# R E V I E W

by Prof. D.Sc. Johann Todorov Davidov
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on a dissertation
for awarding the scientific degree **„Doctor of Sciences in Mathematics“**

area of higher education: *4. Natural Sciences, Mathematics and Informatics*,
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scientific speciality: *Geometry*

**Author**: Assoc. Prof. Dr. Ivan Minchev Minchev
**Topic**: The geometry of quaternionic-contact manifolds
and the Yamabe problem

I am a member of scientific panel for this procedure according to order No. РД 38-113/19.02.2020 of the Rector of Sofia University ”St. Kliment Ohridski” Prof. D.Sc. Atanas Gerdzhikov. As a member of the scientific panel, I have received all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at Sofia University “St. Kliment Ohridski". .

**Personal data**

Asoc. Prof. Dr. Ivan Minchev was born on June 12, 1975. In the period 1995-2001, he was a student in the Faculty of Mathematics and Informatics of the Sofia University ”St. Kliment Ohridski”. He graduated with excellence in 2001 and got masters's degree. From 2002 till 2005 he was a Ph.D. student at the Department of Geometry of the Faculty of Mathematics and Informatics, Sofia. He successfully defended his Ph.D. thesis ”Differential geometry of metric connections with torsion” in 2006. He was an assistant at the Department of Geometry in the period March, 2005 - May, 2012, then a senior assistant. He became an Associate Professoir in 2013. He had post-doc positions at Humboldt University of Berlin (October, 2006 - November, 2008), Philips University Marburg, Germany (November, 2008 - March, 2012) and Masaryk University Brno, Czech Republic (November, 2013 - November, 2016). He defended a habilitation thesis at the Philips University Marburg in 2012, the jury being constituted by well-known experts in differential geometry. There is no direct analog in Bulgaria of a habilitation thesis in Germany, but it corresponds approximately to the Bulgarian degree ”Doctor of Sciences”.

Assoc. Prof. Dr. Ivan Minchev is currently serving as a Vice-dean of the Faculty of Mathematics and Informatics.

**Participation in scientific projects**

Ivan Minchev has taken part in five scientific projects. Four of them has been supported by Bulgarian National Science Fund and one project has been funded by National Academies of USA.

**Educational activity**

In the Faculty of Mathematics and Informatics, Ivan Minchev has conducted problem solving sessions on Linear algebra, Algebra, Geometry, Analytic geometry, Differential geometry. He has had classes on Algebraic topology in English at the Humboldt University and in German at the Philips University. Ivan Minchev has taught Analysis, Differential geometry and Elementary geometry at the Philips University (in German). He has also had lectures and exercise classes on various mathematical subjects at the Faculty of Biology, Faculty of Chemistry and Pharmacy and Faculty of Economics and Business Administration of the Sofia University

**General characterization of the dissertation**.

The dissertation of Ivan Minchev consists of 208 pages with a list of references which comprises 91 items. The material is based on 4 papers joint works with S. Ivanov and D. Vassilev.

A short description of the main problems discussed in the dissertation and the contents of its chapters is given in the introduction (3 pages).

