Referee Report for the Thesis of Prof. Nadezhda Ribarska "Fragmentability and Functional Analytic Approach to Necessary Optimality Conditions"

for obtaining the Scientific Degree "Doctor of Mathematical Sciences"

The thesis of Nadezhda Ribarska, which was sent to me electronically, is written in English on 187 pages, including 103 items of cited references, and contains two chapters. It is based on 9 publications of the author, 6 of which with coauthors, one of which is submitted, and also on an unpublished manuscript. The short description of the thesis (avtoreferat) is written in Bulgarian on 50 pages and has 105 items of cited references. Both the thesis and the avtoreferat are written according to the standards and rules provided by the respective laws and regulations.

Let me start with some biographical notes. Nadezhda Ribarska graduated from Sofia Mathematical High School, which was, and still is, considered as one of the most prestigious schools for mathematically gifted children in Bulgaria. She obtained Baccalaureate and Master degrees in Mathematics at the Sofia University, and almost immediately after that was accepted to the PhD program at the same university. After completing her PhD degree, she was appointed as Assistant Professor, climbed through the ranks over the years, and is currently Full Professor at the Department of Mathematical Analysis, School of Mathematics and Informatics, Sofia University "Kl. Ohridski". She has visited a number of research centers and has taken part with talks in a number of conferences in Bulgaria and abroad. Her research has been funded by various agencies in Bulgaria and abroad. She has been actively involved in teaching at Sofia University and has served in a number of committees.

The results presented in the thesis of Ribarska have been published in well-established journals, some of them highly ranked in the respective area of research, and has attracted the attention of many researchers in Bulgaria and abroad. I will not comment here on citations and impact factors because in my opinion these numerical data are meaningless as criteria for quality of research. I will focus on the scientific contents of the thesis.

The dissertation consists of two chapters. Chapter 1 is devoted to problems of functional analysis, with a topological flavor. The main themes are fragmentability and countable cover by sets of small local diameter. Chapter 2 deals with nonsmooth analysis aimed at developing necessary conditions for optimality, for optimal control problems described by mappings acting in infinite dimensional spaces.

Let me recall that a topological space X is said to be fragmented by a metric ρ if, for every $\varepsilon > 0$ and each nonempty subset Y of X, there is a nonempty relatively open subset of Y whose ρ-diameter is less than ε. This notion, introduced by Jayne and Rogers, is linked to the study of the Radon-Nikodym property in the theory of Banach spaces. In Section 1.2, Ribarska gives a necessary and sufficient condition for a topological space to be fragmentable. By using this condition, she obtains that the Banach space of all continuous functions on a fragmentable compact space is weak Asplund. Recall that a topological space X is of type $(\mathcal{S})$

if for each upper-semicontinuous map φ on a Baire space B that takes nonvoid compact sets in X as values, there exists a cross section $\sigma : B \to V$ which is continuous at each point of a dense G_δ subset of B. There is an internal condition for X to be of type $(\mathcal{S})$: the space X should be fragmented by a metric ρ on X, i.e. each nonempty subset of X should admit nonempty relatively open subsets of arbitrarily small ρ-diameter. Preiss, Phelps and Namioka proved that each smooth Banach space is a weak Asplund space. Ribarska obtains a stronger result: the dual, supplied by the weak topology, of a smooth Banach space is fragmentable.

In Subsection 1.2.2, Ribarska considers the three space property for σ-fragmentability. A topological space X is said to be σ-fragmentable by a metric ρ if for each $\varepsilon > 0$ one has $X = \cup\{X_j, j \in \mathbb{N}\}$, where each X_j has the property that each nonempty subset of X_j has a nonempty relatively open subset of ρ-diameter less than ε. The following theorem is proved: Let E be a Banach space, H be a closed subspace and let $F := E/H$. If F and H are σ-fragmentable, then X is also σ-fragmentable.

In Section 1.3, N. Ribarska considers the property of having a countable cover by sets of small local diameter (the material here is published in the paper [93] which appeared in the highly ranked journal Studia Mathematica). A topological space X with a metric ρ is said to have a countable cover by sets of small local ρ-diameter if for every $\varepsilon > 0$ one can cover X by a countable union of subsets X_n, $n \in \mathbb{N}$, in such a way that for every n and every $x \in X_n$ there exists a neighborhood U of x satisfying $\rho\text{-diam}(U \cap X_n) < \varepsilon$. This property of a Banach space equipped with the weak topology is known to be closely related to the existence of good renormings of X. In her thesis, Ribarska gives necessary and sufficient conditions for existence of a metric ρ in which a given topological space has the property of having a countable cover by sets of small local diameter. These results are applied to show that for Hausdorff compacta X and Y the topological space of all continuous functions on $X \times Y$ equipped with the pointwise topology has the property in question provided that the space has an equivalent p-Kadets norm. On a related subject, in 1987 Gruenhage defined a class of spaces containing Eberlein compacta and more generally Gulko compacta, and proved that members of this class have dense G_δ metrizable subspaces. In Subsection 1.3.1 Ribarska shows that the ordinal space $\omega_1 + 1$ is not a Gruenhage space.

