R E P O R T

on a Thesis for awarding the degree "Doctor of Sciences"

Scientific field: Natural sciences, mathematics and informatics

Professional field: 4.5. Mathematics (Mathematical Logic)

Title: Computable structure theory: jump of structure, coding and decoding

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Overview

The presented thesis is devoted to the investigation of some problems in the computable structure theory. It studies the relations between definability properties of the mathematical structures and their algorithmic behaviour. The main goal of the author is to prove results and develop methods that shall make it possible to estimate the complexity of mathematical structures and make a comparison between them. The theory is developed for objects defined on a countable support.

This thesis is based on the author's research in the last 10-15 years. The main problems tackled in this work are the following:

(1) To investigate various structure properties, in particular, jump inversion theorems, and the possibility of a jump spectrum to be a degree spectrum of a structure.

(2) To determine model-theoretic conditions under which a structure admits jump inversion.

(3) To investigate effective imbeddings of a class of structures in another class of structures. To tackle the problem of the efective decoding for special classes of structures.

(4) To investigate some model-theoretic properties of cohesive powers of linear orders.

(5) To investigate various properties of natural substructures of the structure of the enumeration degrees $\mathcal{D}_e$.

State of current research

My general impression is that the author is well acquainted with the state of the art and the most recent results in the computable structure theory. A big part of the considered problems are considered in the field as important theoretically, as well as for the applications. The author demonstrates deep knowledge of her field of research and capacity to apply her knowledge to the solution of important problems.

Methods

The author uses the language of set theory, as well as combinatorial methods and other methods that are typical for the computable structure theory, such as the method of forcing. The thesis has common points with some areas of abstract algebra.

Brief description of the thesis

The presented thesis amounts 270 pages of text and consists of an introduction, six chapters, and a list of references including 162 items. In what follows, I shall give a short description of the topics covered in this dissertation.

The first chapter is introductory. It contains a brief summary of all the results obtained in the thesis. The author provides some preliminary facts about the evolution of the notions central for this thesis, as well as some ideas that motivate the research that was carried out and that is described in the next few chapters.

Chapter 2 does not contain original results. It deals with the rigorous definition of several notions and methods that are central for the consequent investigations in this thesis: Turing reducibility, genericity and forcing, enumeration reducibilty, degree spectra, definability of a structure.

In **Chapter 3** the author investigates the notion of a jump of a structure introduced by A. Soskova and I. Soskov. It is a generalization of the notion of relative computability of Moschovakis. In the last few years it was developed further by various researchers among which is the author of this thesis.

Some of the more important results in this chapter are the following. The author proves Theorem 3.2.1, according to which every jump spectrum is a degree spectrum, or, in other words, for each structure $\mathcal{A}$ there exists a structure $\mathcal{B}$, for which $DS_1(\mathcal{A}) = DS(\mathcal{B})$. Further on, a jump inversion theorem for algebraic structures is established (Theorem 3.3.9). If $\mathcal{A}$ and $\mathcal{B}$ are structures, for which $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$, then there exists a structure $\mathcal{C}$ with $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_1(\mathcal{C}) = DS(\mathcal{A})$.

The obtained results are applied further to the n-th jump spectra of the structure $\mathcal{A}$ defined as $DS_n(\mathcal{A}) = \{\boldsymbol{d}^n : \boldsymbol{d} \in DS(\mathcal{A})\}$, where $1 \leq n < \omega$. It is proved that if $DS(\mathcal{A}) \subseteq DS_n(\mathcal{B})$, then there exists a structure $\mathcal{C}$, for which $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_n(\mathcal{C}) = DS(\mathcal{A})$ (Theorem 3.4.2). Another important result in this section is Theorem 3.4.8, according to which for every natural number $n \in \mathbb{N}$ and every Turing degree $\boldsymbol{d} \geq \boldsymbol{0}^n$ there is a structure $\mathcal{C}$, with $DS_n(\mathcal{C}) = \{\boldsymbol{x} \mid \boldsymbol{x} > \boldsymbol{d}\}$.

In **Chapter 4** the author considers the notion of strong jump inversion. A structure admits strong jump inversion if it satisfies the following condition: for every oracle X, if X' computes $D(\mathcal{C})'$ for some isomorphic copy $\mathcal{C}$ of $\mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$.

In this chapter the author finds conditions that settle the question of when a structure $\mathcal{A}$ admits a strong jump inversion (Theorem 4.2.5). It is proved that many of the known classes of structures satisfy the conditions of Theorem 4.2.5. These include some special classes of linear orders (Theorems 4.3.2 and 4.3.3); boolean algebras without 1-atoms (Proposition 4.3.6); models of an $\aleph_0$-categorical theory such that the set of all Σ_2-sentences is computably enumarable (Theorem 4.3.13); differentially closed fields of characteristic 0 (Theorem 4.33). It is shown that the saturated model of the theory of differentially closed fields of 0 characteristic 0 has a strongly constructive copy (Corollary 4.3.36).

Chapter 5 consders a different approach to the algorithmic complexity of algebraic structures. Let $\mathcal{K}$ and $\mathcal{K}'$ be two classes of structures, and let Θ be a Turing computable embedding of $\mathcal{K}$ in $\mathcal{K}'$, for which

$$\mathcal{A}_1 \cong \mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{K} \Longleftrightarrow \Theta(\mathcal{A}_1) \cong \Theta(\mathcal{A}_2).$$

Later on, other authors initiate the reasearch on stronger versions of Turing computable embeddings.

The more significant results in this chapter amount to the following:

There exists a graph G, such that for any linear order L, G is not Medvedev reducible to the jump L' (Proposition 5.2.4). On the other hand for every graph H there exists a linear order L, for which H is Medvedev reducible to the second jump $L^{(2)}$ (Proposition 5.2.6).

