SOFIA UNIVERSITY
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COPULAS ON SOBOLEV SPACES AND APPLICATIONS

SUMMARY OF PHD THESIS

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Contents

Introduction

Copulas appear as a modern tool in creating flexible probability models with more than one random variable. Any multivariate model could be constructed by means of copulas. They are widely used in areas as insurance, finance, banking or more general where there are interacting processes (risks) and the need to establish their dependence.

The history of copulas begins in 1959 with Sklar's article [62], where the main result for copulas - the Sklar theorem is conjectured for the first time. It states that copulas allow explicit construction of multivariate distribution functions by one-dimensional distributions playing the role of marginals. Indeed, let X and Y be two random variables with distribution functions $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ respectively and $H(x, y) = P(X \leq x, Y \leq y)$ is the join distribution function of X and Y, such as $F(x) = H(x, +\infty)$, $G(y) = H(+\infty, y)$ are the margins of H. Then there exists a copula C such that for all $(x, y) \in \mathbb{R}^2$

$$H(x, y) = C\big(F(x), G(y)\big).$$

If F and G are continuous, then C is unique. The opposite statement is also true.

The proof of Sklar's theorem was not given in the first article [62], but a sketch of it was provided by Sklar in 1973 and, finally, showed in detail by Schweizer and Sklar in 1974 [59].

Comprehensive theory of copulas is developed in [50], [11] и [32].

The advent of copulas in finance, which is well documented in [26] originated a wealth of different investigations: see, for example [7] where copulas are introduced from the viewpoint of mathematical finance applications. There copulas are used in order to describe major topics such as asset pricing, risk management and credit risk analysis, [43] contains an introduction to the realm of copulas aimed at the quantitative risk manager, [51] reviews the use of copulas in econometric modelling, etc.

While the most common models impose copulas on given data and examine whether the copula (e.g. Gaussian copula, Clayton copula, t-copula, etc.) describes correctly the stochastic dependence (usually copula statistical tests are done), the present dissertation proposes a method for obtaining a specific copula as a solution of a differential equation that corresponds to real data.

Let us recall the definition of a copula. Let $I = [0, 1]$ and $I^2 = I \times I$. A function $C : I^2 \to I$ is called two-dimensional copula, if

1. For all $(u, v) \in I^2$

$$C(u,0) = 0 = C(0,v), \quad C(u,1) = u, \; C(1,v) = v; \tag{0.1}$$

2. C is a 2-increasing, i.e. for every u_1, u_2, v_1, v_2 in I, such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$V_C(B) \equiv C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \tag{0.2}$$

where B is a rectangle $[u_1, u_2] \times [v_1, v_2]$ and the expression (0.2) defines the C-volume of B.

The main two problems related to copulas are

1. How to check if a given function is 2-increasing (usually the boundary conditions are trivial)?

2. How to define a new copula?

In the case $n \geq 3$ the complexity of the issue above is significant and the number of known n-dimensional copulas is very limited.

In the presented dissertation the author gives an answer of the problems posed above considering copulas as functions in the appropriate Sobolev spaces.

Unlike the most papers related to copulas (e.g. [50]), where considerations are restricted to functions which are well defined in each point and it is proven that copulas are Lipschitz functions on their domain ([50], theorem 2.2.4), we consider copulas under more general assumptions and prove a number of nice properties, such as generalized definitions of 2-increasing and n-increasing properties.

Further, the main result is the resolution of the boundary value problem

$$\begin{aligned} &\partial_{uv} C(u,v) = f(u,v) \text{ в } I^2 \text{ (in weak sense)};\\ &C(u,0) = 0 = C(0,v);\\ &C(u,1) = u, \; C(1,v) = v, \text{ for all } u, v \in I, \end{aligned}$$

under certain conditions on f: First we consider the case when f is smooth function and later generalize it to $f \in W^{-1,p}(I^2)$, $p > 2$. This problem can be thought of as a Dirichlet problem for the wave equation due to the considered conditions over the boundary I^2. However, it is important to note here that this is an ill-posed boundary value problem. Nevertheless, we prove existence and uniqueness of the solution by imposing some assumptions over the boundary regarding the right hand side of the equation .

