Sofia University "St. Kliment Ohridski"
Faculty of Mathematics and Informatics

Toroidal compactifications of discrete quotients of the complex two-ball

Abstract
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of Pancho Georgiev Beshkov

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Scientific advisor: prof. Azniv Kasparian

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A bounded domain $\mathcal{D} \subset \mathbb{C}^N$ is symmetric if for any point $z \in \mathcal{D}$ there is a holomorphic involution $\sigma_z : \mathcal{D} \to \mathcal{D}$ with unique fixed point z. Any bounded symmetric domain $\mathcal{D} = G/K$ is a homogeneous space of a reductive Lie group G. A bounded symmetric domain $\mathcal{D}$ is irreducible if it cannot be decomposed into a direct product of bounded symmetric domains of smaller dimension. The complex 2-ball $\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 < 1\}$ is the only irreducible bounded symmetric domain of complex dimension 2. Due to its homogeneity under the unitary group

$$U(1,2) = \left\{g \in \mathrm{GL}(3,\mathbb{C}) \,|\, \mathcal{H}_{1,2}(gz, gz') = \mathcal{H}_{1,2}(z, z'),\, \forall z, z' \in \mathbb{B}\right\}$$

of a Hermitian form $\mathcal{H}_{1,2} : \mathbb{C}^3 \to \mathbb{C}^3$ of signature $(1,2)$, the ball $\mathbb{B}$ admits a lot of quotients $\mathbb{B}/\Gamma$ by discrete subgroups $\Gamma < U(1,2)$. Most interesting are the quotients $\mathbb{B}/\Gamma$ of finite $U(1,2)$-invariant volume, whose associated groups Γ are called lattices of $U(1,2)$. A lattice $\Gamma < U(1,2)$ is arithmetic if there exists a number field $\mathbb{Q} \subseteq k \subset \mathbb{C}$ with integers ring $\mathcal{O}_k$, such that $\Gamma \cap \mathrm{GL}(3, \mathcal{O}_k)$ is of finite index in Γ and in $\mathrm{GL}(3, \mathcal{O}_k)$.

The real rank of a bounded symmetric domain $\mathcal{D} = G/K$ is the maximal dimension of an abelian subgroup $A < G < \mathrm{GL}(N, \mathbb{C})$, whose entries are simultaneously diagonalizable over $\mathbb{R}$. Margulis has shown in [34] that all lattices Γ in the holomorphic isometry group G of an irreducible bounded symmetric domain $\mathcal{D} = G/K$ of real rank ≥ 2 are arithmetic. The complex 2-ball is an irreduicble bounded symmetric domain of real rank 1 and its holomorphic isometry group $U(1,2)$ is known to admit non-arithmetic lattices. The first examples of non-arithmetic lattices $\Gamma < U(1,2)$ appear in Mostow's [38] from 1980. Further constructions are due to Deligne-Mostow's [9], [10], Deraux's [11], [12], Deraux, Parker and Paupert's [13], [14] and etc. Last year, Baldi and Ullmo showed in [4] that if a quotient $\mathbb{B}/\Gamma$ by a lattice $\Gamma < U(1,2)$ contains infinitely many maximal complex totally geodesic submanifolds, then Γ is an arithmetic lattice. Our study of the non-compact ball quotients $\mathbb{B}/\Gamma$ with smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ applies simultaneously to arithmetic and non-arithmetic lattices $\Gamma < U(1,2)$.

The smoth compact ball quotients $\mathbb{B}/\Gamma$ are characterized topologically by an equality of their Chern numbers. Recall that for an arbitrary complex projective surface $X \subset \mathbb{P}^N(\mathbb{C})$, the Chern number $c_2(X) \in \mathbb{Z}$ is the Euler number of X and $c_1^2(X) = K_X^2 \in \mathbb{Z}$ is the self-intersection number of the canonical divisor K_X of X. Van de Ven's [48] from 1966 shows that the minimal complex projective surfaces of general type satisfy the inequality $c_1^2(X) \leq 8c_2(X)$. Later, Bogomolov's [5] improves this bound to $c_1^2(X) \leq 4c_2(X)$. As a consequence of the solvability of the Monge-Ampère equation, Yau's [49] establishes that a smooth minimal surface $X \subset \mathbb{P}^N(\mathbb{C})$ of general type is a ball quotient $X = \mathbb{B}/\Gamma$ if and only if $c_1^2(X) = 3c_2(X)$. The same result is proved independently by Miyaoka in [35]. Many authors refer to $c_1^2(X) = 3c_2(X)$ as to the Bogomolov-Miyaoka-Yau equality.

By the means of the algebraic number theory, Prasad and Yeung have established the existence of finitely many arithmetic lattices $\Gamma < SU(1,2)$ with smooth compact ball quotient $\mathbb{B}/\Gamma$. They show that any such Γ is associated with a totally real number field k, a simple algebraic group $G(k)$ over k, a totally imaginary quadratic extension $l \supset k$ and a division algebra $\mathcal{D}$ with center l. There is a unique real place ν_o of k, such that the group $G(k_{\nu_o}) \simeq SU(1,2)$ over the completion $k_{\nu_o} \simeq \mathbb{R}$ is non-compact. For any other archimedean place $\nu \neq \nu_o$ of k, the group $G(k_\nu) \simeq SU(3)$ is compact. After classifying k, l and $\mathcal{D}$, Prasad and Yeung describe the arithmetic lattices $\Gamma < SU(1,2)$ with smooth compact quotient $\mathbb{B}/\Gamma$. Making use of a computer implementation, Cartwright and Steger obtain in [8] presentations of all the arithmetic lattices $\Gamma < SU(1,2)$ with smooth compact quotient $\mathbb{B}/\Gamma$.

