

REVIEW

under the procedure for public defence of the PhD Thesis entitled:
"Toroidal compactifications of discrete quotients of the complex two-ball"
under the procedure for acquisition of the educational and scientific degree "Doctor"
by Pancho Georgiev Beshkov

In the Scientific field: 4. Natural Sciences, Mathematics and Informatics,
Professional field: 4.5. Mathematics,
Doctoral program: "Algebra, Topology and Applications", Department: "Algebra",
Faculty of Mathematics and Informatics (FMI),
Sofia University "St. Kliment Ohridski" (SU).

The review has been prepared by prof. Maya Miteva Stoyanova, Ph.D., Deputy Dean of FMI, Department of "Algebra", Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", in my capacity as chairman of the scientific jury, according to Order for the defence of this PhD thesis according to Order RD-38-141/04.03.2022 of the Rector of the Sofia University "St. Kliment Ohridski" and decision of the scientific jury (Protocol №1/14.03.2022).

1 General characteristics of the dissertation thesis and the presented materials

The presented PhD thesis is 126 pages long and consists of an introduction, four chapters and a bibliography of 50 titles. The abstract, within 19 pages, summarizes the content of the dissertation, clearly and accurately reflecting both the main contributions of the doctoral student and where the results are tested. The dissertation is based on results published in two publications of the doctoral student, one in the Annual of Sofia University, FMI, and the other in the Proceedings of the Bulgarian Academy of Sciences. Both a list and copies of both publications are attached. The materials and documents presented by PhD student Pancho Beshkov certify that all the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria (ADAS in the Republic of Bulgaria) and its regulations have been met. Pancho Georgiev Beshkov fulfills the minimum national requirements under Art. 2b. para. 2 and para. 3 of the ADAS in the Republic of Bulgaria.

2 Short CV and personal impressions of the candidate

I have known Pancho Georgiev Beshkov first as a part-time lecturer and doctoral student at the Department of Algebra, and now as an assistant at the department. Pancho Beshkov acquired a Bachelor's degree from the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski", majoring in Mathematics in 2010. After that he obtained a Master's degree from the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski", master's program " Mathematics and mathematical physics ". Since 15.02.2017 he has been enrolled as a full-time PhD student in the doctoral program "Algebra, Topology and Applications", Department of "Algebra", FMI-SU. By Order № RD-20-401/06.02.2020 he was dismissed with the right to defence, as of 15.02.2020. From 01.10.2020 Pancho Beshkov is an assistant at the Department of Algebra, FMI-SU.

Pancho Beshkov is a correct and responsible colleague, with a good student opinion as a teacher (as far as I am informed). In personal conversations, Prof. Kasparyan told me that they have a good professional relationship as a supervisor and PhD student, which is evident from the joint scientific results presented by them (in the dissertation of Pancho Beshkov).

3 Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

The present PhD thesis investigates the smooth toroidal compactifications $X = (B/\Gamma)'$ of non-compact quotients B/Γ of the complex two-ball B by a lattice $\Gamma < U(1, 2)$. More precisely, it focuses on the finite unramified coverings $f : X_2 = (B/\Gamma_2)' \rightarrow X_1 = (B/\Gamma_1)'$ of smooth toroidal compactifications, which restrict to finite unramified coverings $f : B/\Gamma_2 \rightarrow B/\Gamma_1$ of the corresponding ball quotients and on some numerical invariants of the smooth toroidal compactifications $X = (B/\Gamma)'$, which are birational to a ruled surface $r : Y \rightarrow E$ with an elliptic base E .

In Chapter 1 and Chapter 2 are introduced all preliminaries, definitions and known facts. Holomorphic vector bundles over a complex manifold, algebraic surfaces, and their Chern numbers are considered, as well as the logarithmic Bogomolov-Miyaoka-Yau equality, characterizing the smooth toroidal compactifications $(B/\Gamma)'$ of ball quotients

B/Γ . Moreover, the construction of a toroidal compactifications $(B/\Gamma)'$ of a quotient of the complex two-ball B by a lattice $\Gamma < U(1,2)$ is described. The addition of the finitely many parabolic points to the quotient leads to a not always smooth complex projective manifold, even with a smooth quotient, and the toroidal compactification is the resolution of the singularities of the parabolic points.

Chapters three and four describe the new original constructions and results obtained.

