REVIEW

by Assoc. Prof. Temenoujka Peneva, Ph.D. – University of Plovdiv Paisii Hilendarski

of the thesis ***On some Diophantine equations and inequalities***,
submitted by *Asst. Zhivko Khristov Petrov*
for awarding the educational and scientific degree *Doctor*,
area of higher education 4. Natural Sciences, Mathematics, and Informatics,
professional field 4.5. Mathematics,
doctoral program in Mathematical Analysis

1. General description of the procedure and the materials presented

By Order No. RD 38-255/30.05.2019 of the Rector of Sofia University St. Kliment Ohridski (SU), according to the Decision of the Faculty Council of the Faculty of Mathematics and Informatics (Protocol No. 6/21.05.2019), I have been nominated as a member of the scientific jury to evaluate a doctoral thesis titled *On some Diophantine equations and inequalities*, for awarding the educational and scientific degree *Doctor* in the area of higher education 4. Natural Sciences, Mathematics, and Informatics; professional field 4.5. Mathematics; doctoral program in Mathematical Analysis. The author of the dissertation is *Asst. Zhivko Khristov Petrov* – a doctoral student in part-time training at the Department of Mathematical Analysis of the Faculty of Mathematics and Informatics (FMI) at SU, with advisor Prof. D.Sc. Doychin Tolev.

Concerning the procedure, I have received: *on paper* – a dissertation and an author's summary; *in electronic form* – the above-listed plus 3 publications on the subject of the dissertation and the curriculum vitae of the doctoral candidate.

2. Brief biographical data of the doctoral candidate

Zhivko Petrov completed his secondary education in 2008 at the School of Mathematics and Natural Sciences Geo Milev in Stara Zagora, specializing in Mathematics and Informatics with English. In 2012 he received a Bachelor's degree in Computer Science at FMI-SU, and in 2014 he completed a Master's degree in Dynamic Systems and Number Theory at the same faculty. On 2 February 2016 he enrolled in the doctoral program (part-time training) at the Department of Mathematical Analysis, FMI-SU.

As a fourth-year student, Zhivko Petrov became a part-time assistant at the Department of Mathematical Analysis, FMI-SU, and in 2014 he was appointed as an assistant in the same department.

3. Topicality of the subject

Diophantine equations and inequalities are probably among the most popular examples of mathematical problems that are very easy to state but difficult to solve, and solutions often resist over centuries.

Such problems are, for example, the well-known Goldbach problems, formulated in 1742. The ternary problem, which states that every odd number greater than or equal to 7 can be written as the sum of three prime numbers, was completely solved only in 2014. The binary problem, which asserts that every even number greater than or equal to 4 is the sum of two prime numbers, remains unsettled. Such a fate has another problem known since antiquity – the twin primes problem. It suggests that there are infinitely many primes p such that $p+2$ is also a prime. Nonetheless, there are some results, which in a sense can be seen as approximations of the unsolved hypotheses mentioned above. For example, by using sieve methods, in 1973 Chen proved that every sufficiently large even integer could be written as the sum of one prime and one almost prime number of order 2; similarly, there exist infinitely many primes p, such that $p+2$ is an almost prime of order 2. As usual, we call an almost prime number of order r any number that has at most r prime factors, counted according to multiplicity; the set of all almost primes of order r is denoted by $\mathcal{P}_r$.

The next two problems also have a long history. In 1770, Lagrange proved that every natural number can be represented as the sum of four squares of integers, a problem first formulated by Claude Bachet in 1621, but known to Diophantus. Also in 1770, Waring considered a generalization of the four-square theorem. He proposed that for every integer $k \geq 2$ there exists a natural number $n = n(k)$, such that every sufficiently large integer N can be written as

$$N = x_1^k + \cdots + x_n^k, \tag{1}$$

where $x_1, \ldots, x_n$ are positive integers. A complete proof of this assertion was found only in 1909 by Hilbert. The problem of finding solutions to the equation (1) in prime variables is known as the Goldbach-Waring problem and was first examined by Vinogradov and Hua in the 40s of the 20th century.