The majority of the lattices $\Gamma < U(1,2)$ have non-compact quotient $\mathbb{B}/\Gamma$. There arises the necessity of constructions of compactifications of such $\mathbb{B}/\Gamma$. A boundary point $z \in \partial\mathbb{B} = \left\{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\right\}$ is Γ-rational if the lattice $\Gamma < U(1,2)$ intersects the stabilizer $\mathrm{Stab}_{U(1,2)}(z)$ in a lattice $\Gamma \cap \mathrm{Stab}_{U(1,2)}(z)$ of $\mathrm{Stab}_{U(1,2)}(z)$. The set $\partial_\Gamma\mathbb{B}$ of the Γ-rational boundary points of $\mathbb{B}$ is acted by Γ with finitely many orbits, called the Γ-cusps. The results of Baily and Borel from [3] imply that adjoining the Γ-cusps $\partial_\Gamma\mathbb{B}/\Gamma$ to $\mathbb{B}/\Gamma$, one obtains a complex projective variety

$$\widehat{\mathbb{B}/\Gamma} = (\mathbb{B}/\Gamma) \coprod (\partial_\Gamma\mathbb{B}/\Gamma)\,.$$

Even if $\mathbb{B}/\Gamma$ is smooth, the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ has singularities at the Γ-cusps $\partial_\Gamma\mathbb{B}/\Gamma$. The resolution of the cuspidal singular points of $\widehat{\mathbb{B}/\Gamma}$ results in the toroidal compactification $(\mathbb{B}/\Gamma)'$ of $\mathbb{B}/\Gamma$. For an arbitrary bounded symmetric domain $\mathcal{D} = G/K$ and an arbitrary lattice $\Gamma < G$ of holomorphic isometries of $\mathcal{D}$ with non-compact quotient $\mathcal{D}/\Gamma$, the toroidal compactifications of $\mathcal{D}/\Gamma$ are constructed by Ash, Mumford, Rapoport and Tai in [1].

In [39] Mumford modifies the Bogomolov-Miyaoka-Yau equality $c_1^2(\mathbb{B}/\Gamma) = 3c_2(\mathbb{B}/\Gamma)$ for smooth compact ball quotients $\mathbb{B}/\Gamma$ to the ones with smooth toroidal compactification $(\mathbb{B}/\Gamma)'$. Let $D := (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ be the toroidal compactifying divisor of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$, $\overline{c_2}(X, D) := e(X \setminus D) = e\,(\mathbb{B}/\Gamma)$ be the Euler number of $\mathbb{B}/\Gamma$ and $\overline{c_1}^2(X, D) := (K_X + D)^2$ be the self-intersection number of the logarithmic canonical divisor $K_X + D$ of (X, D). Mumford shows that if (X, D) is of logarithmic general type, i.e., if a sufficiently large tensor power of the line bundle, asociated with $K_X + D$ provides a projective morphism of X onto a surface, then $\overline{c_1}^2(X, D) \leq 3\overline{c_2}(X, D)$ with equality $\overline{c_1}^2(X, D) = 3\overline{c_2}(X, D)$ if and only if $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification with toroidal compactifying divisor D. From now on, we refer to $\overline{c_1}^2(X, D), \overline{c_2}(X, D)$ as to the logarithmic Chern numbers of (X, D) and call $\overline{c_1}^2(X, D) = 3\overline{c_2}(X, D)$ the logarithmic Bogomolov-Miyaoka-Yau equality.

The present thesis studies the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ of non-compact quotients $\mathbb{B}/\Gamma$ of the complex 2-ball $\mathbb{B}$ by a lattice $\Gamma < U(1,2)$. More precisely, it focuses on the finite unramified coverings $f : (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)'$ of smooth toroidal compactifications, which restrict to finite unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of the corresponding ball quotients and on some numerical invariants of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$, which are birational to a ruled surface $r : Y \to B$ with an elliptic base B. The aforementioned original results of the thesis constitute the third and the fourth chapters. Let $L_1 \simeq \mathbb{P}^1(\mathbb{C})$ be a smooth irreducible rational curve on a surface X, whose contraction does not create a singularity. Then the self-intersection number of L_1 is $L_1^2 = -1$ and L_1 is called briefly a (-1)-curve on X. Any unramified covering $f : X_2 \to X_1$ of degree $d \in \mathbb{N}$ of smooth surfaces restricts to an unramified covering $f : L'' \to L'$ of degree

d of the union L' of the rational (-1)-curves on X_1 by the union L'' of the smooth rational (-1)-curves on X_2. Let $D^{(j)} := (\mathbb{B}/\Gamma_j)' \setminus (\mathbb{B}/\Gamma_j)$ be the toroidal compactifying divisors of $\mathbb{B}/\Gamma_j$ and $\rho_j : (\mathbb{B}/\Gamma_j)' \to Y_j$ be finite sequences of blow downs to minimal surfaces Y_j. The thesis constructs a bijective correspondence between the finite unramified coverings $f : (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)'$ of smooth toroidal compactifications, which restrict to finite unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of the same degree and the finite unramified coverings $\varphi : Y_2 \to Y_1$ of the corresponding minimal models, which restrict to finite unramified coverings $\varphi : \rho_2\left(D^{(2)}\right) \to \rho_1\left(D^{(1)}\right)$ of the same degree. The aforementioned type of unramified coverings endow the set Σ of the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ with a partial order $\succeq$. The minimal entries $(\mathbb{B}/\Gamma_o)' \in \Sigma$ with respect to $\succeq$ are called primitive, while the maximal ones $(\mathbb{B}/\Gamma_1)' \in \Sigma$ are designated as saturated. Chapter 3 of the thesis establishes that any $(\mathbb{B}/\Gamma)' \in \Sigma$ dominates some primitive $(\mathbb{B}/\Gamma_0)' \in \Sigma$. A necessary and sufficient condition for the presence of a saturated $(\mathbb{B}/\Gamma_1)' \in \Sigma$ with $(\mathbb{B}/\Gamma_1)' \succ (\mathbb{B}/\Gamma)'$ is the finiteness of the fundamental group $\pi_1(Y)$ of a minimal model Y of $(\mathbb{B}/\Gamma)'$. The third chapter characterizes the saturated and primitive $(\mathbb{B}/\Gamma)' \in \Sigma$ of non-positive Kodaira dimension $\kappa(\mathbb{B}/\Gamma)' \in \{-\infty, 0\}$. If $\beta : X = (\mathbb{B}/\Gamma)' \to Y$ blows down disjoint smooth rational curves and $D := X \setminus (\mathbb{B}/\Gamma)$ is the toroidal compactifying divisor of X, the relative biholomorphism group $\mathrm{Aut}(X, D)$ is shown to be finite and isomorphic to $\mathrm{Aut}(Y, \beta(D))$.