Similarly, we solve the boundary value problem in $\mathbb{R}^n$ when $n \geq 3$.

The outline of the dissertation is as follows. Chapter 1 considers the case $n = 2$. Section 1.1 contains the generating technique for 2-increasing functions and a number of examples where we demonstrate the applicability of our method. Section 1.2 provides the required knowledge on Sobolev spaces and *a priori* estimates, from which the uniqueness of the solution follows. Section 1.3 discuses the smooth case and in Section 1.4 the general solution is considered. The applicability of our approach is shown in Section 1.5 through an example providing a new proof of Sklar' theorem in the considered space.

Chapter 2 considers the case $n \geq 3$. In Section 2.1 we give two generalisations of the notion of an n-increasing function and demonstrate the applicability of our method to several examples. In Section 2.2 we provide a new proof of the necessary and sufficient condition a function to be an Archimedean copula, first considering the case $n = 2$, and then for every $n \geq 3$. In Section 2.3 we prove a theorem for uniqueness and existence of the solution of a boundary value problem in the case of n-dimensional copulas.

In Chapter 3 we consider a real-life problem for insurance risk assessment and solve it with numerical methods. Section 3.1 provides the required knowledge about the numerical methods used to solve the equation. In Section 3.2 we build the desired copula and analyze the solution.

Presentation of the dissertation paper

In this section we will present the main result with accompanying comments and explanations.

The following notations are used frequently.

For $n > 0$, let us denote $\overline{\mathbb{R}}^n = \overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}$. If $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $(b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, $a_k \leq b_k$, for $k = 1, \ldots, n$, we denote with B the n-box

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

The vertices of the n-box B are the points $(c_1, c_2, \ldots, c_n)$, where each c_k is equal to either a_k or b_k, for $k = 1, \ldots, n$.

With $I^n = [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1]$ we denote the n-dimensional unit cube.

Let G be a domain in $\mathbb{R}^n$ and let $\mathscr{D}$ be the space of tests functions on G and let $\mathscr{D}'$ be the space of distributions on G.

If $\varepsilon > 0$, let J_ε be a mollifier and f_ε – the regularization of f.

In **Chapter 1** we consider the case of bivariate copulas.

In **section 1.1** we study the 2-increasing functions. We prove **lemma 1.1.2**, where we demonstrate,that if a function $H : \mathbb{R}^2 \to \mathbb{R}$ has continuous gradient and continuous mixed derivative H_{xy}, then H is 2-increasing if and only if

$$H_{xy} \geq 0 \text{ в } \mathbb{R}^2.$$

Based on this result we give a new generalized definition of the notion 2-increasing function, when derivatives exist only in a weak sense, precisely

Definition 1.1.3 A distribution $H \in \mathscr{D}'$ is weakly 2-increasing if for any test function $\varphi \geq 0$ in $\mathscr{D}(\mathbb{R}^2)$:

$$\big(H_{xy}, \varphi\big) \geq 0. \tag{0.3}$$

Certainly, we would like both definitions to be equivalent when H is a smooth function. That is why we prove **lemma 1.1.4**, which states that if $H \in \mathscr{D}'(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$ is weakly 2-increasing then H is 2-increasing in the sense of the former definition.

By calculating weak derivatives we demonstrate in a new way using our definition that the following functions $W(u, v) = \max(u + v - 1, 0)$, $M(u, v) = \min(u, v)$ and $C(u, v) = \big[min(u, v)\big]^{-\theta}(uv)^{1-\theta}$ are 2-increasing while the function $\max(u, v)$ is not.

In **Section 1.2** we define the boundary value problem and show uniqueness of the solution proving a priori estimate. At the begining of the section we provide the required knowledge on Sobolev spaces needed for the proof.

