Referee report

according to the procedure for the defense of a PhD Thesis entitled

"Variational analysis without variational principles"

for the acquisition of PhD degree of the Sofia University

by the PhD student: **Stoyan Raychev Apostolov**
in the field of higher education: **4. Natural sciences, mathematics and informatics**
professional direction: **4.5. Mathematics**
doctoral program: **Mathematical analysis of the Faculty of Mathematics and Informatics (FMI) of the SU "St. Kliment Ohridski"**

The referee report is prepared by Prof. DSc. Mihail Ivanov Krastanov as a member of the scientific jury, according to Order No. RD 38-309/01.07.2022 of the Rector of SU "Saint Kliment Ohridski".

1. General description of the PhD Thesis and the presented materials

The presented materials have been prepared in accordance with the Law on the Development of the Academic Staff in the Republic of Bulgaria (LDASRB), the Regulations for the Application of this Law, as well as the Regulations for the Terms and Procedures for Acquiring Scientific Degrees and Holding Academic Positions in SU" St. Cl. Ohridski". These include: 1) Dissertation in English; 2) Abstract in Bulgarian; 3) Copy of bachelor's degree diploma; 4) Copy of master's degree diploma; 5) Copies of printed scientific publications on the subject of the dissertation; 6) Reference from the "Authors" system; 7) List of scientific publications on the subject of the dissertation; 8) Declaration by the co-authors on the doctoral student's contribution in three articles; 9) Autobiography in Bulgarian; 10) Protocol for verification of originality and Opinion in connection with the procedure for the prevention of plagiarism, signed by the scientific supervisor; 11) StrikePlagiarism.com software product report of no plagiarism; 12) Declaration of originality and absence of plagiarism by the doctoral student.

The thesis of the PhD student Stoyan Apostolov has a volume of 72 pages and contains an introduction, preliminary information, three chapters in which the obtained results are presented, a conclusion and a bibliography of 61 titles. It is written in English. The Introduction provides a motivation for the research in the thesis and briefly presents the main results. In the Preliminary chapter, some of the main concepts and assertions used in the thesis are introduced. The third and fourth chapters are devoted to the study of the properties of various generalizations of the concept of transversality, as well as

the existing relationships between them. The fifth chapter investigates conditions that lead to continuity of the Bellman function for optimization problems in metric spaces with the property that every open sphere is connected.

It should be noted that no variational principles are used in the proofs of the PhD thesis. This allows the author to focus on the essential properties of the problems under consideration and to obtain results under natural assumptions.

2. Data and personal impressions about the candidate

From the presented autobiography, it can be seen that Stoyan Apostolov was born on 29.12.1992 in the city of Sofia. He completed his higher education at the Faculty of Mathematics and Informatics of Sofia University, receiving a bachelor's degree in 2017 and a master's degree in 2019. Both are in "Applied Mathematics" specialty. While still a student, he won a silver medal at the 22nd International Mathematics Competition, held from July 27 to August 2 in Blagoevgrad. From July 2019 to June 2022, he was a full-time PhD student in the Mathematical Analysis doctoral program at FMI, SU, with scientific supervisor Prof. Dr. Nadezhda Ribarska.

I know Stoyan Apostolov as a student. He made me a strong impression with his inquisitiveness and quick thinking. Even during the lectures, he tried to understand the meaning of the new concepts, as well as the ideas underlying the proofs. These character traits were manifested in full force during the course on "Variational Analysis" taught by Professor Asen Donchev in 2020 in our faculty for the students of the master's program "Optimization" and the doctoral students from the doctoral programs "Mathematical Analysis" and "Operation's Research". In connection with the exam for this course, Stoyan Apostolov had to prove an assertion related to the continuity of the Bellman function. He constructs an example that shows that Theorem 5 of Section 1 of Chapter IX of the monograph Well-Posed Optimization Problems by Asen Donchev and Tulio Tsolezzi does not hold true if the optimality criterion is a function that is Pompey-Hausdorff continuous. Professor Asen Donchev shared with me that he was impressed by this observation of Stoyan Apostolov and encouraged him to continue working on this problem. As a consequence of this research, chapter 5 of the presented dissertation was written.