In Chapter 3 it is presented an explicit construction of a bijective correspondence between the finite unramified coverings $X_1 = (B/\Gamma_1)' \rightarrow X = (B/\Gamma)'$ of a smooth toroidal compactification $X = (B/\Gamma)'$ and the finite unramified coverings $Y_1 \rightarrow Y$ of a minimal model Y of X .

Next it is presented an explicit construction of a bijective correspondence between the finite unramified coverings $X = (B/\Gamma)' \rightarrow X_1 = (B/\Gamma_1)'$ by a smooth toroidal compactification $X = (B/\Gamma)'$, which are compatible with a sequence $\rho : X \rightarrow Y$ of blow downs to a minimal surface Y and the finite unramified coverings $Y \rightarrow Y_1$, compatible with ρ .

It is shown that the finite unramified coverings $X_2 = (B/\Gamma_2)' \rightarrow X_1 = (B/\Gamma_1)'$ of smooth toroidal compactifications, which are compatible with a sequence $\rho_2 : X_2 \rightarrow Y_2$ of blow downs to a minimal surface Y_2 and induce finite unramified coverings $Y_2 \rightarrow Y_1$ of the minimal model Y_1 of X_1 provide a partial order in the set S of the smooth toroidal compactifications $X = (B/\Gamma)'$ of the quotients B/Γ of the complex 2-ball B by a lattice $\Gamma < U(1,2)$. The minimal elements of S are called primitive, while the maximal ones are saturated. Any $X = (B/\Gamma)' \in S$ dominates some primitive $X_0 = (B/\Gamma_0)' \in S$. A smooth toroidal compactification $X = (B/\Gamma)'$ is dominated by a saturated $X_1 = (B/\Gamma_1)'$ if and only if X has finite fundamental group $\pi_1(X)$. Making use of the properties of the minimal projective surfaces Y of non-positive Kodaira dimension, the thesis characterizes the saturated and the primitive $X = (B/\Gamma)' \in S$ with minimal model Y .

Let $X = (B/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (B/\Gamma)$ and $\beta : X \rightarrow Y$ be a finite sequence of blow downs to a minimal surface Y , whose exceptional divisor $E(\beta) = \bigsqcup_{i=1}^s L_i$ has disjoint irreducible components L_i . In Chapter 4 the group $Aut(X, D) = Aut(X, D, E(\beta))$ is shown to be finite and isomorphic to $Aut(Y, \beta(D)) = Aut(Y, \beta(D), \beta(D)^{sing})$.

Let $\beta : X = (B/\Gamma)' \rightarrow Y$ be a blow down of smooth irreducible rational (-1)-curves L_i , $1 \leq i \leq s$, on a smooth toroidal compactification $X = (B/\Gamma)'$ to a minimal ruled surface $r : Y \rightarrow E$ with an elliptic base E , and $D := X \setminus (B/\Gamma) = \sum_{j=1}^k D_j$ be the toroidal

compactifying divisor of B/Γ with smooth elliptic irreducible components D_j . The PhD thesis expresses explicitly the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) in terms of the intersection numbers $L_i \cdot D$ and the self-intersection numbers $\beta(D_j)^2$ of the smooth elliptic curves $\beta(D_j) \subset Y$. If all $\beta(D_j)$ are sections of $r : Y \rightarrow E$, then the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) is expressed only by $L_i \cdot D$, $1 \leq i \leq s$. When $r|_{\beta(D_j)} : \beta(D_j) \rightarrow E$ is of degree $d_j > 1$ for at least one $1 \leq j \leq k$, the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) implies an inequality on $L_i \cdot D$ for $1 \leq i \leq s$.

By the means of the logarithmic Bogomolov-Miyaoka-Yau equality for (X, D) , respectively, the inequality on $L_i \cdot D$, $1 \leq i \leq s$, described in Chapter 4, are obtained lower bounds on the number k of the cusps of B/Γ , which coincides with the number of the smooth elliptic irreducible components D_j of the toroidal compactifying divisor

$$D = X \setminus (B/\Gamma) = \sum_{j=1}^k D_j.$$

Any ball quotient B/Γ with smooth toroidal compactification $X = (B/\Gamma)'$, whose minimal model is a ruled surface $r : Y \rightarrow E$ with an elliptic base E , is shown to contain a non totally geodesic punctured sphere $L_i \setminus D \subset B/\Gamma$, arising from a smooth irreducible rational (-1) -curve $L_i \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$ on $X = (B/\Gamma)'$.