Besides, the covering relations among smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$, the thesis discusses the number of the cusps and the number of the non totally geodesic punctured spheres $L_i \setminus D \subset \mathbb{B}/\Gamma$ for smooth $(\mathbb{B}/\Gamma)'$, whose minimal model Y admits a ruling $r : Y \to B$ with an elliptic base B. The main tool for studying the aforementioned numerical invariants is the logarithmic Bogomolov-Miyaoka-Yau equality for a pair (X, D), consisting of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ and its toroidal compactifying divisor

$$D := X \setminus (\mathbb{B}/\Gamma) = \sum_{j=1}^{k} D_j$$

with smooth elliptic irreducible components D_j. If $\beta : X \to Y$ is the blow down of the smooth irreducible rational (-1)-curves $L_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$, $1 \leq i \leq s$ on $X = (\mathbb{B}/\Gamma)'$ to a minimal ruled surface $r : Y \to B$ with elliptic base B then $C_j := \beta(D_j)$ are shown to be smooth elliptic curves, on which the ruling $r : Y \to B$ restricts to finite unramified coverings $r|_{C_j} : C_j \to B$ of degree $d_j \in \mathbb{N}$. If $d_j = 1$ for all $1 \leq j \leq k$ and all C_j are sections of $r : Y \to B$, the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) is expressed by the intersection numbers $L_i.D$ for $1 \leq i \leq s$. When there is at least one $d_j > 1$, the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) is expressed by $L_i.D$ for $1 \leq i \leq s$ and by the self-intersection numbers C_j^2 for $1 \leq j \leq k$. That allows to derive an inequality on $L_i.D$ for $1 \leq i \leq s$. In either case is established the existence of a punctured sphere $L_i \setminus D \subset \mathbb{B}/\Gamma$, which is not totally geodesically embedded in $\mathbb{B}/\Gamma$. As another consequence of the logarithmic Bogomolov-Miyaoka-Yau equality are derived lower bounds on the number k of the cusps of $\mathbb{B}/\Gamma$, depending on the existence, respectively, the non existence of $d_j > 1$. For comparatively small k are obtained lower bounds $\mu_k \geq 2$ on the number of the non totally geodesic $L_i \setminus D \subset \mathbb{B}/\Gamma$ in either case.

Let us consider briefly few works of other authors, related to the original results of the thesis. The covering relations between quotients of $\mathbb{B}$ by lattices $\Gamma < U(1,2)$ are studied by Uludağ, Stover, Di Cerbo and Stover, etc. Uludağ constructs an infinite series of ramified coverings of singular ball quotients, which are birational to the projective plane $\mathbb{P}^2(\mathbb{C})$. In [45] Stover shows that there are exactly two singular non-compact ball quotients $\mathbb{B}/\Gamma_1$, $\mathbb{B}/\Gamma_2$ of minimal invariant volume and any smooth non-compact $\mathbb{B}/\Gamma$ of minimal volume or, equivalent, of minimal Euler number 1, is a covering of $\mathbb{B}/\Gamma_1$ or $\mathbb{B}/\Gamma_2$ of degree 72. Di Cerbo and Stover's [18] establishes that there are five ball quotients $\mathbb{B}/\Gamma_j$, $1 \leq j \leq 5$ with smooth toroidal compactifications $(\mathbb{B}/\Gamma_j)'$ and Euler number $e(\mathbb{B}/\Gamma_j) = 1$. One of them, $\mathbb{B}/\Gamma_1 = \mathbb{B}/\Gamma_{\rm Hir}$ is Hirzebruch's example from [25] with an abelian minimal model $A_{\rm Hir}$. The remaining ones - $\mathbb{B}/\Gamma_2, \ldots, \mathbb{B}/\Gamma_5$ are birational to bi-elliptic surfaces. In [16] Di Cerbo and Stover show that any Galois covering $\zeta_G : A \to A_{\rm Hir} = A/G$ with a finite fixed point free group G is associated with an unramified G-Galois covering

$$\zeta_G' : \mathbb{B}/\Gamma_G \longrightarrow \mathbb{B}/\Gamma_{\rm Hir} = (\mathbb{B}/\Gamma_G)/G$$

of ball quotients. The considerations of Chapter 3 generalize the aforementioned result. As an application of the correspondence between the unramified finite Galois coverings of $A_{\rm Hir}$ and the unramified finite Galois coverings of $\mathbb{B}/\Gamma_{\rm Hir}$, [16] shows that for any $n \in \mathbb{N}$ there exist non biholomorphic ball quotients $\mathbb{B}/\Gamma_1, \ldots, \mathbb{B}/\Gamma_n$ with one and a same toroidal compactification $(\mathbb{B}/\Gamma_1)' = (\mathbb{B}/\Gamma_2)' = \ldots = (\mathbb{B}/\Gamma_n)'$. In [17] Di Cerbo and Stover construct a series $(\mathbb{B}/\Gamma_n)'$ of smooth toroidal compactifications with Euler numbers $e(\mathbb{B}/\Gamma_n) = n$ and bi-elliptic minimal models. Finite unramified coverings of Hirzebruch's example $(\mathbb{B}/\Gamma_{\rm Hir})'$ are used in Stover's [46], towards a construction of infinite series $(\mathbb{B}/\Gamma_{n,1})'$, $(\mathbb{B}/\Gamma_{n,2})'$ of smooth toroidal compactifications. The Euler numbers $e(\mathbb{B}/\Gamma_{n,1})$ and the numbers $\nu(\mathbb{B}/\Gamma_{n,1})$ of the cusps of $\mathbb{B}/\Gamma_{n,1}$ tend to ∞ as $n \to \infty$. The second series $\mathbb{B}/\Gamma_{n,2}$ has $\lim\limits_{n\to\infty} e(\mathbb{B}/\Gamma_{n,2}) = \infty$ and bounded $\nu(\mathbb{B}/\Gamma_{n,2})$. Making use of Deligne-Mowtow's non-arithmetic examples from [9], Stover obtains a series $\mathbb{B}/\Gamma_{n,3}$ with linear growth of $e(\mathbb{B}/\Gamma_{n,3})$ and $\nu(\mathbb{B}/\Gamma_{n,3})$ with respect to n.

