OPINION
on the Thesis Submitted for Awarding the Educational and Scientific Degree “Doctor” (Ph.D.) in the Professional Direction 4.5. Mathematics, Doctoral Degree Program “Mathematical Analysis”

Author of the thesis: Zhivko Hristov Petrov, Assistant Professor in the Chair of Mathematical Analysis at the Faculty of Mathematics and Informatics of the Sofia University “St. Kliment Ohridski”.
Title of the thesis: „On Some Diophantine Equations and Inequalities”.
Member of the Scientific Jury: Doctor of Sciences Vesselin Stoyanov Drensky, Professor at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences, Full Member of the Bulgarian Academy of Sciences.

The thesis is in the field of analytic number theory which is a classical part of number theory but uses techniques from mathematical analysis. The thesis has 93 “standard” TEX-pages and consists of an introduction, 5 chapters, and a list of 88 titles of the used literature.

Actuality of the problems studied in the thesis. We shall discuss some of the mile stones in analytic number theory. It seems that the theory starts with Euler who in 1748 used generating functions to find the number of solutions in nonnegative integers of linear equations with nonnegative coefficients. Later, in 1818 his sixth (from eight) proofs of the quadratic reciprocity law Gauss used sums of units of unity which are analogues over finite fields of the Gamma function. The analytic number theory from modern point of view starts with Dirichlet who in 1837 proved his famous theorem for the existence of infinitely many primes in infinite arithmetic progressions. For this purpose he introduced the so called L-functions and used methods from analysis. Later, in 1848 – 1850 Chebyshev in his attempt to prove the asymptotic law for the distribution of primes, and also with analytic methods, proved the Bertrand postulate for the existence of primes between n and $2n$. The next important step is the famous paper by Riemann from 1859 (which is his only paper in number theory), where he introduced the ζ-function, showed its importance for the distribution of primes and stated several conjectures, one of which is the celebrated Riemann conjecture. In 1896 Hadamard and Vallee-Poussin developed further the ideas of Riemann and, using method of complex analysis, proved the theorem for the distribution of primes. We shall mention also the circle method of Hardy and Littlewood developed in 1919-1925 in a series of papers on the Warring problem. In 1934 Vinogradov developed the method of trigonometric sums and used it to solve several important problems in number theory. In particular, in 1937 he proved for sufficiently large integers the ternary version of the Goldbach conjecture. More recently, after 1950, the analytic methods were combined with the classical combinatorial sieve method introduced by Eratosthenes in the third century B.C. Directly connected with the problematics studied in the thesis are three of the most famous problems in number theory:

- The Goldbach conjecture (from 1742 in result of the correspondence of Goldbach with Euler), that every even number greater than 2 can be presented as a sum of two primes. The ternary version of the conjecture states that every odd number greater than 5 is a sum of three primes. After a series of improvements in the bounds in the theorem of Vinogradov, the last step in the proof of the conjecture was made by computer check.
- The conjecture for the existence of infinitely many prime twins, i.e., pairs of primes in the form $(p, p + 2)$.
- The Waring problem from 1770: For every integer $k \geq 2$ there exists an integer n such that for every positive integer N the equation

$$x_1^k + \cdots + x_n^k = N$$

has a solution. (The solution of the Waring problem given by Hilbert in 1909 also uses analytic methods.)

These three classical problems and their combinations have a series of versions and analogues. For example, the mathematicians actively study an analogue of the Waring problem assuming that the degree of the unknowns is an arbitrary real $c > 1$ and the difference $\Delta = x_1^c + \cdots + x_n^c - N$ is small. There is a version of the problem when the solutions of the inequality are primes and the estimate for the error Δ is given in terms N. Also, instead with primes one may ask for the existence of a solution in integers which are products of small number of prime factors. Here, in a natural way one applies versions of the sieve method. In the thesis the author makes an interesting combination of the three classical problems stated above. I am convinced that the problems studied in the thesis are important and of contemporary interest which is confirmed also by the permanent research activity in the area in the latest decades.

2. The level of knowledge of the current status of the problems and creative interpretation of the literature. The author of the thesis knows very well the current status of the problems. The list of 88 references starts in 1900, 17 of the references are published before 1950 and 19 after 2010. The "educational" side of the Ph.D. requires that the Ph.D. student has to prove that she or he has "entered" in the area. I think that Zhivko Petrov has entered in a modern and in the same time classical branch of mathematics, and reading the thesis one obtains a clear idea for the history and the contemporary situation for the problems and the methods for their study. I want especially to stay that the author has acquired a rich arsenal of methods from number theory and mathematical analysis and has applied it practically to obtain the results in his thesis.

3. Scientific contributions. In his thesis the author has solved three problems in analytic number theory. After the first two introductory chapters the main result in the third chapter is the following. For every real c between 1 and 29/28 every sufficiently large integer N can be presented as $[p^c] + [m^c]$, where p is prime and m is a product of prime factors whose number is bounded in terms of c. This result is obtained jointly with the scientific advisor Prof. Dr. Sci. Doychin Tolev. The main result of the fourth chapter is that for any c between 1 and 17/16 every sufficiently large integer N can be presented in the form $[p_1^c] + [p_2^c] + [p_3^c]$, where p_1, p_2, and p_3 are primes, and the integers $p_1 + 2$, $p_2 + 2$, and $p_3 + 2$ have a number of prime factors, again bounded in terms of c. This result is an own achievement of the Ph.D. student. The fifth chapter of the thesis is based on a joint paper with Prof. Angel Kumchev, a Bulgarian mathematician working in the USA. It contains the proof for existence of solutions of inequalities of the form $|p_1^c + \cdots + p_s^c - N| < (\log N)^{-1}$, where p_i are primes of Pyatetzki-Shapiro. (These primes are of the form $p = \left[m^{\frac{1}{\gamma}}\right]$ and were introduced by Pyatetzki-Shapiro in 1953.) I am convinced that the results obtained in the thesis completely satisfy the "scientific" side of the Ph.D. degree.

4. Quality of the publications included in the thesis. The number and the quality of the publications completely satisfy the requirements of the Faculty of Mathematics and Informatics of the University of Sofia. The results in the thesis are based on three journal papers published in 2017 and 2018. Two of the papers are joint with the scientific advisor in the Proceedings of the Steklov Institute and with Prof. Kumchev in Monatshefte fuer Mathematik, and the third paper in the Annual of the University of Sofia is written by the Ph.D. student only. I accept that the joint papers are written with equal contribution by both authors. The paper with the scientific advisor is cited once by a respected foreign mathematician. Additionally, the results of the thesis have been presented at seminars and meetings in Bulgaria and abroad.

5. Recommendations and remarks. Concerning the thesis, I do not have any essential critical remarks and recommendations.

6. The summary of the thesis and the abstract of the contributions (which is contained in the introduction of the thesis and in the description of the results in the summary) are sufficiently detailed and give clear and adequate information for the contents and the main results of the thesis.

Conclusion: The presented thesis is in a classical but continuing to be actively developed branch of mathematics. It is at very high educational and scientific level and satisfies all requirements to a thesis in the area of mathematics and its applications. I recommend to the respectable Scientific Jury to award Assitant Professor Zhivko Hristov Petrov with the educational and scientific degree "Doctor" (Ph.D.) in professional direction 4.5. Mathematics, Doctoral Degree Program "Mathematical Analysis".

Sofia, June 26, 2019 Member of the Scientific Jury:

(Acad. Dr. Sci. Vesselin Drensky)