In recent years, sieve methods have given a new impact on studying classical additive problems which cannot yet be solved in primes. For example, the hypothesis that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be written as the sum of four squares of primes is still open. However, in 2017 Tsang and Zhao showed that any sufficiently large integer N, satisfying the above natural congruence condition, can be represented as $N = x_1^2 + x_2^2 + x_3^2 + x_4^2$, where all $x_i \in \mathcal{P}_4$.

In other cases, additional restrictions are being imposed on some of the variables, and as a consequence new hybrid problems emerge. I will only mention a few important results. In 2009, Zhou proved that for any integer $k \geq 3$ there exist infinitely many arithmetic progressions $p_1, \ldots, p_k$, such that $p_1+2, \ldots, p_k+2 \in \mathcal{P}_2$. Also in 2009, Cai and Lu proved that if $N \equiv 5 \pmod{24}$ is a sufficiently large integer, then there exist infinitely many solutions to the equation $p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = N$ in primes p_1, p_2, p_3, p_4, p_5, such that $p_1 + 2$, $p_2 + 2 \in \mathcal{P}_2$, $p_3 + 2$, $p_4 + 2 \in \mathcal{P}_4$, $p_5 + 2 \in \mathcal{P}_5$. In 2015, Matomaki and Shao established that for every sufficiently large integer $N \equiv 3 \pmod 6$ there exist infinitely many triplets of primes p_1, p_2, p_3, such that $p_1+p_2+p_3 = N$ and p_1+2, p_2+2, $p_3+2 \in \mathcal{P}_2$.

As can be seen from the given examples and from the discussion in the next section, the subject of the thesis, the problems considered in it, and the methods chosen for their solution are all very current and many mathematicians work intensively on them.

4. Content analysis of the thesis and contributions

Zhivko Petrov's thesis is presented on 93 pages and consists of five chapters and a bibliography.

The bibliography includes 88 titles (64 in English, 12 in Russian, 4 in French, 2 in German, 1 in Italian, and 5 in Bulgarian), all of them appropriately cited in the text of the dissertation. Out of these 88 titles, 77 are publications in scientific journals or conference proceedings, of which 30 have been published after the year 2000. All this indicates that the doctoral student is well acquainted with the current research in the field and has entered deep into the subject.

This impression is reinforced by **Chapter 1**. It is introductory and as usual the first paragraph consists of notation and definitions for the most commonly used quantities. The second paragraph, however, contains a very detailed and well organised historical overview of the emergence and study of numerous significant number-theoretic problems. It explains the motivation of the candidate to study the problems in the dissertation. At the end of the chapter, the main new results of the thesis are stated, and some of the applied methods are discussed.

Chapter 2 provides some auxiliary results from the fields of mathematical analysis and number theory, which are used in the proof of the main theorems. A large number of the statements are well-known and therefore, their proof is omitted. Instead, the reader is referred to a bibliographic source that contains the proof. The new statements are provided with proof.

The main contributions of the dissertation are included in the next three chapters, each of them based on one of the publications. The exposition in these chapters has a clear and logical structure, which helps the reader to follow and understand the argument: 1) formulation of the main theorem; 2) beginning of the proof, in which more specific notation and quantities are introduced, the problem is interpreted analytically, and some auxiliary problems are formulated; 4) proof of the auxiliary problems; 5) end of the proof. Chapter 5 contains an additional paragraph with lemmas necessary for the proof of the main theorems.

Chapter 3 is dedicated to a Diophantine equation that can be seen as a modern interpretation of the equation (1) from the Waring problem in case $n = 2$ and the degree $k > 1$ is not an integer.

In 1974, Deshouillers proved that if $1 < c < 4/3$ is given, and N is a sufficiently large integer, then the equation

$$[m_1^c] + [m_2^c] = N$$

has solutions in natural numbers m_1 and m_2 (as usual, $[t]$ denotes the integer part of t). Afterwards, the interval for c is extended several times, and the best result to date, $1 < c < 3/2$, was obtained by Konyagin in 2003.