3. Analysis of the candidate's scientific achievements, contained in the presented thesis and the corresponding publications, included in the procedure

In the introduction of the presented thesis, a brief historical overview of the known results in the field is made and the main results obtained by the author are presented. I must emphasize that Stoyan Apostolov knows very well the state of the subject of the thesis. Perhaps the fact that his scientific supervisor, together with other students of hers, is actively working in the same field and significant scientific results have already been obtained plays a role here. This gives me reason to assume that in time we will be able to talk about the influence of Bulgarian mathematics in the development of this modern direction of variational analysis.

In the second chapter, concepts and statements are introduced, which are used in the exposition further on.

The third chapter contains six sections. In the first section, a characterization of the concept of subtransversality is obtained. This characterization is based on a technical assertion (Lemma 3.1.3) of linearly decreasing distance between two closed sets under appropriate assumptions. It is proved using transfinite induction. In my opinion, this is very appropriate because it allows one naturally to obtain a geometric intuition for its meaning. In the first section, a characterization of the subtransversality property was obtained in terms of the slope of the so-called "pairing function" (Theorem 3.1.6). The second section is devoted to the property of transversality. Using the results of the first section, characterizations of the transversality property in terms of "translated" subtransversality (Theorem 3.2.1) and "translated" tangential subtransversality (Theorem 3.2.2) are obtained. A characterization of the transversality property in terms of the slope of the "pairing function" is also obtained (Proposition 3.2.5). In the third section, the inherent transversality property is investigated. The first statement in this section is a characterization of the tangential transversality property in terms of the slope of the "pairing function" (Theorem 3.3.1). In a 2015 paper by Druzhviatski, Joffe and Lewis, a metric characterization of the intrinsic transversality property is obtained, also in terms of the slope of the "pairing function". This characterization was used in the thesis to define the property of intrinsic transversality in arbitrary metric spaces (Definition 3.3.2). In the thesis, a characterization of the inherent transversality property is proven. From this characterization it follows that the properties intrinsic transversality and tangential transversality are very close in the following sense: For tangential transversality between two closed sets **A** and **B** at a common point **x**, a certain inequality is required for all points of the two sets **A** and **B**, which are located in a suitable neighborhood of the point **x**. For the property of intrinsic transversality, the same is said to hold, only for the points of the sets **A\B** and **B\A**. As a consequence of this result, it follows that the properties tangential transversality, intrinsic transversality, and subtransversality coincide for the case of convex sets. At the end of the third section, it is shown that the resulting metric characterization of intrinsic transversality is equivalent in Hilbert spaces to a characterization proposed in a 2020 paper by Thao, Bui, Cuong, and Verhaegen. In the fourth section it is shown that the subregularity property of a multivalued image between two metric spaces at a point of the graph can be characterized in terms of the subtransversality property of two suitably defined closed sets, one of which is the graph of the multivalued image (Theorem 3.4.1) . As a corollary of this theorem, it is obtained that the regularity property of a multivalued map with a closed graph between two normed spaces at a point of the graph can be characterized in terms of the transversality properties of the same two closed sets. In the fifth section, a characterization of the subregularity property of a multivalued map between two complete metric spaces is obtained (Theorem 3.5.2). This characterization gives a new look to the results of Joffe (2017) and Kruger (2015). The proof is based on the previously obtained characterization of the subtransversality property. In the last section, it is obtained a characterization of the regularity properties of a multivalued map with closed graph between two complete metric spaces in terms of a "linear rate of reduction". Two corollaries are obtained from this result: The first is a characterization of the metric regularity of a multivalued mapping between two complete metric spaces by its contingent variation (Corollary 3.6.2), and the second is a characterization of the metric regularity of a multivalued mapping between two Banach spaces by its contingent derivative (Corollary 3.6.3).