In Section 1.4, Ribarska considers a stability property for locally uniformly rotund renorming. Let X be a compact Hausdorff space and E be a Banach space. Let $C(X, E)$ be the Banach space of all continuous maps $f : X \to E$ (if $E = I$, we write $C(X)$). For a Banach space X, its norm is said to be locally uniformly rotund (LUR) if for x and $\{x_n\}_{n=1}^{\infty}$ on the unit sphere, $\|x + x_n\| \to 2$ implies $\|x - x_n\| \to 0$. A space with this property is strictly between the class of strictly convex spaces and the class uniformly convex spaces. Historically, the notion of local uniform rotundity was important because the dual norm of X^* being LUR implies that the predual norm of X is Fréchet differentiable. As a result, a substantial part of renorming theory has been devoted to constructing LUR norms on dual spaces. A starting point of the research of Ribarska in that direction is a result of J. E. Jayne, I. Namioka and C. A. Rogers, which states that for a set Γ, if $\{X_a\}_{a\in\Gamma}$ is a family of compact Hausdorff spaces such that for each finite subset $F \subset \Gamma$ we have that $C(\prod_{a\in F} X_a)$ admits an equivalent LUR norm, then $C(\prod_{a\in\Gamma} X_a)$ also admits an equivalent LUR norm. Jayne, Namioka and Rogers asked whether that conclusion still holds under the weaker assumption that only each $C(X)$ admits an equivalent LUR norm. Ribarska gives an affirmative answer to this question, and actually prove more, by first showing that if $C(X)$ and $C(Y)$ admit a (pointwise lower semicontinuous) equivalent LUR norm, then also $C(X \times Y)$ admits a (pointwise lower semicontinuous) equivalent LUR norm. This latter result in turn follows from a more general theorem, whose proof is a real *tour de force* based on the characterization of LUR renormability in terms of countable covers by sets of small local diameter due to A. Moltó, J. Orihuela and S. L. Troyanski, a refinement due to M. Raja and the stability result by N. K. Ribarska in Studia mentioned above. The results above are quite similar to the contents of a paper by Babev and Ribarska, which however is not cited in the copy of the thesis I received, which I consider as an unintentional omission; the paper is cited in the avtoreferat.

Chapter 2 of the thesis starts by considering an optimal control problem for an infinite-dimensional system of ODEs in which the state and control variables belong to Banach spaces. The starting point of the trajectory is a given function of a parameter, while the final state belongs to a given closed and convex terminal set in the corresponding Banach space. The cost is in an integral form, and the time interval is fixed. The main assumption is that the algebraic difference between the set of endpoint values of the quasi-linearized variations of the state trajectory, corresponding to some variations of the control and parameter, and the terminal set is quasi-solid, i.e., its closed convex hull has a nonempty interior in its closed affine hull. For this problem, the authors prove a necessary optimality condition in the form of a maximum principle, which consists of a co-state equation, a maximum condition of the Pontryagin type, a transversality condition, and a stationarity condition with respect to the parameter. The proof utilizes on needle-like variations and the Ekeland variational principle. As an example, a model of population dynamics is presented.

In the following subsection, Ribarska considers an optimal control problem with a state variable from a Banach space, with a nonlinear dynamics, a Mayer-type cost functional, and a closed and convex target set. Two types of control variations are studied, the needle and the diffuse variations. For each of them, properties of the corresponding cone of terminal state variations are found. Optimality conditions are then obtained based on these properties and on a non-separation property of two closed sets, one of which is convex and the other is approximated by the Bouligand tangent cone or the tangent Clarke cone. The non-separation property is applied to the case where the convex set is the target constraint set and the other set is the set of terminal values generated by a given family of control variations. A Pontryagin type maximum principle is derived for the class of diffuse control variations. Additional results are obtained for the other class of variations, including the finite-dimensional case. The necessity of some assumptions is claimed.

In Section 2.4 the author presents results mostly from the paper [64]. The main contribution here is a new notion of tangent sets, specifically, uniform tangent and then sequentially uniform tangent sets. The key idea is that these sets remain tangent when defined through the topological closure or the convexification. This new concept leads to a new characterization of the intersection of sets by tangent cones. The ultimate goal is to get an abstract Lagrange multiplier rule. Indeed, when we have the “right” definition of a tangent, then we are well prepared to derive a Lagrange multiplier rule. In Section 2.4.2 a specific optimal control problem is considered, and, by using the new concept of tangency, a new version of the Pontryagin principle is obtained.

I attended a talk of Ribarska at the 2016 Spring Conference of the Bulgarian Mathematical Union, where she presented the new approach to obtaining Lagrange multipliers in a very clear and convincing way. I strongly believe that the new approach proposed by Ribarska and her coauthor has the potential to be applied to optimal control problems governed by partial differential equations. This is a very involved and technical area of research but also very important, not only for applications but also for answering highly nontrivial open questions in the theory of optimal control and beyond. I strongly encourage Ribarska and her co-authors to pursue this line of research and truly believe that this approach may produce important results.

It is my opinion that the thesis of N. Ribarska satisfies all requirements of the respective laws and regulations for scientific degrees in Republic of Bulgaria and specifically at Sofia University. I strongly recommend to the thesis committee to grand Prof. Nadezhda Ribarska the Scientific Degree "Doctor of Mathematical Sciences."

Signature: Dec. 15, 2017

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