It is only natural to ask whether a similar result will be true if m_1 and m_2 are restricted to the set of prime numbers. So far, this problem is beyond the scope of modern methods, and perhaps its difficulty is commensurate with that of Goldbach's binary hypothesis. However, in 2017, Kumchev proved that if $1 < c < 17/16$, then for every sufficiently large integer N, the equation

$$[p^c] + [m^c] = N \tag{2}$$

has solutions in prime p and natural m.

In the discussed chapter of the thesis, the doctoral student succeeds in imposing an additional multiplicative condition on the nubmer m. He proves (Theorem 3.1.1) that if $1 < c < 29/28$, then for every sufficiently large N, the equation (2) has solutions in prime p and almost prime m with no more than $\left[\frac{52}{29-28c}\right] + 1$ prime factors. Note that $\left[\frac{52}{29-28c}\right] + 1$ equals 53 if c is close to 1, and it is large if c is close to 29/28.

The proof begins with the use of the linear sieve (Lemma 2.4.1) that leads to the estimate of an exponential sum, which contains $[p^c]$ in its exponent. Afterwards, the doctoral candidate applies Vinogradov's lemma (Lemma 2.5.3) with suitable parameters and constructs the smooth functions $g_z(t)$ in (3.33). Their original choice allows for the aforementioned exponential sum to be written as a linear combination of similar sums, which have a smooth function in their exponents. For evaluation of those new sums, Vaughan's identity is used, resulting in exponential sums of the first and second types. The technique of estimating these sums may be considered "standard", but it poses many technical challenges and requires careful treatment of the quantities in order to successfully apply the Van-der-Corpus theorem (Lemma 2.2.5) with $k = 2$ or $k = 3$. It is precisely the estimates of the sums of the second type (in particular the amount $\Omega_{4,4}$) that determine the upper limit of 29/28 of the interval for c, in which the theorem is valid, as well as the number of factors of m in (2).

The results are published in [70].

In **Chapter 4**, another Diophantine equation is considered, which is analogous to the problem of Waring-Goldbach in the case when $n = 3$ and $k > 1$ is a non-integer. It is proved (Theorem 4.1.1) that if $1 < c < 17/16$, then for every sufficiently large integer N, the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = N \tag{3}$$

has solutions in primes p_1, p_2, p_3, such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ has at most $\left[\frac{95}{17-16c}\right]$ prime factors, counted with multiplicity. Obviously $\left[\frac{95}{17-16c}\right]$ equals 95 if c is close to 1, and it is large if c is close to 17/16.

The existence of solutions of the equation (3) in prime numbers was first established by Tolev and Laporta in 1995, provided that $1 < c < 17/16$. In 2017, Tolev considered the analogous problem for the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E} \tag{4}$$

and proved that if $1 < c < 15/14$ and $E > 0$ is an arbitrarily large constant, then for every sufficiently large real number N, the inequality (4) has solutions in primes p_1, p_2, p_3, such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$ is an almost prime with at most $\left[\frac{369}{180-168c}\right]$ prime factors, counted with multiplicity.

In the proof of Theorem 4.1.1, the doctoral student combines ideas from the proof of the two results mentioned above, successfully overcoming the difficulties arising from the hybrid nature of the problem. In the beginning, the vector sieve is applied, which allows for the sifting of vectors, whose all components are almost prime numbers. Then the Davenport-Heilbronn method, which is a simplified version of the Hardy-Littlewood circle method, is applied. It deals with bounds of integrals over only one major arc and two minor arcs. Two results from the work of Tolev (2017) play an essential role in the proof. The first is formulated as Lemma 4.2.1 and provides an asymptomatic formula for the sum $\overline{L}(\alpha)$, used to determine the main term in the integral over the major arc. The second is Lemma 4.2.4, which is crucial for the evaluation of the exponential sums in the integrals over the minor arcs. Of particular interest are the estimates of integrals in Lemma 4.2.2 and Lemma 4.2.3.

The results are published in [69].