In the first two sections of chapter 4, the definitions of concepts are given and assertions are formulated that are used in the following sections of the same chapter. The main result of the fourth chapter is formulated in the third section. The essence of this result is that two appropriate variational conditions are sufficient for tangential transversality of two closed sets in a Banach space. Two variants of this abstract result section are formulated (Theorems 4.3.1 and 4.3.2). The consequences obtained from it are interesting: Let one closed set be the set of

all points that are admissible for a given minimization problem, and let the second be the epigraph of the considered function (criterion of the problem) at a given reference point. Three cases are considered: 1. The admissible set is the graph of a continuous linear operator, the function is semicontinuous from below with respect to both variables and locally satisfies the Lipschitz continuity condition with respect to the first variable, uniformly with respect to the second and there exists a uniform tangent set giving rise to the tangent cone of Clarke to the epigraph of the considered function at the reference point (Theorem 4.4.3). 2. The function satisfies the Aubin condition at the reference point and the admissible set is the graph of a compact linear operator (Theorem 4.4.6); 3. The epigraph of the function and the admissible set are collectively massive at the reference point (Proposition 4.4.8). Then, as a consequence of the abstract result, under each of these conditions proves a theorem for the Lagrange multipliers in the case that the reference point is a solution of the minimization problem. Here I want to point out that the considered Aubin condition is an infinite-dimensional variant of the pointwise Aubin condition. And it is the basic assumption in one of the most general necessary conditions for the optimality of the basic problem of the calculus of variations. This condition was proved by Clarke in 2005.

The fifth chapter of the dissertation examines the continuity of the Bellman function

S(p) = inf {g(y): y \in D(p)}

in the metric space **Y** in which every open sphere is connected. In the first section, two definitions of topological semi-continuity above (below) and top semi-continuity (below) with respect to the Pompey-Hausdorff metric are given. In the second section, an example is presented for which the set of admissible points **D(p)** depending on the parameter **p** is continuous in the Pompey-Hausdorff sense at the point p, while the Bellman function is not continuous at the same point **p**. In the same section, the concept of relaxed uniform continuity assumption (RUCA) of the pair **(D,g)** is defined. It was proved in Theorem 5.2.3 that if the set **D(p)** of admissible points is continuous in the Pompey-Hausdorf sense at the point **p**, the function **g** is continuous in the set **D(p)** and the pair **(D,g)** satisfies the condition (RUCA), then the Bellman function **S** is continuous at the point **p**. In the third section, two sufficient conditions for the continuity of the Bellman function are presented. In the first statement (Theorem 5.3.1) it is assumed that the set **D(p)** of admissible points is continuous in the topological sense at the point **p** and the function **g** is continuous in the set **D(p)**, and in the second (Theorem 5.3.2) requires that the set **D(p)** of admissible points is Pompey-Hausdorff continuous at the point **p** and that the function **g** is uniformly continuous in the set **D(p).** An interesting fact is that the assumption (RUCA) is necessary to obtain continuity of the Bellman function in the following sense: A multivalued image **F** is topologically continuous at a point **x** if and only if the pair **(f,F)** satisfies the condition (RUCA) for any function f that is continuous on the set **F(x)** (Theorem 5.3.4).

The used 61 references show excellent knowledge of the scientific field of the thesis by the Stoyan Apostolov. The standard lexicographical arrangement in ascending order of the surnames of the first author has been used.

4. **Approbation of the results**

The results of the PhD thesis have been published in 3 articles, all in well-known journals with impact factor:

S. Apostolov, M. Krastanov and N. Ribarska, Sufficient Condition for Tangential Transversality, Journal of Convex Analysis, 27, 2020, 19-30, WoS Mathematics Q3 (2020)

S. Apostolov, On continuity of optimal value map, Comptes rendus de l'Academie bulgare des Sciences, Vol 74, 2021, No 4, 506-513, WoS Multidisciplinary sciences Q4 (2021)

S. Apostolov, M. Bivas, and N. Ribarska, Characterizations of Some Transversality-Type Properties, Set-Valued and Variational Analysis, 30, 2022, Issue 3, 1041-1060, WoS Mathematics applied Q2 (2021)

It makes me a strong impression that one of these articles is a standalone. For the other two articles, there is a Declaration by the co-authors that all authors have contributed equally.

