Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 61, No 12, 2008

SCIENCES ET INGENIERIE
Automatique et informatique

UTILITY FUNCTION DERIVED FROM AFFECTIVE BALANCE THEORY

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(*Submitted by Corresponding Member M. Hadjiiski on October 28, 2008*)

Abstract

We propose a novel way to define a utility function which is based on the differential equations of Affective Balance Theory. Introduced in 1987 that theory is more general than the Nobel Prize winning Prospect Theory, but remained poorly understood by the decision-making community due to its bottom-up approach to choice behaviour involving biophysical models of neuron interactions. Under the assumptions of equation equilibrium, and equivalence between utility and satisfaction, we obtain an analytical formula for the agent's utility function. Into that formula we set constants to values determined in advance from a mix of a psychological experiment with 129 participants and a computational optimizing procedure. The new utility function is asymmetric with respect to gains and losses, exactly as in Cumulative Prospect Theory.

Key words: utility function, Affective Balance Theory, Cumulative Prospect Theory

Introduction. There are many ways to define a utility function. Historically, DANIEL BERNOULLI [1] was the first to do so, writing in 1738, "a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount." Bernoulli put this idea in mathematical form, reasoning that an infinitesimally small gain dW would bring to its recipient utility, proportional to that gain, and inversely proportional to that person's total wealth W. Therefore, the infinitesimal gain's utility would be $a.dW/W$, where a is a positive proportionality constant. A received amount of money which raises one's wealth from W_1 to W_2 (where $0 < W_1 < W_2$) would bring this utility

$$\int_{W_1}^{W_2} \frac{a.dW}{W} = a \ln \frac{W_2}{W_1}.$$

VON NEUMANN and MORGENSTERN [14] needed only monotonicity with respect to gains to define axiomatically their utility function, but later researchers found it necessary to invest much effort in developing complex analytical models. That activity culminated perhaps in KEENEY and RAIFFA's work [6], who suggested different formulae for a variety of risk attitudes. Seeking a mathematically complicated utility function is justified only when the researcher extracts expert knowledge. Such efforts continue to this day, and interesting results have been published also by some Bulgarians. For example, PAVLOV [9,10] has developed recurrent stochastic algorithms for evaluation of expert utilities; TENEKEDJIEV et al. [11,12] have introduced an Arctg approximation of one-dimensional utility functions.

Meanwhile, a psychological streak was added to the field, bringing insight into the mental processes of judgment and decision making [5]. This work was eventually recognized with a Nobel Prize for economics to Daniel Kahneman, one of the area's most prominent figures. TVERSKY and KAHNEMAN's Cumulative Prospect Theory [13] posits this utility function

$$v(\xi) = \begin{cases} \xi^{\alpha} & \text{if } \xi \geq 0, \\ -\lambda(-\xi)^{-\beta}, & \text{if } \xi < 0, \end{cases} \tag{1}$$

where ξ is the experienced gain or loss, and α, β, λ are constants. For a sample of 25 American students these authors found $\alpha = \beta = 0.88$ and $\lambda = 2.25$, which produced the famous curve much steeper for losses than for gains (Fig. 1, right). However, that model was purely phenomenological offering no theoretical understanding of the mental mechanism behind those revealed attitudes.

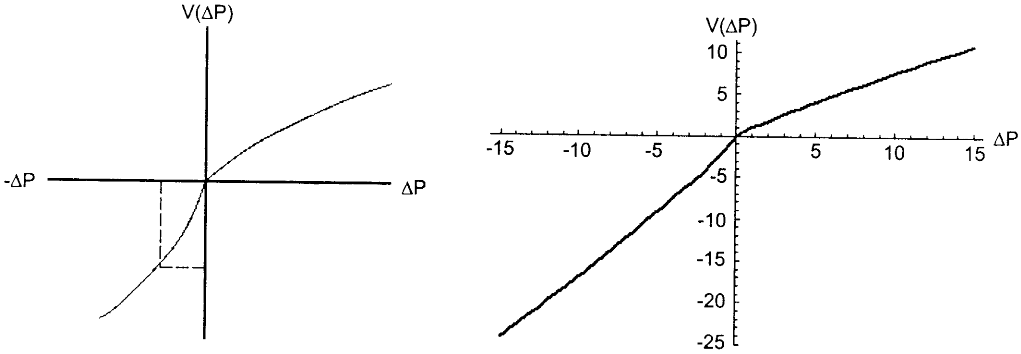

Fig. 1. Utility functions according to Prospect Theory (left) and Cumulative Prospect Theory (right)

