R E P O R T

on a Thesis for awarding the degree "Doctor"

Scientific field: 4. Natural sciences, mathematics and informatics

Professional field: 4.5. Mathematics

Title: Toroidal compactifications of the discrete quotients of the complex two-ball

Author: Pancho Georgiev Beshkov

Overview

The presented thesis deals with the a problem which can be related to the area of algebraic geoemtry and algebraic toplogy. The main problems treated in the thesis are concerned with the investigation of toroidal compactifications of discrete factors of the complex two-ball. This problem is of special interest since the factors of the complex two-ball and their compactifications generalize in a natural way the Riemann surfaces of genus at least 2.

The thesis is based on the reserch carried out by the author in the past few years.

State of the current research

My general impression is that the author is well acquainted with the state of the art and the most recent results in the research on the treated problems. A good deal of the investigations that are carried out are considered in the field as important theoretically. The author demonstrates deep knowledge of his field of research and capacity to apply his knowledge to the solution of important problems.

Methods

In his investigations the author uses a wide spectrum of mathematical methods that can be related to the algebraic geometry, algebraic topology, group theory and the general linear algebra.

Brief description of the thesis

The presented thesis amounts 123 pages of text and consists of an introduction, four chapters, and a list of references including 50 items. In what follows, I shall give a short description of the topics covered in this dissertation.

Chapter 1 is introductory and contains a brief description of the more significant definitions and theoretical facts needed for the thesis. The main focus is set on important special results like the classification of the minimal compact complex surfaces by Enriques and Kodaira, the logarithmic Chern numbers of smooth complex projective planes, logarithmic Bogomolov-Miyaoka-Yau equality, describing the smooth toroidal compactifications and their toroidal compactifying divisor.

Chapter 2 contains a description of a construction of the toridal compactification $(\mathbb{B}/\Gamma)'$ of a factor $\mathbb{B}/\Gamma$ of the complex two-ball by a lattice $\Gamma < U(1,2)$. Using the action of $U(1,2)$ on the two-ball, on its boundary and on the remaining part of $\mathbb{P}^2(\mathbb{C})$, the author arrives at the definition of the toroidal compactification $(\mathbb{B}/\Gamma)'$. The chapter contains an explanation of additional questions related the intersection indices $L_i \cdot D$.

The original contributions of this thesis are contained in chapters 3 and 4.

Chapter 3 is devoted to the properties saturation and primitivity of toroidal comptactifications. In this chapter the author establishes a a bijective mapping of finite unramified coverings $(X, D, E(\rho))$ and $(\rho(X), \rho(D), \rho(E(\rho))$. Here $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification of a factor of the complex 2-ball, D is the toroidal compactifying divisor, $\rho : X \to y$ is a composition of of blow downs of (-1)-curves to a minimal surface Y and $E(\rho)$ is the exceptional divisor of ρ. The main results are contained in a series of statements in section 3.3 – these are Propositions 3.16–3.20 and 3.22. All they are related to the presence of the properties primitivity and sturation of smooth toroidal compactifications of a factor of the complex two-ball with Kodaira dimension $\kappa(X) \leq 0$.

In chapter 4, the author considers again a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ of a discrete factor $\mathbb{B}/\Gamma$ of the two-ball and a blow down of (-1)-curves to a minimal ruled surface with an elliptic base. The goal here is to express the logarithmic equality of Bogomolov-Miyaoka-Yau for (X, D) through the intersection indices of the irreducible component of the exceptional divisor of β and the smooth irreducible components of the toroidal compactifying divisor D.

The chapter starts with two lemmas (4.1 and 4.2). In the first one it is proved that the β-images of the elliptic components of the toroidal compactifying divisor of $\mathbb{B}/\Gamma$ are smooth elliptic curves. If the intersection indices satisfy $L_i \cdot C_j \geq 2$, then the curve C_j is singular. Thus it follows by these two lemmas that the indices are 0 or 1. In the second lemma, the author considers a ruled surface $r : Y \to B$ with an elliptic base B and two smooth elliptic curves C_1, C_2, on which r is restricted to a unramified covering of degree d_i. Then if the self-intersection number is 0 or 1, it follows that $C_1 \sim B_o + a_1 F$ is proportional to the canonic divisor K_Y of Y or is an intersection of r with a_1. The latter holds true if the self-intersection number is smaller than 0. Apart from this, it is proved that the equality $d_1 d_2 (C_1^2 + C_2^2) = 2 C_1 C_2$ holds true.