The scientific metrics of these articles, compared with the minimum requirements for the educational and scientific degree "doctor", according to Resolution No. 26 of February 13, 2019 on the amendment and addition of the Regulations for the implementation of the Law on the development of the academic staff in the Republic of Bulgaria, adopted with Decree No. 202 of the Council of Ministers of 2010 (promulgated, SG No. 75 of 2010; amended and supplemented, No. 19 of 2011, No. 9 of 2012, No. 62 of 2013, No. 60 of 2014, No. 57 of 2015 and No. 56 of 2018) are as follows: All publications fall into **Group G7** and collect a **total of 141 points,** with a minimum requirement of **30 points** for the acquisition of the educational and scientific degree "doctor" in the scientific field **4. Natural sciences, mathematics and informatics**, professional direction **4.5 Mathematics**. The first publication is in Q3 and is valued at 45 points, the second is in Q4 and is valued at 36 points, and the third is in Q2 and is valued at 60 points. The obtained **141 points** significantly (**more than 4.5 times**) exceed the minimum requirements for obtaining the educational and scientific degree "doctor" in the scientific field and professional direction of the procedure

Based on the submitted materials, the reviewer assumes that there is no proven plagiarism in the submitted thesis and research papers under this procedure.

The results of the dissertation have been presented by the author so far in the following reports:

1. "Sufficient conditions for tangential transversality", 47th Winter School in Abstract Analysis, Svratka, Czech Republic, 2019, https://www2.karlin.mff.cuni.cz/ lhota/ (based on a joint work with Mikhail Krastanov and Nadezhda Ribarska)

2. "Intrinsic transversality and tangential transversality", 15-th International Workshop onWell-Posedness of Optimization Problems and Related Topics, June 28 - July 2, 2021, Borovets, Bulgaria, http://www.math.bas.bg/ bio/WP21/ (based on a joint work with Mira Bivas and Nadezhda Ribarska)

3. "Intrinsic transversality and tangential transversality", The 13th International Conference on Large-Scale Scientific Computations LSSC 2021, June 7 - 11, 2021, Sozopol, Bulgaria (based on a joint work with Mira Bivas and Nadezhda Ribarska)

4. "Intrinsic transversality and tangential transversality", Spring Scientic Session, Faculty of Mathematics and Informatics, Sofia University, 27 March 2021 (based on a joint work with Mira Bivas and Nadezhda Ribarska)

5. ”On continuity of optimal value map“, Spring Scientic Session, Faculty of Mathematics and Informatics, Sofia University, 26 March 2022

5. Qualities of the Resume

The Resume is written in Bulgarian in a volume of 39 pages. It accurately reflects the content of the dissertation work and corresponds to the requirements of LDASRB and the Regulations on the terms and conditions for acquiring scientific degrees and holding academic positions at SU "Kliment Ohridski".

Unfortunately, the numbering of the assertions, definitions, references and others in the auto-reference does not match their numbering in the thesis. In several places there are wrong chapter’s numbers and section’s numbers. There are also spelling mistakes. There are inconsistencies with the reference from the thesis. All this makes it difficult to read and interferes with establishing consistency with the thesis.

7. Conclusion

Having read this PhD thesis and the accompanying publications presented in the procedure and based on the analysis of their significance and the scientific results contained in them, I confirm that the **presented PhD thesis and the scientific publications to it**, as well as the quality and originality of the presented in them, results and achievements, **meet the requirements of LDASRB**, the Regulations for the application of LDASRB and the Regulations for the conditions and procedures for acquiring scientific degrees and holding academic positions in the SU for the candidate's acquisition of the educational and scientific degree "doctor" in the scientific field: 4. Natural sciences, mathematics and informatics, professional direction: 4.5. Mathematics (Mathematical Analysis). I want to emphasize that the qualities of the thesis **significantly exceed the minimum national requirements** in the professional field. Here I must note that no plagiarism was found in the scientific papers submitted to the competition.

Based on the above, I **strongly recommend that the scientific jury award** Stoyan Raichev Apostolov the educational and scientific degree "doctor" in a scientific field: **4.Natural sciences, mathematics and informatics**, professional direction: **4.5. Mathematics (Mathematical Analysis).**

27.09. 2022. Prepared the referee report:

/prof. DSc. Mikhail Ivanov Krastanov/