Theorem 1.2.6 Let $C \in W^{1,p}(I^2)$, $p > 2$ be a solution of the following problem

$$\partial_{uv} C = f(u,v), \ \ (u,v) \in I^2,$$

where $f \in L_p(I^2)$ and the above equality holds only in a weak sense, i.e.

$$\big(C_{uv}, \varphi\big) = (f, \varphi),$$

for all $\varphi \in \widetilde{W}^{1,p}(I^2) = \big\{ w \in W^{1,p}(I^2) \,\big|\, w|_{u=0} = w|_{v=0} = 0 \big\}$. Also, let

$$\begin{cases} C(0,v) = 0 = C(u,0) \\ C(u,1) = u, \ C(1,v) = v, \end{cases}$$

where $u, v \in I$.

Then there exists a constant M, which does not depend on f, such that

$$\|C\|_{W^{1,p}(I^2)} \leq M \, \|f\|_{L_p(I^2)}.$$

From the above *a priori* estimate directly follows the uniqueness of the solution.

Corollary 1.2.7 The solution of the problem is unique.

Let's note that when the solution $C \in W^{1,p}(I^2)$ and $C_{uv} \in W^{-1,p}(I^2)$, the uniqueness theorem still holds, since the right side of the equation becomes $f = 0 \in L_p(I^2)$.

And last but not least **theorem 1.2.6** actually gives us continuity of the solution C in regards to f. This result we use in **Chapter 3** when we construct the copula from the considered real data.

In **Section 1.3** we prove the existence of the solution to the considered boundary value problem, where the function $f(u,v)$ in the right hand side of the equation $\partial_{uv} C = f(u,v)$ satisfies the following conditions:

$$f \in L_p(I^2), \ p \in (1, +\infty);$$

$$f(u,v) \geq 0, \ \text{for all } (u,v) \in I^2;$$

$$\int\limits_{B_{u,1}} f(\xi,\eta) d\xi \, d\eta = u, \ \text{for all } u \in [0,1], \ \text{where } B_{u,1} = [0,u] \times [0,1];$$

$$\int\limits_{B_{1,v}} f(\xi,\eta) d\xi \, d\eta = v, \ \text{for all } v \in [0,1], \ \text{where } B_{1,v} = [0,1] \times [0,v].$$

The proof is based on [66], §15, where the Goursat problem for the hyperbolic equation $\partial_{uv}W + a\partial_u W + b\partial_u W + cW = f(u,v)$ is solved, with the initial data being prescribed differentiable functions on the characteristics $\{(u,v) \in \mathbb{R}^2 \,\big|\, u = 0\}$ and $\{(u,v) \in \mathbb{R}^2 \,\big|\, v = 0\}$ and continuous right hand side $f(u,v)$. The existence and uniqueness of the C^1-solution is proved for any rectangle with two sides being the mentioned characteristics with smooth initial data on them.

In **Section 1.4** we extend the notion of solution of the boundary value problem by using the concept of a weak derivative. We prove in that case the existence of $W^{1,p}(I^2)$-solution, which is unique due to the estimate from **theorem 1.2.6**, when the right hand side $f \in W^{-1,p}(I^2)$ and satisfies (respectively modified) conditions (by analogy of the smooth case) together with additional condition for its Fourier transform $\widehat{f} \equiv \mathscr{F}(f)$.

Theorem 1.4.1 Let $f \in W^{-1,p}(I^2)$, $p > 2$ and $f \geq 0$ in a weak sense. Suppose the conditions

$$\lim_{\varepsilon \to 0^+} \left(\widetilde{f}, \chi_{B_{u,1}} * J_\varepsilon \right) = u, \text{ for all } u \in I,$$
$$\lim_{\varepsilon \to 0^+} \left(\widetilde{f}, \chi_{B_{1,v}} * J_\varepsilon \right) = v, \text{ for all } v \in I,$$

are satisfied, where $\widetilde{f} \in \left(W^{1,p}(\mathbb{R}^2)\right)'$ is an extension of f.