Chapter 5 is devoted to another analog of the Waring-Goldbach problem: the existence of solutions in prime numbers $p_1, p_2, \ldots, p_n$ of the Diophantine inequality

$$|p_1^c + p_2^c + \cdots + p_n^c - N| < \varepsilon, \tag{5}$$

where $c > 1$ is a non-integer, N is a sufficiently large real number, and $\varepsilon > 0$ is a fixed arbitrarily small number. This problem was first considered in 1952 by Piatetski-Shapiro, who found an upper estimate for the quantity $H(c)$ defined as the least number n, such that the inequality (5) is soluble for all sufficiently large N. Approximately at the same time Piatetski-Shapiro considered the question whether the sequence

$$\mathcal{N}_\gamma = \{n \in \mathbb{N} : n = [m^{1/\gamma}] \text{ for some } m \in \mathbb{N}\}$$

contains infinitely many primes. He proved that if $\pi_\gamma(N)$ denotes the number of primes $p \le N$ that belong to $\mathcal{N}_\gamma$, one has the asymptotic formula $\pi_\gamma(N) \sim N^\gamma/\log N$, provided that $11/12 < \gamma < 1$. Over the years, the interval of γ for which $\pi_\gamma(N) \to \infty$ has been extended several times. The best result in this direction, $205/243 < \gamma < 1$, was obtained in 2001 by Rivat and Wu. Besides, many researchers started considering the solubility of classical additive problems in prime numbers of Piatetski-Shapiro (primes of the form $\mathcal{N}_\gamma$). One particularly interesting result belongs to Balog and Friedlander. In 1992, they proved that if $20/21 < \gamma < 1$, then for all sufficiently large integers N the equation $p_1 + p_2 + p_3 = N$ has solutions in primes $p_1,\, p_2,\, p_3 \in \mathcal{N}_\gamma$.

In the chapter under discussion, Theorem 5.1.1 plays a central role. It states that if $c > 5$ is a non-integer, $1 - \rho < \gamma < 1$ with $\rho = (8c^2 + 12c + 12)^{-1}$, $n \ge 4c \log c + \frac{4}{3}c + 10$, and N is a sufficiently large real number, then the inequality

$$|p_1^c + p_2^c + \cdots + p_n^c - N| < (\log N)^{-1} \tag{6}$$

has solutions in primes $p_1,\, p_2, \ldots, p_n \in \mathcal{N}_\gamma$.

The next three theorems are concerned with variants of the inequality (6) with $n =$ 3, 4, and 2, respectively, provided that γ and c are close to 1.

Theorem 5.1.2 asserts that if $\gamma < 1 < c$ and $15(c-1) + 28(1-\gamma) < 1$, then for sufficiently large real numbers N the inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-1}$$

is soluble in primes $p_1,\, p_2,\, p_3 \in \mathcal{N}_\gamma$.

Theorem 5.1.3 states that if $\gamma < 1 < c$ and $8(c-1) + 21(1-\gamma) < 1$, then for all sufficiently large real N the inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < (\log N)^{-1}$$

has solutions in primes $p_1,\, p_2,\, p_3,\, p_4 \in \mathcal{N}_\gamma$.

Finally, for large real Z, denote by $\mathcal{E}(Z)$ the set of $N \in (Z/2, Z]$, such that the inequality

$$|p_1^c + p_2^c - N| < (\log N)^{-1}$$

is not soluble in primes $p_1,\, p_2 \in \mathcal{N}_\gamma$. Then Theorem 5.1.4 states that if $\gamma < 1 < c$ and $8(c-1) + 21(1-\gamma) < 1$, then the Lebesgue measure of $\mathcal{E}(Z)$ is $O(Z \exp(-(\log Z)^{1/4}))$.

Before commenting on the proof of the theorems, I must emphasize that it depends to a large extent on the estimates of exponential sums in the second paragraph of the chapter. This paragraph can be seen as a quick introduction to the methods for evaluation of exponential sums. Various tools have been used – Vaughan's identity, Heath-Brown's identity, Van-der-Corpus's method. (In [59], the candidate and his co-author remark that it is possible to improve the obtained results by a more careful treatment of the exponential sums, albeit not large enough to be worth the effort.)