All smooth compact ball quotients $\mathbb{B}/\Gamma$ are known to be of general type. Hirzebruch's [25] and Holzapfels's [28], [29] provide examples of smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ with abelian minimal model. Momot's [37] constructs smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ of Kodaira dimension $\kappa(\mathbb{B}/\Gamma)' = 1$. Di Cerbo and Stover's [17] establishes the existence of smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ with a bi-elliptic minimal model. The majority of the smooth toroidal compactifications are of general type. In [29] Holzapfel shows that any smooth toroidal compactification $(\mathbb{B}/\Gamma_0)'$ with abelian minimal model has a smooth finite ramified cover $(\mathbb{B}/\Gamma)'$ of general type. The existence of smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ of Kodaira dimension $\kappa(\mathbb{B}/\Gamma)' = -\infty$ is an open problem. Non-toroidal rational compactifications of ball quotients are studied by Holzapfel, Pineiro and Vladov in [27] and by Uludağ in [47]. Part of these compactifications have isolated cyclic quotient singularities. Kasparian and Kotzev's [30] provides a ball quotient compactification with isolated cyclic quotient singularities, which is birational to a minimal ruled surface with an elliptic base. Chapter 4 of the thesis establishes various numerical restrictions on the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$, whose minimal model Y is a ruled surface $r : Y \to B$ with an elliptic base B. That questions the existence of such $\mathbb{B}/\Gamma$.

The ball quotient surfaces $\mathbb{B}/\Gamma$ and their compactifications generalize naturally the Riemann surfaces of genus ≥ 2, which are smooth compact quotients of the unit disc $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$. Another motivation for studying $\mathbb{B}/\Gamma$ is the presence of moduli spaces of complex projective varieties, which are ball quotients. Let N be a complex manifold. Fixing the structure of a real analytic manifold on N and varying the complex structure on N, one obtains a family $\pi : \mathfrak{N} \to \mathfrak{M}$ over a complex analytic variety $\mathfrak{M}$, parameterizing the isomorphism classes of the complex structures on N. Under some additional technical restrictions, $\mathfrak{M}$ is called a moduli space of N. Let

$$V := \{a = (a_1, \ldots, a_4) \in \mathbb{C}^4 \mid a_1 + \ldots + a_4 = 0\} \simeq \mathbb{C}^3$$

and

$$V_o := \{a \in V \mid a_i \neq a_j, \ \forall 1 \leq i \neq j \leq 4\}.$$

Any point $a \in V_o$ is associated with a smooth Picard curve

$$C_3(a_1, \ldots, a_4) := \left\{ x = [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) \,\middle|\, x_0 x_2^3 = \prod_{i=1}^{4} (x_1 - a_i x_0) \right\},$$

which is of genus 3. For any permutation $\sigma \in S_4$, $C_3(a_1, \ldots, a_4)$ and $C_3(a_{\sigma(1)}, \ldots, a_{\sigma(4)})$ coincide. If $a' = (a'_1, \ldots, a'_4) = \lambda a = (\lambda a_1, \ldots, \lambda a_4)$ for some $\lambda \in \mathbb{C}^*$ then $C_3(a'_1, \ldots, a'_4)$ is isomorphic to $C_3(a_1, \ldots, a_4)$. Thus, the open subset

$$\mathfrak{M}_o := (V_o/S_4)/\mathbb{C}^* = (V_o/\mathbb{C}^*)/S_4 \subset (V/\mathbb{C}^*)/S_4 = \mathbb{P}(V)/S_4 = \mathbb{P}^2(\mathbb{C})/S_4$$

of the singular projective variety $\mathbb{P}^2(\mathbb{C})/S_4$ parameterizes the isomorphism classes of the smooth Picard curves. In [26] Holzapfel shows that $\mathfrak{M}_o = \mathbb{B}/\Gamma_o$ is a quotient of the complex 2-ball $\mathbb{B}$ by a lattice $\Gamma_o < U(1,2)$. The closure $\mathrm{Cl}(\mathfrak{M}_o) = \mathbb{P}^2(\mathbb{C})/S_4$ is a singular rational variety. The points of $\mathrm{Cl}(\mathfrak{M}_o) \setminus \mathfrak{M}_o$ parameterize degenerate, singular complex plane projective curves. Another example of a moduli space, covered by $\mathbb{B}$, is provided by Dolgachev and Kondo's [20]. Let $p_i := [a_i : 1] \in \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq 5$ be five different points on the projective line and

$$C(p_1, \ldots, p_5) := \left\{ x = [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) \,\middle|\, x_0^6 = x_0 \prod_{i=1}^{5} (x_1 - a_i x_2) \right\}.$$

The cyclic group $\mathbb{C}_5 = \langle e^{\frac{2\pi i}{5}} \rangle = \left\{ e^{\frac{2\pi i s}{5}} \,\middle|\, 0 \leq s \leq 4 \right\}$ of order 5 acts on $\mathbb{P}^2(\mathbb{C})$ by the rule

$$\mathbb{C}_5 \times \mathbb{P}^2(\mathbb{C}) \longrightarrow \mathbb{P}^2(\mathbb{C}), \quad \left(e^{\frac{2\pi i s}{5}}, [x_0 : x_1 : x_2] \right) \mapsto \left[e^{\frac{2\pi i s}{5}} x_0 : x_1 : x_2 \right]$$

and leaves invariant $C(p_1, \ldots, p_5)$ for all $p_1, \ldots, p_5 \in \mathbb{P}^1(\mathbb{C})$. The double cover

$$\widehat{S}(p_1, \ldots, p_5) \longrightarrow \mathbb{P}^2(\mathbb{C}),$$

ramified over $C(p_1, \ldots, p_5) \subset \mathbb{P}^2(\mathbb{C})$ has a K3 resolution of the singularities $S(p_1, \ldots, p_5)$, endowed with an action of $\mathbb{C}_5$. The moduli space $\mathfrak{M}$ of such $S(p_1, \ldots, p_5)$ is isomorphic to the moduli space $\mathfrak{M}_1$ of the subsets $\{p_1, \ldots, p_5 \,|\, p_i \neq p_j, \forall 1 \leq i \neq j \leq 5\} \subset \mathbb{P}^1(\mathbb{C})$ and $\mathfrak{M}_1 = \mathbb{B}/\Gamma_1$ is shown to be a ball quotient.