Then there exists a unique solution $C \in W^{1,p}(I^2)$ of the problem:

$$\begin{aligned} &C_{uv}(u,v) = f(u,v) \text{ в } I^2 \text{ (in a weak sense) };\\ &C(u,0) = 0 = C(0,v);\\ &C(u,1) = u,\ C(1,v) = v, \text{ for all } u, v \in I, \end{aligned}$$

under the conditions

$$\left\| \mathscr{F}^{-1} \left\{ \frac{\chi_1(\xi,\eta)|\eta|}{\xi} \cdot \frac{\widehat{f}(\xi,\eta)}{(1+\xi^2+\eta^2)^{\frac{1}{2}}} \right\} \right\|_{L_p} < +\infty,$$
$$\left\| \mathscr{F}^{-1} \left\{ \frac{\overline{\chi}_1(\xi,\eta)|\xi|}{\eta} \cdot \frac{\widehat{f}(\xi,\eta)}{(1+\xi^2+\eta^2)^{\frac{1}{2}}} \right\} \right\|_{L_p} < +\infty,$$

where χ_1 и $\overline{\chi}_1$ are smooth regularization functions with the following properties:

a) $supp\, \chi_1 \subset \{\text{cone neighbourhood of } (0, \pm 1)\} \backslash \{ \text{ neighbourhood of } (0,0)\}$;

b) $supp\,\overline{\chi}_1 \subset \{$cone neighbourhood of $(\pm 1, 0)\} \backslash \{$ neighbourhood of $(0,0)\}$.

The proof is based on the following reasoning. As two of the boundary conditions are obtained from the right hand side of the equation we will focus on the following Goursat problem in a weak sense

$$\begin{cases} \text{Find a solution } h \in W_0^{1,p}(K_1), \text{such that } h_{uv} = f \in W^{-1,p}(\mathbb{R}^2), \\ \text{in a weak sense, i.e. } (h_{uv}, \varphi) = (f, \varphi), \text{ for all } \varphi \in \mathscr{D}(K_1), \\ \text{where } supp\, f \text{ is bounded and } K_1 = \big\{(u,v) \in \mathbb{R}^2 \mid u > 0,\ v > 0\big\}. \end{cases}$$

Next step in our considerations is to give analogous formulation of the problem on $\mathbb{R}^2$, not only on K_1. Thus Fourier transforms and corresponding Sobolev spaces are applicable.

We demonstrate that the above problem is not the equivalent but follows from

$$\begin{cases} \text{Find a solution } H \in W^{1,p}(\mathbb{R}^2),\ p > 2,\ supp\, H \subset \overline{K_1}, \\ \text{such that } (H, \varphi_{uv}) = (f, \varphi), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2). \end{cases}$$

The advantage of the new formulation is obvious: if we find such an H, then his traces vanish on ∂K_1, ant hence this will be the wanted solution h.

To show the existence of a weak solution H of the last problem, i.e.

$$(H, \varphi_{uv}) = (f, \varphi), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2),$$

we follow the procedure based on Hahn-Banach theorem (see [47], §4.2). Note that such a solution is not unique but we prove uniqueness thanks to the additional conditions imposed.

In **Section 1.5** we use a simple example to show how our approach works. We prove in a new way the Sklar's theorem in the case when the corresponding probability density functions are continuous and do not vanish.

Even in this simple case, choosing different probability density functions (suitable for our considerations) allows us to generate a variety of copulas not observed in ([50]).

In **Chapter 2** we study n-dimensional copulas.

In **Section 2.1** we generalize the definition of n-increasing function, when $n \geq 3$. Again based on the fact that a function $H : \mathbb{R}^n \to \mathbb{R}$ having continuous derivatives is n-increasing if and only if

$$H_{x_1 \ldots x_n} \geq 0 \text{ в } \mathbb{R}^n,$$

we give a new generalized definition of the notion n-increasing function when the derivatives are weak, i.e.