The proof of Theorem 5.1.1 is given a central role in this chapter. It begins with the application of the Davenport-Heilbronn form of the circle method, counting the solutions of the inequality (6) in Piatetski-Shapiro primes in suitable diminishing ranges. A result of Brüdern and Kumchev (2001) in Fourier analysis is applied to construct a kernel $K(x)$, and to reduce the problem to finding a lower bound for the integral for $R(N)$ in (5.36). Then the integral is written as the sum of 5 integrals over one major arc, two minor arcs and two infinite intervals (trivial region), correspondingly, and their evaluation is inevitably related to the lemmas in the second paragraph. It is worth pointing out, that the minor arc terms are estimated by considering the number of natural solutions of other Diophantine inequalities with fewer variables.

The proofs of Theorems 5.1.2 and 5.1.3 are similar to that of Theorem 5.1.1. Theorem 5.1.4 is proved by following the structure of Theorem 5.1.3 (where $n = 4$), applying Plancherel's theorem (Lemma 2.1.3) and replacing some pointwise bounds with means of squares of exponential sums.

The results are published in [59].

5. Publications on the subject of the thesis

The publications on the subject of the doctoral thesis are three: one of them is individual and was published in the Annual of Sofia University in 2017. The other two are joint works and are published in highly-regarded international journals. The paper with co-author Doychin Tolev was published in 2017 in Proceedings of the Steklov Institute of Mathematics (IF 0.623/Q3, SJR 0.43/Q2), while the joint work with Angel Kumchev from Towson University, USA, appeared in 2018 in Monatshefte für Mathematik (IF 0.807/Q2, SJR 0.701/Q2). I accept that the contributions of the authors to the joint papers are equivalent.

Since all publications are in peer-reviewed journals, they satisfy the requirements of the *the Regulations for the terms and conditions for awarding academic degrees and occupying academic positions at FMI-SU* for at least two reviewed publications, at least one from which to appear in a scientific journal.

In my opinion, the presented publications contain original results of significant scientific value and are at a very high scientific level. Their importance is also confirmed by the fact that the article *On an equation involving fractional powers with one prime and one almost prime variables*, Proc. Steklov Inst. Math., 298 (2017), Suppl. 1, 38–56, joint work with Doichin Tolev, has already been cited in a publication in the respected journal Acta Arithmetica (IF 0.476/Q4, SJR 0.68/Q2). No citations of the other two articles have been noticed, but given that they have been published very recently, it can be expected that in the future they will also attract the attention of mathematicians working in this field.

Between 2016 and 2018, Zhivko Petrov has given talks at seven scientific forums, one of them being the prestigious European number theory conference Journées Arithmétiques (France).

6. Critical remarks and recommendations

I have no critical remarks and comments. The inevitable inaccuracies of editorial nature are few and easily overcome.

7. Author's summary

The author's summary, along with the bibliography, consists of 20 pages and fully reflects the content and contributions of the dissertation. It includes a historical overview of the problems; a brief description of the structure of the dissertation with statements of the new results and some elements of their proof; a list of publications on the subject of the thesis; a list of citations; a list of the scientific forums on which the results have been reported; a bibliography.

Conclusion

After getting acquainted in detail with the dissertation of Zhivko Petrov, I am convinced of his profound theoretical knowledge, his ability to formulate new problems and do independent research. My assessment of the doctoral thesis, the author's summary, the scientific publications, and the contributions of Zhivko Petrov is positive. The thesis complies with the requirements, conditions, and criteria established under the *Law for Academic Staff Development in the Republic of Bulgaria* and *the Regulations for its implementation*, as well as *the Regulations for the terms and conditions for awarding academic degrees and occupying academic positions at SU and FMI-SU.*

The above considerations give me reason to propose to the Honorable Scientific Jury to award to Zhivko Khristov Petrov the educational and scientific degree *Doctor* in the area of higher education 4. Natural Sciences, Mathematics, and Informatics; professional field 4.5. Mathematics; doctoral program in Mathematical Analysis.

26/06/2019 Reviewer: ....

(Assoc. Prof. T. Peneva)