The thesis consists of an introduction, four chapters and a bibliography of 50 titles. The Chapters 1 and 2 collect some preliminaries on Chern numbers of algebraic surfaces, the logarithmic Bogomolov-Miyaoka-Yau equality, characterizing the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ of ball quotients $\mathbb{B}/\Gamma$ and the construction of $(\mathbb{B}/\Gamma)'$. The last two chapters cover the original results of the author from [6], respectively, [7]. More precisely, Chapter 1 starts by recalling the notions of a holomorphic vector bundle $\mathcal{E}$ over a complex manifold M and the transition functions of $\mathcal{E}$. It explains why the holomorphic line bundles on M are classified by the cohomology group $H^1(M, \mathcal{O}^*)$. The next topic under consideration is the bijective correspondence between the divisors and the line bundles on M. Chapter 1 recalls the notions of a Hermitian metric h on a holomorphic vector bundle $\pi : \mathcal{E} \to M$ and a connection D on $\mathcal{E}$. Making use of Cartan's method of moving frames (cf.[21]), it focuses on the unique connection on $\pi : \mathcal{E} \to M$, which is compatible with the Hermitian metric h and the complex structure on $\mathcal{E}$. The curvature matrix Θ of this connection with respect to an orthonormal frame of $\mathcal{E}$ is described in detail. The Enriques-Kodaira classification of the minimal smooth complex projective surfaces is briefly recalled. The Chern classes of holomorphic vector bundles are introduced as the cohomology classes of the elementary symmetric polynomials of the entries of the curvature matrix Θ. That allows to define the Chern numbers of a smooth complex projective surface and to formulate the Bogomolov-Miyaoka-Yau equality, characterizing the compact smooth ball quotients $\mathbb{B}/\Gamma$, as well as the logarithmic Bogomolov-Miyaoka-Yau equality, describing the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ and their toroidal compactifying divisors $D := (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$.

The second chapter is devoted to the construction of the toroidal compactification $(\mathbb{B}/\Gamma)'$ of a quotient $\mathbb{B}/\Gamma$ of the complex 2-ball $\mathbb{B}$ by a lattice $\Gamma < U(1,2)$. Starting from a scratch, it describes the transitive action of $U(1,2)$ on $\mathbb{B}$, $\partial\mathbb{B}$ and $\mathbb{P}^2(\mathbb{C}) \setminus (\mathbb{B} \cup \partial\mathbb{B})$, identifying the corresponding stabilizers with $U_1 \times U_2$, a maximal parabolic subgroup P of $U(1,2)$ and, respectively, with $U(1,1) \times U_1$. Further, P is shown to be a maximal proper subgroup of $U(1,2)$. The group P is solvable and, therefore, a minimal parabolic subgroup of $U(1,2)$. The second chapter describes the refined Langlands decomposition of the standard maximal parabolic subgroup $P_o := \mathrm{Stab}_{U(1,2)}(1,0) < U(1,2)$, stabilizing $(1,0) \in \partial\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$. That allows to discuss the horospherical decomposition of $\mathbb{B}$, associated with P_o, as well as the corresponding Siegel domain realization of $\mathbb{B}$. A maximal parabolic subgroup P of $U(1,2)$ is Γ-rational if $\Gamma \cap P$ is a lattice of P. Let us denote by $\mathrm{MPar}(\Gamma)$ the set of the Γ-rational maximal parabolic subgroups of $U(1,2)$ and consider the centre $Z(N_P)$ of the unipotent radical N_P of some $P \in \mathrm{MPar}(\Gamma)$. Then $\Gamma \cap Z(N_P)$ is a lattice of the real 1-dimensional Lie group $Z(N_P)$ and the quotient $\mathbb{B}/[\Gamma \cap Z(N_P)]$ is a family of punctured discs of variable radii, parameterized by the complex line $N_P/Z(N_P) \simeq \mathbb{C}$. Adjoining the origins of these discs, one obtains the partial compactification $(\mathbb{B}/[\Gamma \cap Z(N_P)])'$ at $P \in \mathrm{MPar}(\Gamma)$. Note that the complex analytic space $(\mathbb{B}/[\Gamma \cap Z(N_P)])'$ is not compact, regardless of its name. The lattice Γ acts by conjugation on $\mathrm{MPar}(\Gamma)$ with finitely many orbits, corresponding to the Γ-cusps $\partial_\Gamma \mathbb{B}/\Gamma$.

For any $\gamma \in \Gamma$ the conjugation

$$\psi_\gamma : \mathbb{B}/[\Gamma \cap Z(N_P)] \longrightarrow \mathbb{B}/[\Gamma \cap Z(N_{\gamma P \gamma^{-1}})]$$

by γ extends to a biholomorphism

$$\psi_\gamma : (\mathbb{B}/[\Gamma \cap Z(N_P)])' \longrightarrow \big(\mathbb{B}/[\Gamma \cap Z(N_{\gamma P \gamma^{-1}})]\big)'$$

and provides a Γ-action on the disjoint union

$$\coprod_{P \in \mathrm{MPar}(\Gamma)} (\mathbb{B}/[\Gamma \cap Z(N_P)])' \tag{0.1}$$

of the partial compactifications at the Γ-rational maximal parabolic subgroups $P < U(1,2)$. The toroidal compactification $(\mathbb{B}/\Gamma)'$ is defined as the Γ-quotient of (0.1). Let $\beta : X = (\mathbb{B}/\Gamma)' \to Y$ be a blow down of the smooth rational (-1)-curves $L_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$, $1 \leq i \leq s$ on a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ and $D := X \setminus (\mathbb{B}/\Gamma)$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$. Chapter 2 explains briefly why the intersection numbers $L_i.D \geq 4$ for all $1 \leq i \leq s$ with equality $L_i.D = 4$ if and only if the punctured sphere $L_i \setminus D$ is totally geodesically embedded in $\mathbb{B}/\Gamma$.

