R E V I E W

of the dissertation entitled "'Variational analysis without variational principles"'

for the acquisition of PhD degree of the Sofia University

by Stoyan Raychev Apostolov

in Mathematics (Mathematical Analysis)

The review is prepared by Prof. DSci Nadia Peycheva Zlateva, member of the scientific jury for the procedure, according to Order No. RD 38-309/01.07.2022 of the Rector of the Sofia University.

1. General description of the dissertation

The dissertation presented by Stoyan Apostolov has a volume of 72 pages and is written in English. It consists of Introduction, Preliminaries, three essential chapters (Chapters 3, 4, 5), Conclusion and Bibliography of 61 titles. On page 65, three author publications on the dissertation and one preprint are listed.

The dissertation contains original results. The reference to results already known results is exhaustive and correct. The content is very well presented. The dissertation contains scientific results that are an original contribution to the scientific field and that can be continued in future research.

In working on the dissertation, the use of variational principles was deliberately avoided. This is somewhat justified because it demonstrates a unified approach to the topic and a possession of other techniques for proving the main results.

2. Data and personal impressions about the candidate

Stoyan Apostolov is a graduate of Faculty of Mathematics and Informatics (FMI) of Sofia University (SU). He graduated from a Bachelor programme in Applied Mathematics in 2017 and a Master programme in Optimization in 2019. As a student, he won a silver medal at the 22-nd International Mathematics Competition, 27 July – 2 August, 2015, Blagoevgrad, Bulgaria. From July 2019 to June 2022, he is a full-time doctoral student in the "Mathematical Analysis" doctoral program of FMI, SU with scientific supervisor Prof. DSci Nadezhda Ribarska.

I have known Stoyan Apostolov since he was my student. I have good impressions of him as a colleague with a clear understanding and a good sense of Mathematics. I know his work from the talks he gave at various scientific forums, which also show his ability to adequately present his new results.

3. Analysis of the dissertation

The **first chapter** is introductory. It provides an overview of the currently known results and briefly presents the results obtained in the dissertation. The overview shows a very good knowledge of the state of the field. Previous results by other authors are duly cited. The contributions in the dissertation are clearly highlighted and the concepts introduced by the author and the results obtained by him are indicated.

The central theme of the dissertation is the notion of transversality. The classical definition of transversality at a common point of two smooth manifolds in Euclidean space $\mathbb{R}^n$ is that the sum of the tangent spaces at the common point is the entire space. H. Sussmann in a 2006 paper summarized the definition of transversality for closed convex cones C^A and C^B – tangential to sets A and B in Euclidean space – they are transversal when $C^A - C^B \equiv \mathbb{R}^n$ and are strongly transversal if they also satisfy $C^A \cap C^B \neq \{0\}$. In the finite dimensional case, the strong transversality of tangent cones of the same type, e.g. of F. Clark, is a sufficient condition for local inseparation of sets. The sets A and B are locally separable at a point $x \in A \cap B$ if there exists a neighborhood U of x such that $U \cap A \cap B = \{x\}$. In the infinite dimensional case, the strong transversality of the tangent cones does not always entail local inseparation of the sets. Example 1.0.1 illustrates this.

Different properties of the transversality type exist, depending on the different possible applications. Many concepts summarizing classical transversality and transversality of cones have been discussed in the literature. Central among these are transversality and subtransversality, also discussed in A. D. Ioffe's monograph *Variational Analysis of Regular Mappings: Theory and Applications* (2017) and the reason for this is their relation to the metric regularity and subregularity of multivalued maps. As a concept, subtransversality was introduced in 2015 in a paper by D. Drusvyatskiy, A. D. Ioffe, and A. S. Lewis (in connection with the proof of linear convergence of the altering projections algorithm), although it has appeared under different names for more than 20 years. Bivas, Krastanov and Ribarska in their 2020 publication introduced the concept of tangential transversality and explored its properties.

In the **second chapter** of the dissertation, some necessary definitions and preliminaries are given.

In § 3.1 of the **Chapter 3**, a characterization of subtransversality in a complete metric space is given – Theorem 3.1.4, as the technical result allowing the transition from local to global inequality is contained in Lemma 3.1.3. Here, a characterization of the subtransversality by means of the slope of the so-called *coupling function*. In § 3.2, the transversality property is considered. A characterization of transversality is given in terms of the so-called translated subtransversality. Characterizations of transversality using the slope of the coupling function are also obtained.

An intermediate notion between subtransversality and transversality is the notion of intrinsic transversality, introduced by D. Drusvyatskiy, A. D. Ioffe, and A. S. Lewis in their aforementioned work in finite dimensional space. They also gave a characterization of intrinsic transversality in a finite-dimensional space in terms of the slope of the coupling function. In § 3.3 this characterization is used as the definition of intrinsic transversality in metric spaces – Definition 3.3.2. In Definition 3.3.3, the property $(\mathcal{LT})$ is introduced, which is proved in Corollary 3.3.4 to be a necessary and sufficient condition for intrinsic transversality of two sets at their common point. This gives an answer to the question posed by A. D. Ioffe about finding a metric characterization of intrinsic transversality.