Chapter 1 (32 pages) provides basic facts about geometry of quaternionic-contact (shortly "QC") manifolds necessary for understanding the results obtained and their proofs. Recall that a quaternionic contact structure on a $(4n+3)$-dimensional smooth manifold $M$, a notion introduced by O. Biquard, consists of a rank $4n$ subbundle $H$ of $TM$, a positive definite metric $g$ on $H$ and a rank 3 subbundle $Q$ of $End(H)$ such that, in a neighbourhood $U$ of each point of $M$, there are 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and a triple $\vartheta = (I_1, I_2, I_3)$ of sections of $Q$ with the following properties: (1) $H|U$ is the kernel of $\eta$; (2) The endomorphisms $(I_1, I_2, I_3)$ of $H$ are three almost complex structures locally generating the bundle $Q$ and satisfying the identities of the imaginary unit quaternions: $I_1^2 = I_2^2 = I_3^2 = -Id_H$, $I_1I_2 = -I_2I_1 = I_3$; (3) $d\eta_s(X,Y) = 2g(I_sX,Y)$ for $X, Y \in H|U$. Any two triples of sections of $Q$ satisfying condition (2) constitute frames of $Q$ which induce the same orientation, thus the bundle $Q$ has a canonical orientation. This bundle is endowed with the restriction of the standard metric on End(H) induced by the metric $g$ of $H$. It is a result of Biquard that if $dim\, M > 7$ there are a unique complementary bundle $V$ of $H$ and a unique connection $\nabla$ on $M$ satisfying certain conditions which we omit here for brevity and refer to the first chapter of the dissertation or the original paper by Biquard. One of the properties of $\nabla$ is that it preserves the bundle $Q$. If $dim\, M = 7$, it is not always possible to find a supplement $V$ to $H$ and a connection $\nabla$ such that $Q$ is preserved by the connection, and one has to assume additionally that $\nabla$ has this property. So, dimension 7 is a special case in the theory of quaternionic-contact manifolds. The bundles $H$ and $V$ are often called "horizontal" and "vertical", respectively. Let the form $\eta = (\eta_1, \eta_2, \eta_3)$ and the almost complex structures $\vartheta = (I_1, I_2, I_3)$ satisfy conditions (1), (2), (3), and let $(\xi_1, \xi_2, \xi_3)$ be the frame of $V$ dual to $(\eta_1|V, \eta_2|V, \eta_3|V)$. The vector fields $\xi_1, \xi_2, \xi_3$ are called Reeb vector fields and have the property that $\imath_{\xi_s} d\eta_t|H = -\imath_{|xi_t} d\eta_s|H$, $s,t = 1,2,3$ (this property is missing in dimension 7 without the additional assumption for $\nabla$). The assignment $\xi_s \to I_s$, $s = 1,2,3$, determines a bundle isomorphism $V \to Q$ which does not depend on the particular choice of $\eta$ and $\vartheta$ . The metric and orientation of $Q$ can be transferred to a metric and orientation on $V$ via this isomorphism. Thus we can define a Riemannian metric on the whole bundle $TM$, again denoted by $g$.