The third chapter reflects the results of the article [6]. More precisely, it discusses the bijective correspondence between the unramified coverings $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of degree d of smooth toroidal compactifications and the unramified coverings $\varphi : Y_2 \to Y_1$ of degree d of their minimal models, which are compatible with a finite sequence $\rho_2 : X_2 \to Y_2$ of blow downs. Let $f : M \to f(M)$ be a surjective holomorphic map of complex manifolds, N be a complex analytic subspace of M or an open subset of M and $f(N) \cap f(M \setminus N) = \emptyset$. As a preparation, it suffices two of the maps

$$f : N \longrightarrow f(N), \quad f : M \setminus N \longrightarrow f(M \setminus N), \quad f : M \longrightarrow f(M)$$

to be unramified coverings of degree $d \in \mathbb{N}$, in order the third one to be an unramified covering of the same degree d. Let $f : X \to X'$ be an unramified covering of degree d of smooth projective surfaces and $D = \coprod_{j=1}^{k} D_j$ be a divisor on X with disjoint smooth irreducible components D_j, such that $f : D \to f(D)$ is an unramified covering of degree d. Then $f : D_j \to f(D_j)$ are shown to be unramified coverings of degree d_j of smooth curves $f(D_j)$ and $f(D_j)$ intersect each other if and only if they coincide. For an arbitrary smooth irreducible rational curve $C' \subset X'$, the complete pre-image $f^{-1}(C') = \coprod_{i=1}^{d} C_i$ turns to consist of d disjoint smooth irreducible rational curves $C_i \subset X$, on which f restricts to biholomorphisms $f : C_i \to C'$, $\forall 1 \leq i \leq d$. Let $\rho_1 : X_1 \to Y_1$ be a finite composition of blow downs and $\varphi : Y_2 \to Y_1$ be an unramified covering of degree d. Then the fibered product commutative diagram

$$\begin{tikzcd} X_2 = X_1 \times_{Y_1} Y_2 \arrow[r, "\rho_2"] \arrow[d, "f"] & Y_2 \arrow[d, "\varphi"] \\ X_1 \arrow[r, "\rho_1"] & Y_1 \end{tikzcd} \tag{0.2}$$

provides a surface X_2 with an unramified covering $f : X_2 \to X_1$ of degree d and a finite composition of blow downs $\rho_2 : X_2 \to Y_2$, such that f and φ are compatible with ρ_2. Moreover, for any (possibly reducible) divisors $D^{(i)} \subset X_i$, which do not contain an irreducible component of the exceptional divisor of $\rho_i : X_i \to Y_i$, $f : D^{(2)} \to D^{(1)}$ is an unramified covering of degree d if and only if the restriction $\varphi : \rho_2\left(D^{(2)}\right) \to \rho_1\left(D^{(1)}\right)$ is an unramified covering of degree d. In particular, if $\rho_1 : X_1 = (\mathbb{B}/\Gamma_1)' \to Y_1$ is the blow down of the smooth irreducible rational (-1)-curves on a smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ then any unramified covering $\varphi : Y_2 \to Y_1$ of degree d pulls back X_1 to a smooth toroidal compactification $X_2 = (\mathbb{B}/\Gamma_2)'$ with unramified coverings $f : X_2 \to X_1$, $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of degree d. Conversely, suppose that $\rho_1 : X_1 \to Y_1$ is a finite composition of blow downs and $f : X_2 \to X_1$ is an unramified covering of degree d. Then the Stein factorization of $\rho_1 f : X_2 \to Y_1$ provides a fibered product commutative diagram (0.2), where ρ_2 is a finite sequence of blow downs, φ is an unramified covering of degree d and f, φ are compatible with ρ_2. As a result, any unramified covering $f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'$ of degree d of a smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ and any finite composition of blow downs $\rho_1 : X_1 \to Y_1$ onto a minimal surface Y_1 induce a commutative diagram (0.2), where $X_2 = (\mathbb{B}/\Gamma_2)'$ is a smooth toroidal compactification, $\rho_2 : X_2 \to Y_2$ is a finite sequence of blow downs onto a minimal surface Y_2 and the morphisms $\varphi : Y_2 \to Y_1$, $\varphi : \rho_2(X_2 \setminus (\mathbb{B}/\Gamma_2)) \to \rho_1(X_1 \setminus (\mathbb{B}/\Gamma_1))$ are unramified coverings of degree d. In such a way, the first section of Chapter 3 establishes that an arbitrary finite unramified covering of the source or of the target of $\rho_1 : X_1 = (\mathbb{B}/\Gamma_1)' \to Y_1$ induces a fibered product commutative diagram (0.2). The second section of the third chapter shows that the compatible finite unramified coverings by the source or by the target of $\rho_2 : X_2 = (\mathbb{B}/\Gamma_2)' \to Y_2$ give rise to a fibered product commutative diagram (0.2). More precisely, let $\rho_2 : X_2 = (\mathbb{B}/\Gamma_2)' \to Y_2$ be a finite composition of blow downs from a smooth toroidal compactification $X_2 = (\mathbb{B}/\Gamma_2)'$ to a minimal surface Y_2 and $f : X_2 \to f(X_2) = X_1$ be an unramified covering of degree d, which is compatible with ρ_2 and restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ of degree d. The thesis shows that the aforementioned assumptions suffice for the existence of a fibered product commutative diagram (0.2). Conversely, if $X_2 = (\mathbb{B}/\Gamma_2)'$ is a smooth toroidal compactification with toroidal compactifying divisor $D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2)$, $\rho_2 : X_2 = (\mathbb{B}/\Gamma_2)' \to Y_2$ is a finite sequence of blow downs to a minimal surface Y_2 and $\varphi : Y_2 \to \varphi(Y_2)$ is an unramified covering of degree d, which is compatible with ρ_2 and restricts to an unramified covering $\varphi : \rho_2\left(D^{(2)}\right) \to \varphi\rho_2\left(D^{(2)}\right)$ of degree d, then there is a fibered product commutative diagram (0.2). The final, third section of the third chapter interprets the non-trivial finite unramified coverings $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of smooth toroidal compactifications, subject to (0.2) as a partial order $X_2 \succeq X_1$ on the set $\mathcal{S}$ of the smooth toroidal compactifications of non-compact ball quotients $\mathbb{B}/\Gamma$. The maximal elements of $\mathcal{S}$ with respect to $\succeq$ are called saturated, while the minimal ones are referred to as primitive. Any smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is shown to dominate a primitive one $X_0 = (\mathbb{B}/\Gamma_0)'$. In a vast distinction, a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is dominated by a saturated $X_1 = (\mathbb{B}/\Gamma_1)' \in \mathcal{S}$ if and only if X has finite fundamental group $\pi_1(X)$. The last section of Chapter 3 discusses the saturation and the primitiveness of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ of Kodaira dimension $\kappa(X) \in \{-\infty, 0\}$. It establishes that such $X = (\mathbb{B}/\Gamma)'$ is saturated if and only if X is a rational surface or X has a K3 minimal model. If $X = (\mathbb{B}/\Gamma)'$ is a rational surface or has an Enriques minimal model Y, then X is primitive. Let $\rho : X = (\mathbb{B}/\Gamma)' \to Y$ be a finite composition of blow downs, which transforms a smooth toroidal compactification onto an abelian or a K3 minimal surface Y, and $D := X \setminus (\mathbb{B}/\Gamma)$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$. Then X is non-primitive if and only if Y admits a non-identical fixed point free automorphism $g : Y \to Y$, which leaves invariant $\rho(D)$. Moreover, in the case of a K3 surface Y, g is to be of order 2. After observing that in some cases the non primitiveness of $X = (\mathbb{B}/\Gamma)'$ concerns the presence of a fixed point free element of the relative biholomorphism group $\mathrm{Aut}(Y, \rho(D))$, the last section of Chapter 3 establishes that if the exceptional divisor $E(\rho) = \coprod_{i=1}^{s} L_i$ of $\rho : X = (\mathbb{B}/\Gamma)' \to Y$ is a disjoint union of smooth irreducible rational (-1)-curves L_i, then the relative biholomorphism groups $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\rho))$ and $\mathrm{Aut}(Y, \rho(D)) = \mathrm{Aut}(Y, \rho(D), \rho(D)^{\mathrm{sing}})$ admit a natural isomorphism, transforming the fixed point free entries of $\mathrm{Aut}(X, D)$ onto the fixed point free elements of $\mathrm{Aut}(Y, \rho(D))$. Moreover, $\mathrm{Aut}(X, D)$ is shown to be a finite group.