In an article by N. Thao et al. since 2020, a generalization of intrinsic transversality in Hilbert spaces has been made, which is based on the structure of the space. The property $(\mathcal{P})$ was also introduced in this article, which is also proved there to be equivalent to the generalization in question in Hilbert spaces. Main results in § 3.3 are Theorem 3.3.9, in which it is shown that in a normed space the property $(\mathcal{P})$ at a common point of closed sets entails the property $(\mathcal{LT})$ for them at the point, and Theorem 3.3.10, in which it is shown, that in the case of a Hilbert space the opposite is also true, i.e. that the property $(\mathcal{LT})$

entails the property ($\mathcal{P}$) into the common point. The proofs is very technical. Establishing the exact relationship between intrinsic transversality and tangential transversality allows one to obtain characterizations of transversality and subtransversality close in nature to tangential transversality. They from theirs side are used to show that

$$\text{transversality} \Rightarrow \text{tangential transversality} \Rightarrow$$

$$\Rightarrow \text{intrinsic transversality} \Rightarrow \text{subtransversality}$$

and that none of the implications are reversible. This result provides new insight on the subject.

In § 3.4 it is shown that a multivalued mapping between metric spaces is metrically subregular at a point of its graph if and only if two suitably chosen sets, one of which is the graph of the mapping, are subtransversal at that point (Theorem 3.4.1), and if the spaces are normed, it is metrically regular at a point on its graph if and only if these same sets are transversal in it (Corollary 3.4.2). In § 3.5, in Theorem 3.5.2, a subregularity characterization of a multivalued map at a point of the graph obtained in works of A. D. Ioffe and A. Y. Kruger, is proved in a new way. In § 3.6, in Theorem 3.6.1, a *rate-of-descent* type characterization of a metric regularity of a multivalued map between complete metric spaces X and Y at a point on its graph is given, and characterizations equivalent to it are obtained in the case when Y is Banach – using the first-order variation of the map and the graphical derivative (Corollary 3.6.2).

The main result in **Chapter 4** is presented in § 4.3 and represents a sufficient condition for tangential transversality in an abstract form in two versions — an easier and more intuitive one (Theorem 4.3.1) and a more general one (Theorem 4.3.2). The usefulness of the obtained theoretical results is illustrated in § 4.4 with some applications, among which it can be noted that if the epigraph of a function satisfies the well-known J.-P. Aubin condition and the constraints of the problem are of a special kind, there exist Lagrange multipliers for the problem – Theorem 4.4.9. As corollaries, some known sufficient conditions for tangential transversality of jointly massive sets are also obtained.

In **Chapter 5**, the continuity of the optimal value of an optimization problem in a metric space is investigated with respect to parametric changes of the feasible, i.e. it is investigated the continuity of the function

$$S_{\text{val}}(p) := \inf\{g(y) : y \in D(p)\},$$

where D is a multi-valued map from a metric space X to a metric space Y, and $g : Y \to \mathbb{R}$ is a function. Similar results find wide application in Mathematical Economics and Optimal Control. In the thesis is examined the interaction between the continuity of g and D to guarantee the continuity of the S_{val} function. A condition depending simultaneously on g and D is formulated and it is shown that it is sufficient for continuity of S_{val}. It is also discussed on how the earlier results fit naturally into the proposed approach.

Counterexample 5.2.1 is given in § 5.2 to illustrate that Hausdorff continuity of g and D is not sufficient to guarantee continuity of S_{val}. A *relaxed uniform continuity assumption* for the pair (D, g) is introduced – Definition 5.2.2, which the pair from the counterexample does not satisfy. In Theorem 5.2.3, it is proved that if D is Hausdorff continuous at p, g is continuous on $D(p)$, and (D, g) is a relaxed uniformly continuous pair, then S_{val} is continuous at p. Some partial cases are discussed in § 5.3.

The dissertation **bibliography** is comprehensive and shows a thorough knowledge of the field. The titles in it are arranged alphabetically by the family name of the first author.

4. Approbation of the results

In § 6.1 of the last chapter, the main contributions of the dissertation are explicitly stated in a very good style. § 6.2 lists three articles in which the results of the dissertation are published:
1. S. Apostolov, M. Krastanov and N. Ribarska, Sufficient Condition for Tangential Transversality, Journal of Convex Analysis, 27, 2020, 19-30, WoS Mathematics Q3 (2020),
2. S. Apostolov, On continuity of optimal value map, Comptes rendus de l'Academie bulgare des Sciences, Vol 74, 2021, No 4, 506-513, WoS Multidisciplinary sciences Q4 (2021),
3. S. Apostolov, M. Bivas, and N. Ribarska, Characterizations of Some Transversality-Type Properties, Set-Valued and Variational Analysis, 30, 2022, Issue 3, 1041-1060, WoS Mathematics applied Q2 (2021),
and one preprint.

It makes an excellent impression and is a very good certificate for the qualities of Stoyan Apostolov that he has an separate publication. I have no reason to doubt his at least equal contribution in the others. The submitted dissertation and related publications contain original results and are free of any plagiarism.

5. Qualities of the Resume

The **Resume** is in Bulgarian, with a volume of 39 pages and 61 titles of cited literature. It comprehensively and correctly reflects the results in the dissertation. In the author's Annotation, part of the Resume, the contributions of the dissertation work are clearly indicated. Unfortunately, the numbering of theorems, definitions, etc. in the Resume does not match their numbering in the dissertation and this makes difficult to trace the correspondences.

6. Conclusion

Stoyan Apostolov's dissertation is an original study in the field of variational analysis. The obtained results are new, interesting and have potential for future development.

Based on the above analysis, **I confirm** that the submitted dissertation work and the scientific publications related to it, as well as the quality and originality of the results presented in them, meet the requirements of the Bulgarian legal regulations for the candidate's acquisition of the PhD in Mathematics. In particular, the candidate satisfies the minimal national requirements in the professional field and no plagiarism was found in the scientific works submitted by him within the procedure.

Based on the above, **I strongly recommend the scientific jury to award Stoyan Raychev Apostolov PhD in Mathematics (Mathematical Analysis)**.

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/Prof. DSci Nadia Zlateva/