kliment Ohridski" for the candidate's acquisition of the educational and scientific degree "doctor" in professional field 4.5 Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional field and no plagiarism has been found in the scientific works submitted.

Based on the above, I recommend the scientific jury to award Matey Boyanov Konstantinov the educational and scientific degree "doctor" in Scientific Area 4. Natural sciences, Mathematics and Informatics, Professional field 4.5. Mathematics, doctoral program "Operations Research".

31.05.2023

(Prof. N. Ribarska, DSci)