The second chapter of the dissertation comprises sections 4 - 9 (95 pages). At the beginning of the chapter, the author obtains certain results of technical character. Then he shows that the restriction of the Ricci tensor of the Biquard connection $\nabla$ to the horizontal space $H$ is determined roughly speaking by the torsion $T$ of $\nabla$ via an explicit formula. The restriction of the Ricci tensor to $H$ is called the QC-Ricci tensor; its trace is called QC-scalar curvature. A QC-manifold is said to be QC-Einstein if the QC-Ricci tensor is proportional to the metric $g$. The factor of proportionality is, of course, the QC-scalar curvature divided by $4n$. The explicit formula for the QC-Ricci tensor implies that a manifold is QC-Einstein exactly when, for every $\xi \in V$, the endomorphism $T_\xi : H \to H$ defined by $T_\xi(X) = T(\xi, X)$ vanishes. An important result (Theorem 5.9) about the QC-Einstein manifolds proved by the author states that the QC-scalar curvature of a QC-Einstein manifold of dimension bigger than 7 is constant (in dimension 7, this is proved in Chapter 3). Moreover, the vertical distribution $V$ is integrable. The latter property is a necessary and sufficient condition for the constancy of the QC-scalar curvature in dimension 7. Examples of QC-Einstein manifolds are the quaternionic Heisenberg group $G(\mathbb{H})$ with its natural QC-structure and every 3-Sasakian manifold. The Biquard connection on $G(\mathbb{H})$ is flat and any QC-manifold with flat Biquard connection is locally isomorphic to $G(\mathbb{H})$. The QC-scalar curvature of a 3-Sasakian manifold is positive. Conversely, if the QC-scalar curvature of a QC-Einstein manifold is a positive constant, then the QC-structure on the manifold is locally determined by a 3-Sasakian structure (hence the manifold is Einstein with positive scalar curvature). This is the contents of Theorem C, one of the main results proved in the thesis. The author studies also the problem how the QC-Ricci tensor changes under conformal transformations of the QC-structure. In particular, he finds all functions for which the corresponding conformal transformation of the standard QC-Einstein structure on the Heisenberg group $G(\mathbb{H})$ is again a QC-Einstein structure (Theorem A). By analogy with the notion of a pseudo-Einstein structure on a Cauchy-Riemann manifold introduced by John M. Lee, the author introduces the notion of a QC-pseudo-Einstein structure. This is a QC-structure for which a certain component of the trace-free QC-Ricci tensor vanishes. According to the explicit formula for the QC-Ricci tensor mentioned above, this condition is equivalent to the vanishing of a certain component of the torsion. As J. Lee has remarked, pseudo-Einstein CR-structures are closely related to CR-pluriharmonic real-valued functions, i.e. functions that are the real part of a CR-holomorphic function. It turns out that in the case of a QC-manifold not the real-valued, but the quaternionic-valued functions are of interest. An analog of holomorphic functions for smooth quaternionic-valued functions on the space $\mathbb{H}$ of quaternions has been introduced by R. Feuter in the 30's of the last century. These functions are called quaternionic regular, and the functions that are analog of anti-holomorphic functions are called anti-regular. A smooth quaternionic-valued function on $\mathbb{H}^n$ is said to be regular (anti-regular) if it is regular (anti-regular) with respect to each quaternionic variable. Combining the idea of the Feuter definition of regular functions and the usual definition of CR-holomorphic functions, the author introduces the notion of a regular (anti-regular) Cauchy-Riemann-Feuter (shortly "CRF") quaternionic-valued function on a QC-manifold. Several properties of anti-regular functions on $\mathbb{H}^n$ and anti-CRF-functions are discussed by the author. One of the reasons for this is that any conformal transformation of the QC-structure of a 3-Sasakian manifold determined by the real part of an anti-regular CRF-function is a pseudo-Einstein QC-structure. Next, the author considers vector fields on a QC-manifold whose flow consists of (local) conformal quaternionic-contact automorphisms; these a called QC-vector fields for short. He proves that on a QC-manifold with positive QC-scalar curvature, assumed constant in dimension 7, the Reeb vector fields are QC-vector fields if and only if the QC-structure is homothetic to a 3-Sasakian structure; in this case the Reeb vector fields preserves the metric as well. The last topic in Chapter 2 is the Yamabe problem for QC-manifolds. This is the problem of finding conformal changes of a QC-structure which lead to a QC-structure with constant QC-scalar curvature. The author gives the following partial solution of the Yamabe problem for the sphere $S^{4n+3}$ (Theorem B). Let $\eta = f\overline{\eta}$ be a conformal transformation of the standard QC-structure on the sphere $S^{4n+3}$ to a QC-structure with constant QC-scalar curvature. Assume in addition that the vertical distribution of the new QC- structure $\eta$ is integrable. Then the new QC-structure is Einstein and, up to a constant factor, $\eta$ is obtained from $\overline{\eta}$ by a conformal quaternionic-contact automorphism $\phi$, i.e., $\phi^*\overline{\eta} = const.f\overline{\eta}$, where $\phi^*\overline{\eta} = \mu\Psi.\overline{\eta}$ for a smooth function $\mu$ and a $so(3)$-matrix $\Psi$. Moreover, such a $\phi$ exists, if $f$ is the real part of an anti-regular CRF-function provided $n > 1$ without any assumption on the vertical distribution. It is well-known that the group of conformal quaternionic-contact automorphisms of the standard QC-structure of $S^{4n+3}$ is the group $Sp(n+1,1)$ and every such an automorphism leads to constant QC-scalar curvature. So, Theorem B tell us that these are all transformations for which the new QC-scalar curvature is constant provided the new vertical distribution is integrable. Note also that the QC-manifold $S^{4n+3}$ with one point deleted is equivalent to the Heisenberg group $G(\mathbb{H})$, and this implies that conformal factor has the form given in Theorem A.