The final, fourth chapter of the thesis presents the results of the article [7]. Let us suppose that $\beta : X = (\mathbb{B}/\Gamma)' \to Y$ is a blow down of the smooth irreducible rational (-1)-curves L_i, $1 \leq i \leq s$ on a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ to a minimal ruled surface $r : Y \to B$ with an elliptic base B and $D := X \setminus (\mathbb{B}/\Gamma) = \sum_{j=1}^{k} D_j$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$. Then $C_j := \beta(D_j)$ are shown to be such smooth irreducible elliptic curves on Y that the restrictions $r : C_j \to B$ are finite unramified coverings of degree $d_j \in \mathbb{N}$. Let $B_0 \subset Y$ be a section of $r : Y \to B$ with minimal self-intersection number $\delta = B_0^2$. By a theorem of Nagata from [40], $\delta \leq g(B) = 1$ does not exceeed the genus $g(B) = 1$ of B. For $\delta < 0$ any smooth irreducible elliptic curve $C_j \subset Y$ is shown in [7] to be a section of $r : Y \to B$. In the case of $\delta = B_0^2 \in \{0, 1\}$, there could exist smooth irreducible elliptic curves $C_j \subset Y$ with $\deg \left[r|_{C_j} : C_j \to B \right] = d_j > 1$ and $C_j^2 = 0$. Chapter 4 reduces the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) to

$$\sum_{i=1}^{s} (L_i.D - 4) = \sum_{j=1}^{k} C_j^2.$$

If all C_j are sections of $r : Y \to B$, this is proved to be equivalent to

$$(k-1) \left[\sum_{i=1}^{s} (L_i.D - 4) \right] = \sum_{i=1}^{s} L_i.D(L_i.D - 1).$$

When $\deg \left[r|_{C_j} : C_j \to B \right] = d_j > 1$ for at least one $1 \leq j \leq k$, there follows the inequality

$$(k-2) \left[\sum_{i=1}^{s} (L_i.D - 4) \right] \geq \sum_{i=1}^{s} (L_i.D - 1)(L_i.D - 2).$$

As a result, at least one of the punctured spheres $L_i \setminus D$, arising from a (-1)-curve $L_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$ on $X = (\mathbb{B}/\Gamma)'$, is to be non totally geodesically embedded in $\mathbb{B}/\Gamma$. If all

C_j are sections of $r : Y \to B$ then the number k of the cusps of $\mathbb{B}/\Gamma$ is shown to be $k \geq 15$. The Euler number of $\mathbb{B}/\Gamma$ turns to be $e(\mathbb{B}/\Gamma) = s \geq 14$. For any $15 \leq k \leq 62$ is computed an explicit lower bound $\mu_k \geq 2$ on the number of the non totally geodesic $L_i \setminus D \subset \mathbb{B}/\Gamma$. In a similar vein, if there exists $1 \leq j \leq k$ with $\deg \left[r|_{C_j} : C_j \to B \right] = d_j > 1$ and $C_j^2 = 0$ then $\mathbb{B}/\Gamma$ is proved to have $k \geq 12$ cusps and Euler number $e(\mathbb{B}/\Gamma) = s \geq 11$. For any $12 \leq k \leq 44$ are obtained explicit lower bounds $\mu_k \geq 2$ on the number of the non totally geodesic $L_i \setminus D \subset \mathbb{B}/\Gamma$.

Scientific Contributions

According to the suthor's opinion, the scientific contributions of the thesis are as follows:

1. Explicit construction of a bijective correspondence between the finite unramified coverings $X_1 = (\mathbb{B}/\Gamma_1)' \to X = (\mathbb{B}/\Gamma)'$ of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ and the finite unramified coverings $Y_1 \to Y$ of a minimal model Y of X.