In contrast, Grossberg has adopted a bottom-up approach, whereby simple interactions between neurons in the brain are described with differential equations from biophysics. Going this way, one can develop models with increasing complexity, and reach the level of observable behaviour. In particular, Affective Balance Theory [3] contains a model of six neurons interacting with neurotransmitters, and accounts for positive and negative emotions resulting from gains and losses.

It is a generalization of the classical Prospect Theory [5]. Even more general is the mathematical theory of Pavlovian conditioning [4], which allows experiences to be stored and used in future judgments. Both these theories guided an experiment in consumer behaviour [7], in which customer satisfaction as well as customer choice were 80–90% correctly predicted by a neural model. In the present paper we show how that model can yield a new analytical expression for the utility function.

From neurons to emotions and utility. The bottom-up theoretical approach is based on three elements: neural activity, neurotransmitters, and local memory, each described by a differential equation. First is the classical Hodgkin-Huxley membrane equation of a single neuron activity. We cite it in a form typically used in gated dipoles [2]

$$\frac{dx_i}{dt} = -A_1 x_i + (A_2 - x_i)J_i^+ - (x_i + A_3)J_i^-. \tag{2}$$

Here x_i is the bioelectric activity of the i-th neuron, $i = 1, \dots, M$, that is, the signal it sends to other neurons; J_i^+ and J_i^- are the sums of excitatory and inhibitory signals that other neurons send to the i-th; A_1, A_2, A_3 are real positive constants ($A_2 \gg x_i$). All neuron equations in this paper are special cases of Eq. (2). Henceforth, we shall use as synonyms the notions 'neuron activity', 'activation', 'neural signal' and denote them all by x with a subscript. In addition, all equations here are in dimensionless form which is done for simplicity and helps avoid unresolved biophysical issues.

A neuron sends signal by emitting neurotansmitters – mediating molecules causing biochemical change in the receiving neuron. The next equation describes neurotransmitter depletion and regeneration in the sending neuron. Here, y_i is quantity of mediator, B and C are constants

$$\frac{dy_i}{dt} = B(1 - y_i) - C.x_i y_i. \tag{3}$$

Finally, some signals get remembered, i.e., get coded in a memory element z_{ij}, whose biophysical substrate is the synapse between two communicating neurons, say, i and j. Henceforth, by z_{ij} we shall denote a dimensionless variable $z_{ij} \in [0, 1]$ with initial value zero (meaning no learning has happened). A signal x_k, sent to i by neuron k, is remembered only after a lasting biochemical change in a connection z_{ij} between i and j, has occurred

$$\frac{dz_{ij}}{dt} = x_i(-D_1 z_{ij} + D_2 x_k). \tag{4}$$

Constants B, C, D_1, D_2 in Eqs. (3)–(4) are also real positive. Grossberg has implemented variations of these three equations to build his gated dipole neural networks, which are capable of accounting for emotions and reflex conditioning. In our experiment [7] we adapted those equations to account for consumer behaviour.

In particular, if a customer of a mobile phone service had chosen between one of two suppliers A and B, and had expected to pay the sum P_a levs (price advertised), but at the end of the month, the final price was P_f, then a difference $\Delta P = P_a - P_f$ has occurred. (No real money was paid in this experiment; it was based on imagined situations.) In most cases, a saved amount ΔP^+ would provoke satisfaction; similarly, a surplus charge ΔP^- would be disappointing. However, a discount felt to be too small would also provoke disappointment, and an unexpectedly small surplus charge might cause slight satisfaction. The model we discuss here is capable of accounting for exactly those effects. Its adapted equations are:

$$\begin{aligned} \frac{dx_1}{dt} &= -A.x_1 + P_a + \delta.\Delta P^+ + M.x_7; \\ \frac{dx_2}{dt} &= -A.x_2 + P_a + \delta.\Delta P^- + M.x_8; \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{dy_1}{dt} &= B_1(1 - y_1) - C_1x_1y_1; \\ \frac{dy_2}{dt} &= B_2(1 - y_2) - C_2x_2y_2; \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{dx_3}{dt} &= -A.x_3 + D.x_1y_1; \\ \frac{dx_4}{dt} &= -A.x_4 + D.x_2y_2; \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{dx_5}{dt} &= -A.x_5 + (E - x_5)x_3 - (x_5 + E)x_4; \\ \frac{dx_6}{dt} &= -A.x_6 + (E - x_6)x_4 - (x_6 + E)x_3; \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{dx_7}{dt} &= -A.x_7 + G[x_5]^+ + L(S_A.z_{7A} + S_B.z_{7B}); \\ \frac{dx_8}{dt} &= -A.x_8 + G[x_6]^+ + L(S_A.z_{8A} + S_B.z_{8B}); \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{dz_{7A}}{dt} &= S_A(-K.z_{7A} + H[x_5]^+); \\ \frac{dz_{7B}}{dt} &= S_B(-K.z_{7B} + H[x_5]^+); \\ \frac{dz_{8A}}{dt} &= S_A(-K.z_{8A} + H[x_6]^+) \\ \frac{dz_{8B}}{dt} &= S_B(-K.z_{8B} + H[x_6]^+); \end{aligned} \tag{10}$$

$$\begin{aligned} o_1 &= [x_5]^+; \\ o_2 &= [x_6]^+. \end{aligned} \tag{11}$$

Here we can afford only a brief discussion on these equations and refer to the original works [4] and [7] for more detail. The $x_1, \ldots, x_8$ variables are neuron activities, and y_1 and y_2 are neurotransmitters. The four $z_{7A}, \ldots, z_{8B}$ are memories. Signal S_A in Eqs. (9) and (10) is equal to one when supplier A is active, and is zero otherwise. Signal S_B is the opposite. The operator $[\cdot]^+$ denotes rectification $[\xi]^+ = \max\{\xi, 0\}$. Variables o_1, o_2 are outputs of the system and comprise a couple of opposing emotions. In our case, those are customer satisfaction and disappointment.

In the experiment, a variety of prices P_a, P_f, was shown to 129 participants, and their reactions recorded by a software application. The latter allowed a person in front of a screen to choose a supplier from A and B, and to report one's own satisfaction or disappointment with that supplier's prices in a psychometric scale. We also measured and logged the time needed for each action. We mapped all these data onto Eqs. (5)–(11) by selecting suitable values for constants A, δ, M, B_1, B_2, C_1, C_2, D, E, G, L, K, and H for each person. Because that system of differential equations could not be solved analytically, at least with today's available mathematical knowledge and its software implementation, we needed stochastic optimization to find an approximate solution. The particular optimizing method we used was a fast version of Simulated Annealing. This algorithm generates random values within certain intervals of the constants and produces many solutions at each iteration. An error function containing all differences between the empirical data (participants' satisfactions and choices), and the relevant system variables is computed for each solution. The currently best solution is retained. The intervals of all constants shrink slightly according to a rule, paying respect to the best solution so far. In theory, continuing this process for infinitely many iterations with infinitely small interval shrinkages leads to the error function's global minimum. In practice, we did about 30 000 solutions for each participant to achieve those 80–90% correct predictions on the entire sample of 129 people. The differential equations were numerically solved by a Runge-Kutta-Felberg 4/5 method.

Because we expected asymmetric human reactions to gains and losses, we introduced in Eqs. (6) not one constant B, and one C, but two separate B_1 and B_2, and also C_1 and C_2. Had people reacted differently to gains, associated in this model with constants B_1 and C_1, than to losses, associated with B_2 and C_2, then one could expect statistically significant differences of the type $B_1 \neq B_2$ or $C_1 \neq C_2$. And indeed, we found $B_1 > B_2$, as shown by the Wilcoxon Matched Pairs Test ($Z = 3.3838$; $p = 0.00072$) and the Sign Test ($Z = 3.1696$; $p = 0.00153$). These tests did not reject the hypothesis $C_1 = C_2$, but due to B_1 and B_2 we found a real asymmetry in people's reactions. This result for Eqs. (6) has direct implications for Eqs. 8. Consider a thought experiment assuming equal gain and loss experienced by someone at the same moment. In other words, the same signal $x(t) = x_1(t) = x_2(t)$ is submitted to the system input. Next, let $x(t)$ be maintained long enough ($t = \infty$) for the neurotransmitters to reach equilibriums. Then Eqs. (6) yield $B_1/(B_1 + C.x_\infty)$ and $B_2/(B_2 + C.x_\infty)$. From