The third chapter of the dissertations consists of sections 10-13 (20 pages). In the first part of the chapter, the author proves that the QC-scalar curvature of a 7-dimensional QC-Einstein manifold is constant (Theorem D). This result and Theorem 5.9 in the preceding chapter complete the proof of one of the main results in the thesis, namely that, in any dimension, a QC-Einstein manifold has a constant QC-scalar curvature. Moreover, the vertical distribution of such a manifold is integrable. Then the author has the nice idea to introduce a metric connection $\widetilde{\nabla}$ on the vertical bundle which is flat if and only the manifold is QC-Einstein. Thus a property in the sub-Riemannian geometry turns out to be equivalent to a property in Riemannian geometry. Using the connection $\widetilde{\nabla}$, the author gives a description of the QC-Einstein condition via equations involving the contact form $\eta = (\eta_1, \eta_2, \eta_3)$, its differential $d\eta$ and a certain 2-form determined by the QC-structure. As a consequence of Theorems C and D and a result of S. Ivanov - D. Vassilev, the author observes that every QC-Einstein manifold of dimension 7 whose QC-scalar curvature is nowhere vanishing is locally QC-homethetic to a 3-Sasakian manifold in a wide sense. It should also be mentioned here Proposition 12.3 which states that a QC-Einstein manifold with zero QC-scalar curvature is a bundle over a locally hyper-Kähler manifold under certain condition on the vertical bundle $V$. This holds, for example, if the leaves of the foliation generated by the distribution $V$ (recall that $V$

is integrable) are compact.

The main goal of the fourth chapter (Sections 15-16, 26 pages) is the solution of the Yamabe problem for the standard QC-structure on the sphere $S^7$. The result, Theorem E, is similar to that of Theorem B, but unlike Theorem B, no additional assumption is imposed. The proof is technical and relies on a divergence formula (Theorem 15.4) whose integration implies that any conformal transformation of the QC-structure of $S^7$ with constant QC-scalar curvature is QC-Einstein. The author also computes the Yamabe constant for $S^7$, the minimum of the Yamabe functional. As an application, he finds (Theorem F) the best constant in the Folland-Stein inequality on the quaternionic Heisenberg group $G(\mathbb{H})$ as well as the functions for which we have equality in this inequality. The crucial observation is that the reciprocal value of the best constant coincides with the Yamabe constant for the QC-manifold $G(\mathbb{H})$, the latter being equivalent to $S^7 \setminus \{point\}$.

In the last chapter, Chapter 5 (Sections 17 and 18, 5 pages), the author extends Theorems E and F to any dimension. The corresponding result is stated as Theorem G. Its proof, in contrast to that of Theorems E and F, is purely analytic.

**Publications on the dissertation**.

The dissertation of Ivan Minchev is based on four papers published in prestigious mathematical journals with high impact factor. It makes an impression that pdf copies of only two of them are posted at the CD provided to the members of the scientific panel. And only these two papers are included in the Reference for meeting the minimum national requirements under of ADASRB. The reason, as the author has explained me, is that the additional papers are not formally involved in the procedure, and the results of them are included in the text of the thesis in order only to obtain completeness of the dissertation topic. I have thoroughly checked all documents concerning ADASRB and can say that this is definitely not a violation of the law.

The papers included in the dissertation have not been used for awarding the educational and scientific degree "Doctor". They have been quoted 13 times by other authors.

**Author's personal involvement**.

I have known Ivan Minchev and his mathematical work for more than 15 years. I have been a member of the scientific panel when he had applied for the academic position of associate professor. I have no doubt that his contribution in the joint papers is of equal worth with the other co-authors. From formal point of view, this has been confirmed by Ivan Minchev in a written statement sent to me by e-mail.

I would like also to mention that I have not discovered any plagiarism.

**Author's summary**.

The author's summary is prepared according to the requirements and correctly presents the content of the dissertation.

**CONCLUSION**

The documents and materials presented by Assoc. Prof. Dr. Ivan Minchev meet the requirements of the Act on the Development of the Academic Staff in the Republic of Bulgaria , the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at Sofia University “St. Kliment Ohridski".

The results in Ivan Mintchev’s dissertation are important contributions to a difficult field of the contemporary differential geometry.

Based on the comments above, I give a positive assessment of the dissertation of Assoc. Prof. Dr. Ivan Minchev Minchev. I am proposing to the scientific panel to vote for awarding him the scientific degree „Doctor of Sciences in Mathematics“, area of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics, scientific speciality ”Geometry”.

02.05.2020 Reviewer:

(Prof. D.Sc. Johann Davidov)