The main result in this chapter is Theorem 1. In this theorem the author considers again a blow down $\beta : X \to Y$ of (-1)-curves L_i, $i = 1, \ldots, s$, on a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ to a ruled surface $r : Y \to B$ with an elliptic base B and a toridal compactifying divisor $D = \sum_{j=1}^{k} D_j$. It is proved that:

(1) the canonicl divisor of X is

$$K_X = \beta^{-1}(K_Y) + \sum_{i=1}^{s} L_i$$

(2) the logarithmic equality of Bogomolov-Miyaoka-Yau takes the form

$$\sum_{i=1}^{s}(L_i \cdot D - 4) = \sum_{i=1}^{k} C_j^2.$$

From Theorem 1 the author obtains three corollaries (4.3–4.5), which contain further interesting results. So, for instance, if the intersection number B_o^2 is $\delta < 0$ or $\delta \in \{0, 1\}$, then the factor $\mathbb{B}/\Gamma$ has at least 15 parabolic points; furthermore, if the numebr of the parabolic points is $15 \leq k \leq 62$, then $\mathbb{B}/\Gamma$ has at least two non-totally geodesic punctured spheres $L_i \setminus D$ and the exact number of the spheres is given in Table 4.2 (Corollary 4.4). Similarly, if $\delta \in \{0, 1\}$ and there exists a curve C_j, for which $d_j \geq 2$, then $\mathbb{B}/\Gamma$ has at least 12 parabolic points and in the case where this number is between 12 and 44 there exist at least two non-totally geodesic punctured spheres (Corollary 4.5). The exact values are given in Table 4.4.

Main results

The main contributions of this thesis amount to the following:

(1) A construction is given of a bijective mapping between the finite unramified coverings of a smooth toroidal compactification X of a factor of the complex two-ball and the finite unramified coverings $Y_1 \to Y$ of a minimal model Y of X.

(2) A characterization of the saturated and the primitive smooth toroidal compactifications with a non-negative Kodaira dimension is made.

(3) Various results concerning the automorphism group $\mathrm{Aut}(X, D)$ of a smooth toroidal compactification are proved.

(4) An explicit expression for the logarothmic equality of Bogomolov-Miyaoka-Yau through the intersection numbers $L_i \cdot D$ is obtained.

(5) Lower bounds on the number of the parabolic points of the factor ball $\mathbb{B}/\Gamma$ is found, which coinsides with the number of the smooth elliptic irreducible components of the toroidal compactifying divisor D.

(6) The existence of a non-totally geodesic punctured sphere is proved, which arises from a smooth irreducible rational (-1)-curve.

Remarks and comments

I have the following remarks, questions and comments related to this thesis:

(1) Why is the main result in chapter 4 called Theorem 1? It does not follow the general rule of numbering results which is chapter number-statement number.

(2) On page 16: Lemma should be in Bulgarian.

(3) Typing mistake in the title of section 4. The same error appears in the running head.

(4) The condition in Corollary 4.5 should be $d_j := \deg[r|_{C_j} : C_j \to B] \geq 2$? The same question in contribution 8 of the author's summary.

(5) In contribution 5 of the summary the author refers to chapter 5. The author has in mind Theorem 1 of chapter 4.

(6) Some of the results are called propositions and some are called theorems. What is the difference between a theorem and a proposition?

(7) The results of this thesis are reported at two workshops in coding theory and two spring sessions of the Faculty of Mathematics and Informatics. It would be desirable to report them also at more specialized conferences, where they could get a more adequate assessment.

Publications related to the thesis

The results in this thesis are published in two papers. The respective journals are as follows:

- Annuaire Universite de Sofia – 1 paper
- Comptes Rendus de l'Academie Bulgare des Sciences - 1 paper
  IF 0.378(2020), SJR 0.24(2020)

One of the papers is in a journal with an impact factor. The other paper is in a journal which is refereed in Zentralblatt.
One of the papers is with one and one is with two coauthors.

Authorship of the obtained results

I have been following the scientific output of the author for several years. That is why I have no doubt that his contribution in this research is significant.

Citations

The candidate has not attached a list of citations of the papers used in this thesis. Since the papers on which the thesis is based appeared shortly before the completion of the thesis, I have no doubt that citations of the author's research will appear in the future.

Authors summary

The author's summary is made according to the existing regulations and reflects properly the main results and contributions of this thesis.

Conclusion

This thesis is focused on problems from algebraic geometry and algebraic topology that are of great importance for the theory. This work does not only answer open problems of principal importance, but also motivates new directions for an ongoing research. I am deeply convinced that the presented thesis **"Toroidal compactifications of the discrete quotients of the complex two-ball"** by **Pancho Georgiev Beshkov** contains results that are an original contribution to the algebraic geometry. The candidate demonstrates deep knowledge of the theory and capacity to develop it in new and important ways. With this, he meets the legal national requirements prescribed by the law, as well as the specific requirements of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences for the professional field 4.5 "Mathematics". I assess **positively** the presented PhD Thesis and recommend to this panel to award **Pancho Georgiev Beshkov** the scientific degree "Doctor" in the scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5 "Mathematics".

Sofia, 09.05.2022 Member of the Scientific Panel:

(Prof. DSc Ivan Landjev)