Definition 2.1.7 Let $G \subset \mathbb{R}^n$ be a domain. A distribution $H \in \mathscr{D}'(G)$ is called a weakly n-increasing distribution in G, if for any non-negative test function $\varphi \in \mathscr{D}$,

$$\left(H_{x_1,\ldots,x_n}, \varphi\right) \geq 0.$$

Similarly to the 2-dimensional case we prove equivalence of the definition with the former one when $H \in \mathscr{D}'(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$.

Then we noticed that we can give a second generalized definition, which in some cases is more convenient. In fact, a disadvantage of **definition 2.1.7** relates to the search of weak derivatives of non-smooth functions (i.e. derivatives in the sense of Distribution theory). Such examples are already given in Chapter 1. Apart from that this definition does not take into account the fact that H belongs to a suitable Sobolev space in the cases under consideration.

Definition 2.1.10 Let $G \subset \mathbb{R}^n$ be a domain, such that its boundary ∂G satisfies the segment condition (see [1]). Then we say that $H \in W^{1,p}(G)$ is a weakly n-increasing function in G, if

$$(-1)^n (H, f_{x_1 \cdots x_n}) \geq 0,$$

for all non-negative $f \in W^{n-1,p}(G)$.

If we assume that $(-1)^n (H, f_{x_1 \cdots x_n}) \geq 0$ holds only for smooth functions f, then by theorem 3.22 in [1] we obtain again **definition 2.1.10**.

Again we prove that if H is continuous, then **definition 2.1.10** is equivalent to the common definition.

We demonstrate in **remark 2.1.14**, that **definition 2.1.7** and **definition 2.1.10** are equivalent, as starting from an arbitrary function $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$ we obtain a function g which vanishes together with its derivatives up to order m on the sides of the cube I^n, passing through the vertex $(1, \ldots, 1)$.

Finally, using our approach we prove in a new way that the function Fréechet-Hoeffding lower bound

$$W^n(x_1, \cdots, x_n) = \max[x_1 + x_2 + \cdots + x_n - n + 1, 0]$$

is not n-increasing in I^n and hence is not a copula when $n \geq 3$.

In **Section 2.2** we consider an important class of copulas – Archimedean copulas, which is studied in details in many works (see chapter 4 in [50], as well as [33], [11], [44], [24]). These copulas find a wide range of applications as they are easy to construct and assess many nice properties.

We provide a new proof of the necessary and sufficient condition a given function to be an Archimedean copula. Since boundary conditions are easily verified a key point is whether the obtained function is n-increasing.

In the case of bivariate copulas we prove:

Theorem 2.2.9 Let φ be a continuous and strictly decreasing function from $[0,1]$ to $[0,+\infty]$, such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ, given by $\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t), & \text{if } 0\le t\le \varphi(0),\\ 0, & \text{if } \varphi(0)\le t\le +\infty.\end{cases}$

Then the function C^2, given by

$$C^2(u,v)=\varphi^{[-1]}\big(\varphi(u)+\varphi(v)\big),$$

is a bivariate copula if and only if φ is convex.

In the case $n\ge 3$, we prove:

Theorem 2.2.12 Let φ a continuous and strictly decreasing function from $[0,1]$ to $[0,\infty]$, where $\varphi(0)=\infty$ and $\varphi(1)=0$. Let φ^{-1} denote the inverse of φ. Then the function $C^n : I^n \longrightarrow I$, given by

$$C^n(x_1,\dots,x_n)=\varphi^{[-1]}\big(\varphi(x_1)+\dots+\varphi(x_n)\big),$$

is an n-copula for all $n>2$, if and only if φ^{-1} is completely monotonic on $[0,+\infty]$, i.e. when the expression $(-1)^k\frac{d^k}{dt^k}g(t)\ge 0$ holds in $[0,+\infty]$.