In a paper from the late 80's Freedman and Stanley constructed a Turing computable embedding Φ from the class of all graphs to the class of all linear orders. It is proved in Theorem 5.2.7 that there exist no $L_{\omega_1,\omega}$ formulas that interpret G inside $\Phi(G)$.

Section 5.3. deals with with the subject of the interpretation of a field in the Heisenberg group. They extend a result by Mal'tsev, according to which for every field F there exists a copy of his defined within the Heisenberg group $H(F)$ via existential formulas with parameters. It is proved in Theorem 5.3.13 that there exist existential formulas without parameters that define an effective interpretation of F in $H(F)$. Furthermore, the author proves sufficient conditions for eliminating parameters from interpretations (Theorem 5.4.9).

Towards the end of the chapter, the author proves (Proposition 5.5.1) that every algebraically closed field C with characteristic 0 is interpretable in $SL_2(C)$ via existential formulas with two parameters.

Chapter 6 considers effective versions of some model theoretic constructions. A central problem here is to clarify the question of when isomorphic computable linear orders induce isomprphic cohesive degrees. The more significant results here are contained in Theorems 6.4.4, 6.6.2, 6.6.4, 6.6.5, as well as in Corollaries 6.4.6 and 6.6.2.

Chapter 7 contains results on the enumeration degrees of susbsets of $\mathbb{N}$. There exists a natural imbedding of the structure of Turing degrees $\mathcal{D}_T$ into the structure of enumeration degrees $\mathcal{D}_e$. A central problem here is the investigation of the structure properties of other natural substructures of $\mathcal{D}_e$. Important results are obtained in Theorems 7.5.4 and 7.6.1, where it is demonstrated that the chain of inclusions "graph cototal – cototal – weakly cototal" sets is strict. Anothr notable results in this chapter are Proposition 7.3.1, Theorem 7.4.3 and Theorem 7.4.17.

Main results

The main contributions of this DSc Thesis are the following:

(1) It is proved that every jump spectrum is a degree spectrum.

(2) Results concerning the n-th jump spectra are proved.

(3) Jump inversion theorems are proved.

(4) Conditions are proved for a structure to admit a strict jump inversion.

(5) Effective versions of some model-theoretic constructions are proved.

(6) Interpretation results are proved that interpret fields of characteristic 0 within the Heisenberg group.

(7) It is proved that the chain of implications "graph cototal – cototal – weakly cototal" for sets is strict.

(8) Structure properties of some substructures of the structure of enumeration degrees are investigated.

Remarks and comments

I have the following remarks, questions and comments related to this thesis:

(1) The thesis is extremely well-written. The considerable efforts to produce an extensive and in the same time readable text of this size are clearly seen. The very few typos here and there cannot spoil the overall positive impression.

(2) I could not figure out what is the difference between a proposition and a theorem? Both are used in the text.

(3) A part of the publications related to this thesis is witten in a co-authorship. It would be good to make a distinction at the appropriate places which result belongs to whom. On the other hand, I am aware that in most cases this is hardly possible: results are oftentimes proved in parallel and the final publication assumes the most appropriate one.

Publications related to the thesis

The results in this thesis are published in eight papers. Four of the papers are in journals with an impact factor. These are:

- Journal of Logic and Computation (2008,2019) – IF 0.789; 0.86

- Journal of Symbolic Logic (2019) – IF 0.642

- Transactions of the American Mathematical Society (2019) – IF 1.422

The remaining four papers are published in volumes with the proceedings of scientific conferences: two of them appear in the Springer Verlag sequence Lecture Notes in Computer Science and the other two are in the proceedings of the 6-th and the 7-th Panhellenic Logic Symposium, respectively.

Results from this thesis are also contained in two further papers that are submitted to mathematical journals and await refereeing.

The list of publications meets the minimal national criteria required in paragraph 7.

Authorship of the obtained results

The authorship of the eight papers on which this thesis is based can be described as follows: in one of them the candidate is the sole author; three of them are with one co-author; one is with two co-authors; one is with five and two are with six co-authors. Towards the end of the author's summary, it is pointed out that in all co-authored papers the contribution of the candidate is the same as that of the remaining collaborators. I accept this explanation. Moreover, in many cases the contribution of the different collaborators is very hard to distiguish exactly.

Citations

The candidate has attached a list of 74 citations of the papers used in this thesis. It is beyond any doubt that the results of Alexandra Soskova are well-known and highly valued in her professional community.

Authors summary

The author's summary is made according to the existing regulations and reflects properly the main results and contributions of this thesis.

Conclusion

This thesis is focused on problems from the computable structure theory that are of great importance for the theory and as well as for the applications. They shed light on the problem of how the structrue properties of the objects determine their algorithmic behaviour. In this thesis the author proves results and develops methods that allow to compare properties related to the computability in different structures. This work does not only answer open problems of principal importance, but also motivates new directions for an ongoing research.

I am deeply convinced that the presented thesis "Computable structure theory: jump of structure, coding and decoding" by Alexandra Andreeva Soskova contains results that are an original contribution to the computable structure theory. The candidate demonstrates deep knowledge of the theory and capacity to develop it in new and important ways. With this, she meets the legal national requirements prescribed by the law, as well as the specific requirements of the University of Sofia and the Faculty of Mathematics and Informatics for the professional field 4.5 Mathematics. I assess **positively** the presented DSc Thesis and recommend to this panel to award **Alexandra Andreeva Soskova** the scientific degree "Doctor of Sciences" in the scientific field 4. Natural sciences, Mathematics and Informatics Professional field 4.5 "Mathematics" (Mathematical Logic).

Sofia, 17.03.2021 Member of the Scientific Panel:

(Prof. Ivan Landjev)