$B_1 > B_2$ it follows that $y_1(x_\infty) > y_2(x_\infty)$ and the asymmetry is propelled to Eqs. (7), and then further to Eqs. 8, which, after the rectification in Eqs. 11, yield $o_1(x) > o_2(x)$.

The last result means that for the majority of people satisfaction has been greater than disappointment in the particular experiment, somewhat in contrast to Prospect Theory's prediction. However, two points are relevant here. First, as we have shown [8] that result is context specific, and is explained by our participants' experience with Bulgaria's mobile phone providers, with whom surplus charge was the norm, and discounts were rare unexpected events at the time of the study (May 2007). Second, the two differing constants B_1, B_2 are stochastically optimized in conjunction with other 11 constants, and therefore should not be analysed in isolation. It turned out that for some participants $B_1 > B_2$ and still, due to all 13 constants, their utility curve was steeper for losses in accord with Prospect Theory. This finding is the subject of the next section.

Emotional balances and a new utility function. As already stated, o_1 and o_2, the rectified versions of neural activities x_5 and x_6 in the model, were directly related to the actual customer satisfaction and disappointment as measured in the experiment. We have numerical values of the constants in Eqs. (5)-(11) obtained from the numerical solutions for each participant. Therefore, if we could derive a formula for the utility function in general, we could fully specify it for each person from our sample.

To do so we reason as follows. It is natural to regard a utility function as something stable over at least a short period of time; something, which could not possibly be influenced by momentary neural dynamics. We used this assumption in the preceding section already, where we hypothesized reaction asymmetry represented by constants B_1 and B_2. Now we continue with this method. If, in a thought experiment, we assume that all Eqs. (5)–(11) have reached equilibrium, the problem becomes tractable and we can obtain analytical expressions for x_5 and x_6. Of course, such a halt in neural activities is impossible in practice – we only claim that it could be considered a reasonable, though fictitious, statistical approximation of that person's overall experience with utilities. Under this assumption, we set the derivatives in Eqs. (5)–(8) equal to zero, and solve the resulting algebraic equations, using the software package Mathematica. Thus, the utility function derived is

$$V(\Delta P) = \begin{cases} x_5(\Delta P) - x_5(0), & \text{if } \Delta P > 0, \\ 0, & \text{if } \Delta P = 0, \\ -x_6(-\Delta P) + x_6(0), & \text{if } \Delta P < 0, \end{cases} \tag{12}$$

where

$$x_5(\Delta P) = [DE(B_1 C_2 P_a \tilde{P} - B_2(-AB_1\delta.\Delta P + C_1 P_a \tilde{P}))]/[A^4 B_1 B_2 + A^2 C_1 C_2 P_a \tilde{P} \\ + (B_2 C_1 + B_1 C_2) D P_a \tilde{P} + AB_1 B_2 (P_a + \tilde{P}) + A^3 (B_1 C_2 P_a + B_2 C_1 \tilde{P})],$$

$$x_6(\Delta P) = [DE(-B_1C_2P_a\tilde{P} + B_2(AB_1\delta.\Delta P + C_1P_a\tilde{P}))]/[A^4B_1B_2 + A^2C_1C_2P_a\tilde{P}$$
$$+ (B_2C_1 + B_1C_2)DP_a\tilde{P} + AB_1B_2(P_a + \tilde{P}) + A^3(B_1C_2P_a + B_2C_1\tilde{P})].$$

and for simplicity we introduced the substitution $\tilde{P} = P_a + \delta.\Delta P$.