At the of the section we present an counterexample of the necessity of the convexity condition for the function φ in the 2-dimensional case.

Section 2.3 generalizes our approach of the 2-dimensional case described in chapter 1, sections §1.2, §1.3 and §1.4, as we obtain a copula as a solution of a boundary value problem in n-dimensional unit cube I^n.

Key result is the paper [5] to the Goursat problem over the unit cube. The main statement we prove in this section is the next theorem.

Theorem 2.3.1 Let the function $f\in W^{1-n,p}(I^n)$, $p>n$, be such that:

a) f satisfies the conditions

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n}\widetilde{f}_\varepsilon(\xi_1,\dots,\xi_n)\chi_{B_k}\,d\xi_1\dots d\xi_n=x_k,$$

for all $k=1,\dots,n$ and for each n-box
$B_k=[0,1]\times\dots\times[0,x_k]\times\dots\times[0,1]\subset I^n$;

б) f is non-negative in the sense of the theory of distributions, i.e.

$$(f,\varphi) \geq 0, \text{ for all } \varphi \in W_0^{n-1,q}(I^n);$$

в) f satisfies the regularity condition $(\mathscr{R})$ (which we will formulate below).

Then there exists a unique solution $C \in W^{1,p}(I^n)$ of the problem:

1) on I^n holds

$$(-1)^n \big(C, \varphi_{x_1 \ldots x_n}\big) = (f,\varphi),$$

for all $\varphi \in W_0^{n-1,q}(I^n)$, where q is the exponent conjugate to p;

2) with boundary conditions

$$C(x_1,\ldots,x_n) = 0, \text{ if } x_k = 0 \text{ for at least one index } k = 1,\ldots,n.$$

Each function $f \in W^{1-n,p}(I^n)$ has the representation

$$\big(f,u\big) = \sum_{0\leq|\alpha|\leq n-1} \big(\partial^\alpha u, f_\alpha\big),$$

where $f_\alpha \in L_p(I^n)$, $|\alpha| \leq n-1$, for each $u \in W^{n-1,q}(I^n)$.

We formulate the regularity condition $(\mathscr{R})$ as follows.

$(\mathscr{R})$: Let us assume that the functions f_α, representing f, satisfy

$$\partial_{x_i}\mathscr{D}^{\alpha'} f_\alpha \in L_p(I^n),\ i = 1,\ldots,n,\ |\alpha| \leq n-1,$$

where

$$\partial_i \mathscr{D}^{\alpha'}\varphi = \begin{cases} \varphi & , \text{ if } \alpha'_i = -1 \\ \partial_{x_i}\mathscr{D}^{\alpha'_i}\varphi & , \text{ if } \alpha'_i \geq 0. \end{cases}$$

For each multi-index $\alpha \in \mathbb{N}^n$, $\mathbb{N} = \{0,1,2,\ldots\}$ we set

$$\begin{cases} \alpha' = ({\alpha'}_1,\ldots,{\alpha'}_n) = (\alpha_1 - 1,\ldots,\alpha_n - 1) \\ \mathscr{D}^{\alpha'} = \mathscr{D}_1^{\alpha'} \ldots \mathscr{D}_n^{\alpha'}, \end{cases}$$

where for each $i = 1,\ldots,n$,

$$\mathscr{D}_i^{\alpha'_i}\varphi = \begin{cases} \displaystyle\int_0^{x_i} \varphi(x_1,\ldots,x_n)dx_i & , \text{ when } \alpha'_i = -1, \\ \varphi(x_1,\ldots,x_n) & , \text{ when } \alpha'_i = 0, \\ \partial_i^{\alpha'_i}\varphi & , \text{ when } \alpha'_i > 0. \end{cases}$$

At the end of the section we comment that the local conditions enforced in the 2-dimensional case in **theorem 1.4.1** are equivalent of the regularity condition ($\mathscr{R}$).