Let $\beta : X = (B/\Gamma)' \rightarrow Y$ be a blow down of smooth irreducible rational (-1) -curves L_i , $1 \leq i \leq s$, on a smooth toroidal compactification $X = (B/\Gamma)'$ to a minimal ruled surface $r : Y \rightarrow E$ with an elliptic base E , and $D := X \setminus (B/\Gamma) = \sum_{j=1}^k D_j$ be

the toroidal compactifying divisor of B/Γ . If $r|_{\beta(D_j)} : \beta(D_j) \rightarrow E$ are biholomorphisms for all $1 \leq j \leq k$, Chapter 4 assumes that $k \leq 62$. If there exists $1 \leq j \leq k$ with $\deg[r|_{\beta(D_j)} : \beta(D_j) \rightarrow E] > 1$, suppose that $k \leq 44$. Under these conditions (in the last Chapter of the PhD thesis) explicit lower bounds $\mu_k \geq 2$ on the number of the non totally geodesic $L_i \setminus D \subset B/\Gamma$ are derived, depending on $\deg(r|_{\beta(D_j)}) = 1$ for all $1 \leq j \leq k$, or on the existence of $\deg(r|_{\beta(D_j)}) > 1$ for some $1 \leq j \leq k$.

The results presented above give me reason to claim that the candidate Pancho Georgiev Beshkov has in-depth knowledge of the PhD thesis, and that his original contributions are sufficient to acquire the educational and scientific degree "Doctor".

4 Approbation of the results

From the presented documents it is evident that PhD student Pancho Georgiev Beshkov has designed his PhD thesis on the basis of results published in two publications

[6] and [7], according to the bibliography in the PhD thesis. Both publications are co-authored with his supervisor Prof. Azniv Kasparian, and publication [6] is co-authored with G. Sankaran. From the submitted declarations for co-authorship it is evident that the contribution of Pancho Beshkov and the other co-authors is equal. Publication [6] was published in the Annual of Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics in 2019, and according to the Regulations for the implementation of the ADAS in the Republic of Bulgaria, this brings the candidate 18 points to fulfilment the minimum national requirements for the acquisition of the educational and scientific degree "Doctor". The results of the second publication were published in 2021 in the Comptes rendus de l'Academie bulgare des Sciences, which is an impact factor magazine (IF: 0.378 for 2020) and is available on the Web of Science Quartile: Q_4 , i.e. adds another 36 points to cover the minimum national requirements. As a result, with 54 points (if 30 points are needed), the candidate Pancho Beshkov covers and exceeds the minimum national requirements under Art. 2b, para. 2 and 3 of the ADAS in the Republic of Bulgaria, required for the acquisition of the educational and scientific degree "Doctor" in Professional field 4.5. Mathematics. I have no information and no suspicions of plagiarism in the presented dissertation and scientific papers on this procedure. I have no information about citations to the publications. The results were presented by Pancho Beshkov at the Spring Scientific Session of the Faculty of Mathematics and Informatics in 2019 and 2021, as well as at the National Seminar on Coding Theory "Prof. Stefan Dodunekov" in 2018 and 2019, respectively.

5 Qualities of the abstract

The abstract in Bulgarian is 20 pages and the one in English is 19 pages. There are both prepared according to all requirements and correctly reflect the content of the dissertation and scientific contributions of the PhD student.

6 Critical notes and recommendations

The presented PhD thesis is in the field of the algebraic geometry, as the high style of the described content requires quite deep mathematical knowledge in advance to understand the results presented in it. This makes the PhD thesis difficult to read by other young scientists and PhD students, which is a small drawback in my opinion. I would recommend in the future Pancho Beshkov to try to describe in more detail the preliminary data and definitions with illustrations through some examples to be more

attractive and accessible to a larger group of readers and really promote the discussed in the PhD thesis, although difficult, very interesting and current tasks. There are also some technical (spelling) errors. My remarks and recommendations above do not in the least substantially change my good impressions of the PhD thesis and the scientific knowledge of the candidate on the topic of the PhD thesis.

7 Conclusion

Having become acquainted with the PhD thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, **I confirm** that the presented PhD thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, fulfils the requirements of the ADAS in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by the candidate of educational and scientific degree "Doctor" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics. In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, **I strongly recommend** the scientific jury to award to Pancho Georgiev Beshkov, the educational and scientific degree "Doctor" in the Scientific field: 4. Natural Sciences, Mathematics and Informatics, Professional field: 4.5. Mathematics, Doctoral program: "Algebra, Topology and Applications", Department of "Algebra", Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski".

May 7, 2022
Sofia

Signature:
Prof. Maya Stoyanova, PhD