2. Explicit construction of a bijective correspondence between the finite unramified coverings $X = (\mathbb{B}/\Gamma)' \to X_1 = (\mathbb{B}/\Gamma_1)'$ by a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$, which are compatible with a sequence $\rho : X \to Y$ of blow downs to a minimal surface Y and the finite unramified coverings $Y \to Y_1$, compatible with ρ.

3. The finite unramified coverings $X_2 = (\mathbb{B}/\Gamma_2)' \to X_1 = (\mathbb{B}/\Gamma_1)'$ of smooth toroidal compactifications, which are compatible with a sequence $\rho_2 : X_2 \to Y_2$ of blow downs to a minimal surface Y_2 and induce finite unramified coverings $Y_2 \to Y_1$ of the minimal model Y_1 of X_1 provide a partial order in the set $\mathcal{S}$ of the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ of the quotients $\mathbb{B}/\Gamma$ of the complex 2-ball $\mathbb{B}$ by a lattice $\Gamma < U(1,2)$. The minimal elements of $\mathcal{S}$ are called primitive, while the maximal ones are saturated. Any $X = (\mathbb{B}/\Gamma)' \in \mathcal{S}$ dominates some primitive $X_0 = (\mathbb{B}/\Gamma_0)' \in \mathcal{S}$. A smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is dominated by a saturated $X_1 = (\mathbb{B}/\Gamma_1)'$ if and only if X finite fundamental group $\pi_1(X)$. Making use of the properties of the minimal projective surfaces Y of non-positive Kodaira dimension, the thesis characterizes the saturated and the primitive $X = (\mathbb{B}/\Gamma)' \in \mathcal{S}$ with minimal model Y.

4. Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$ and $\beta : X \to Y$ be a finite sequence of blow downs to a minimal surface Y, whose exceptional divisor $E(\beta) = \coprod_{i=1}^{s} L_i$ has disjoint irreducible components L_i. The group $\mathrm{Aut}(X, D) = \mathrm{Aut}(X, D, E(\beta))$ is shown to be finite and isomorphic to $\mathrm{Aut}(Y, \beta(D)) = \mathrm{Aut}(Y, \beta(D), \beta(D)^{\mathrm{sing}})$.

5. Let $\beta : X = (\mathbb{B}/\Gamma)' \to Y$ be a blow down of smooth irreducible rational (-1)-curves L_i, $1 \leq i \leq s$ on a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ to a minimal ruled surface $r : Y \to B$ with an elliptic base B and $D := X \setminus (\mathbb{B}/\Gamma) = \sum_{j=1}^{k} D_j$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$ with smooth elliptic irreducible components D_j. Chapter 4 obtains explicitly the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) in terms of the intersection numbers $L_i.D$ and the self-intersection numbers $\beta(D_j)^2$ of the smooth elliptic curves $\beta(D_j) \subset Y$. If all $\beta(D_j)$ are sections of $r : Y \to B$, then the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) is expressed only by $L_i.D$, $1 \leq i \leq s$. When $r|_{\beta(D_j)} : \beta(D_j) \to B$ is of degree $d_j > 1$ for at least one $1 \leq j \leq k$, the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) implies an inequality on $L_i.D$ for $1 \leq i \leq s$.

6. By the means of the equality, respectively, the inequality on $L_i.D$, $1 \leq i \leq s$, described in 5. are obtained lower bounds on the number k of the cusps of $\mathbb{B}/\Gamma$, which coincides with the number of the smooth elliptic irreducible components D_j of the toroidal compactifying divisor $D = X \setminus (\mathbb{B}/\Gamma)$.

7. Any ball quotient $\mathbb{B}/\Gamma$ with smooth toroidal compactification $(\mathbb{B}/\Gamma)'$, whose minimal model is a ruled surface $r : Y \to B$ with an elliptic base B is shown to contain a non totally geodesic punctured sphere $L_i \setminus D \subset \mathbb{B}/\Gamma$, arising from a smooth irreducible rational (-1)-curve $L_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$ on $(\mathbb{B}/\Gamma)'$.

8. Let $\beta : X = (\mathbb{B}/\Gamma)' \to Y$ be a blow down of smooth irreducible rational (-1)-curves L_i, $1 \leq i \leq s$ on a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ to a minimal ruled surface $r : Y \to B$ with an elliptic base B and $D = X \setminus (\mathbb{B}/\Gamma) = \sum\limits_{j=1}^{k} D_j$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$. If $r|_{\beta(D_j)} : \beta(D_j) \to B$ are biholomorphisms for all $1 \leq j \leq k$, let us assume that $k \leq 62$. If there exists $1 \leq j \leq k$ with $\deg \left[f|_{\beta(D_j)} : \beta(D_j) \to B \right] > 1$, suppose that $k \leq 44$. Chapter 4 of the thesis provides explicit lower bounds $\mu_k \geq 2$ on the number of the non totally geodesic $L_i \setminus D \subset \mathbb{B}/\Gamma$, depending on $\deg \left(r|_{\beta(D_j)} \right) = 1$, $\forall 1 \leq j \leq k$ or on the existence of $\deg \left(r|_{\beta(D_j)} \right) > 1$ for some $1 \leq j \leq k$.

Approbation of the results

The results of the thesis are published in the following two articles:

- Beshkov P., Kasparian A., Sankaran G.: Saturated and primitive smooth compactifications of ball quotients, *Ann. Sofia Univ., Fac. Math. and Inf.*, **106**, 2019, 53–77.
- Beshkov P., Kasparian A.: Lower bounds on the number of cusps of a toroidal compactification with a ruled minimal model, *C. R. Acad. Bulg. Sci.*, **74(8)**, 2021, 1120–1127.

The scientific contributions of the thesis are reported at:

1. National Coding Seminar "Professor Stefan Dodunekov", 2018.
2. National Coding Seminar "Professor Stefan Dodunekov", 2019.
3. Spring Scientific Session of Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski", 2019.
4. Spring Scientific Session of Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski", 2021.

Declaration of the authenticity of the presented results

I declare that the presented Ph.D. thesis contains original results, obtained from research conducted by myself (with the help and guidance of my scientific advisor and co-authors). The results obtained by other scientists have been thoroughly and clearly cited in the bibliography.

Signature:

(Pancho Georgiev Beshkov)

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