In **Chapter 3** we build a copula using our approach, i.e. we obtain a copula as a solution of a boundary value problem using real data coming from a Bulgarian insurance company. We apply a spectral numerical method, based on the Chebyshev polynomials, to numerically solve the differential equation.

There exist many models describing the dependence between *claim amount* and *severity of the claim occurrence*. In the dissertation we propose a method which allows to study the dependence between *claim amount* and *moment of claim occurrence* (towards the date of subscribing policy).

We consider CASCO motor insurance policies for a company operating on the Bulgarian market. Data represents company's portfolio for 5 years period with 87 917 number of claims.

Section 3.1 provides the required information about the numerical methods used to solve the equation.

In **Section 3.2** we build the desired copula and analyze the solution. We determine that the accuracy of the numerical method is 10^{-4} due to the data we have.

Finally, we conclude that the observed copula is very close to $\Pi(u, v) = uv$, i.e. the difference between both copulas is 10^{-4}. Distribution functions: *claim amount* and *moment of claim occurrence*, could be considered as independent for the observed portfolio.

Main contributions

These are the main accomplishments in the thesis due to the author:

In Chapter 1 – Bivariate copulas on Sobolev Spaces – we consider the case $n = 2$, and give a new generalized definition of the property 2-increasing function. We demonstrate equivalence of the definition with the former one in the case of smooth functions. We consider a number of examples.

Next we formulate the boundary value problem. We prove the existence and uniqueness of the solution under additional conditions on the right hand side of the equation.

In Chapter 2 – n-dimensional copulas – we consider the general case when $n \geq 3$. We give two new definitions equivalent to the former definition of an n-increasing function.

As a corollary we prove in a new way the necessary and sufficient condition a given function to be an Archimedean copula.

An the end of the chapter we formulate the boundary value problem and prove the existence and uniqueness of the solution under additional conditions on the right hand side of the equation.

In Chapter 3 – Practical application in an example from the insurance industry – we study how our method can practically be used by an insurance company to assess the insurance risks and exposure. We construct a copula using our approach, i.e. we obtain a bivariate copula as a solution of a boundary value problem using real data coming from an insurance company present on the Bulgarian market. Distribution functions: *claim amount* and *moment of claim occurrence*, could be considered as independent for the observed portfolio.

Publications related to the thesis

1. Iordanov I., N. Chervenov. Copulas on Sobolev spaces, Comptes rendus de l'Académie bulgare des Sciences, Vol 68, No1, pp.11-18, 2015.

2. Iordanov I., N. Chervenov. Copulas on Sobolev spaces, Serdica Math. J. 42, 335 - 360, 2016.

3. Chervenov N., I. Iordanov, B. Kostadinov. Goursat problem over unit cube in first quadrant of $\mathbb{R}^n$ (with applications to existence of copulas), AIP Conference Proceedings 2048, 040022 (2018); doi: 10.1063/1.5082094.

4. Chervenov N., B. Kostadinov. Generalisation of the Notion of an n-increasing function. Archimedean Copulas. Comptes rendus de l'Académie bulgare des Sciences, Vol 72, No3, pp.292-300, 2019.

5. Chervenov N., I. Iordanov, B. Kostadinov. n-dimensional copulas and week derivatives, Serdica Math. J. 44, 2018.

6. (pre-press) Stoilov N., N. Chervenov, Spectral approach to application of copulas in actuarial science.

Dissemination of the results, connected to the dissertation

Some of the results in the dissertation have been presented on several conferences:

1. "On the Sklar's theorem", Spring Scientific Session of FMI, 16.03.2013.

2. "Generalized 2-increasing functions", Spring Scientific Session of FMI, 29.03.2014, https://www.fmi.uni-sofia.bg/bg/proletna-nauchna-sesiya-na -fmi-2014.

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Declaration of originality

The author declares that the dissertation contains original results obtained by him or in joint work with his scientific advisor and/or other coauthors of common papers of the considered in the dissertation topics. Results of other scientists were properly cited.

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