We now turn to the properties of the utility function in Eq. (12). First, it must be stressed that its complicated form with eight parameters (A, δ, B_1, B_2, C_1, C_2, D, E) is due to the nature of the differential equations, and has nothing to do with curve-fitting considerations. Each parameter has its biophysical meaning related to neural activities. Second, the empirical procedure to assess those parameters was quite different from the traditional revealed preference approach. In our case, the emotions of satisfaction and disappointment, caused by expected and final prices to be paid, became proxies for utility and disutility. Finally, the two-part function in Eq. (12) was adjusted by $x_5(0)$ and $x_6(0)$ to offset nonzero neural activities. Such linear transformations of utility functions are common ever since von Neumann and Morgenstern, and reflect the mere fact that we do not know enough about the absolute zero of utility.

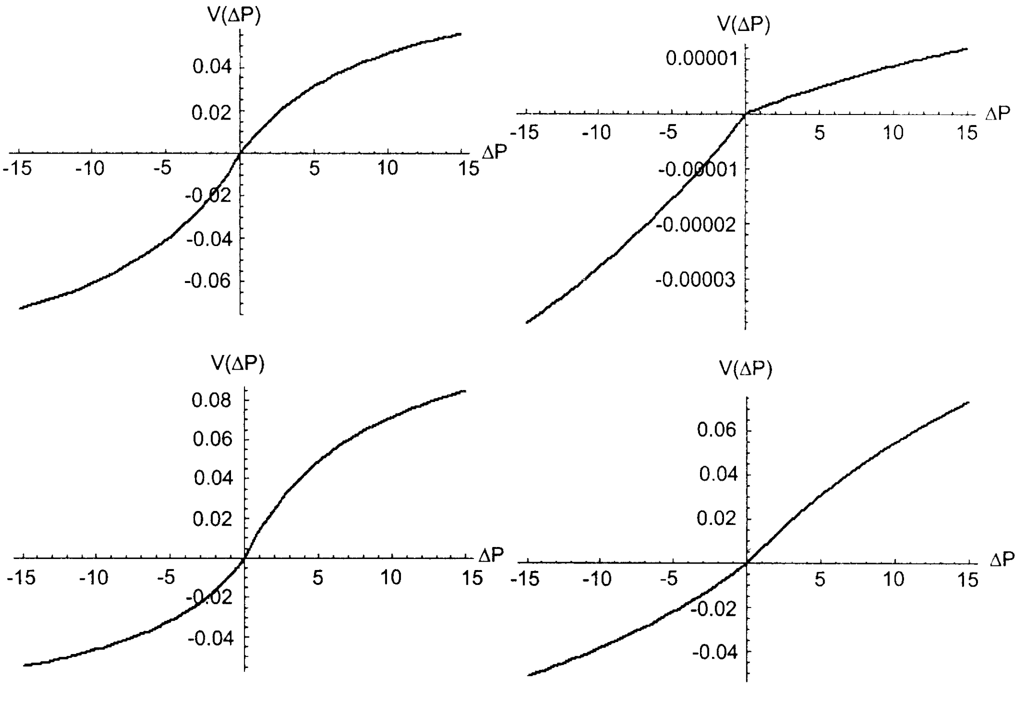

Fig. 2. Affective Balance Theory-based utility functions of four participants in the experiment

Figure 2 shows typical functions for four participants in our experiment. They all are computed with Eq. (12) and apparently are in agreement with the form established in Prospect Theory. For example, the top right plot of Fig. 2 is virtually identical with the function shown in Fig. 1 (right) and computed by Eq. (1), up to a scaling coefficient. In the second row of Fig. 2, we show utility functions steeper for gains than for losses, typical of people accustomed to surplus charges in their phone bills, and surprised to receive a discount. Again, this attitude does not contradict Prospect Theory because in its original form the latter admits shifts of reference point. For example, getting used to losses moves one's reference point to the left of the neutral position, as shown with dotted line in Fig. 1 (left plot). Now the steepest part of the function remains in the gains area, and that is exactly what we observe in the second row of Fig. 2.

Conclusion. We presented a new theoretical approach to defining utility functions. An achievement of mathematical psychology, such as Affective Balance Theory, offered a way to develop analytically a formula for personal utility, which relates easily to that of Cumulative Prospect Theory. Our formula's complexity is, we believe, offset by its elements' clear neurobiological meaning and therefore, our approach is not unprincipled. Finally, our experimental procedure was very different from the traditional revealed preference approach, yet it arrived at very similar results.

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