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COMPUTABLE STRUCTURE THEORY : JUMP OF STRUCTURE, CODING AND DECODING

ABSTRACT

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Contents

Chapter 1

Introduction

This work is in computable structure theory, and considers the relationship between definability and computability in mathematical structures. Computable structure theory studies the interplay between complexity and structures. It is an area inside computability theory and logic that is concerned with the computable aspects of mathematical objects and constructions. We are interested in questions like the following: How difficult is it to represent a certain structure? Which structures can be represented computably? How difficult is it to compute certain relations on a structure, or perform certain constructions on it? We are particularly interested in answers that connect computational properties with algebraic or combinatorial properties of the structure.

We all know that in mathematics there are proofs that are more difficult than others, constructions that are more complicated than others, and objects that are harder to describe than others. The objective of computable mathematics is to study this complexity, to measure it, and to find out where it comes from. This work concentrates on the complexity of structures. By structures, we mean objects like rings, graphs, or linear orderings, which consist of a domain on which we have relations, functions, and constants.

The goal of computable mathematics is to find the extent to which certain classical results of mathematics are effectively true. In algebra this investigation based on the intuitive notion of effectiveness dates back to van der Waerden who in his 1930 book "*Modern Algebra*" defined an explicitly given field as one the elements of which are uniquely represented by distinguishable symbols with which we can perform the field operations algorithmically. In his pioneering paper [vdW30] on non-factorability of polynomials from 1930, van der Waerden essentially proved that an explicit field $(F;+,\cdot)$ does not necessarily have an algorithm for splitting polynomials in $F[x]$ into their irreducible factors. The work of Church, Gödel, Kleene, Markov, Post, Turing and others in the next decade established the rigorous mathematical foundations for the computability theory. In the 1950s, a famous problem, involving the interplay of algebra and computability, the word problem, was resolved. It was shown independently by Novikov [Nov55] and Boone [Boo59] that there exists a finitely presented group G such that the word problem for G is undecidable. In 1956, Fröhlich and Shepherdson [FS56] used the precise notion of a computable function to obtain a collection of results and examples about explicit rings and fields. Rabin [Rab60] and Maltsev [Mal61, Mal62] studied more extensively computable groups and other computable (also called recursive or constructive) algebraic structures. Another spectacular negative solution to a famous problem, which involves the interplay of number theory and computability, Hilbert's TenthProblem, was completed by Matiyasevich [Mat70] in 1970. Building on work of Davis, Putnam, and J. Robinson (see [Mat93]), he established that there is no effective procedure to decide whether a given Diophantine equation has a solution in integers. In the 1970s, Metakides and Nerode [MN77, MN79] and other researchers from the West initiated a systematic study of computability in mathematical structures and constructions by using modern computability-theoretic tools, such as the priority method, forcing method and various coding techniques. At the same time and independently, computable model theory was developed in the he Russian school in Moscow, and Siberian school of constructive mathematics. In the past few decades there has been increasing interest in computable structure theory, in connections with the algebra, analysis, topology, computer science, etc. Detailed accounts of the history of the subject and the state of the art can be found in [AK00] and [Mon].

In our work we want to measure the complexity of a structure, so we attach to every structure a set of degrees that describes them: *the degree spectrum of a structure*. Since computability theory is developed on the natural numbers we need to work with structures with countable domains, whose elements can be enumerated by natural numbers. Given a structure $\mathcal{A}$, *a presentation* (copy) of $\mathcal{A}$ is an isomorphic (or homomorphic) copy of $\mathcal{A}$ whose domain is either the set of the natural numbers $\mathbb{N}$ or an initial segment of $\mathbb{N}$. The degree spectrum $DS(\mathcal{A})$ is the set of all Turing degrees of the atomic diagrams of all presentations of the structure $\mathcal{A}$, a notion, introduced by Richter [Ric81] and investigated by Knight [Kni86] and many others. Let J be the set of all Turing jumps of the elements of the degree spectrum of a structure $\mathcal{A}$, a natural question is: if there is a structure $\mathcal{A}'$ with a degree spectrum J. Thus, we come to the definition of the notion of jump of a structure. It is an analogue of the notion of Turing jump. One can compare the complexity of structures using Muchnik reducibility between structures — for structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \leq_w \mathcal{B} \iff DS(\mathcal{B}) \subseteq DS(\mathcal{A})$, i.e. since the degree spectrum is upwards closed, every presentation of $\mathcal{B}$ computes a presentation of $\mathcal{A}$. So, for the jump $\mathcal{A}'$ of the structure $\mathcal{A}$, we always have $\mathcal{A} <_w \mathcal{A}'$. The jump $\mathcal{A}'$ has more computational power that the structure $\mathcal{A}$. The Σ^c_2 definable relations in $\mathcal{A}$ are exactly the Σ^c_1 definable in $\mathcal{A}'$. Since the jump of a structure is an analogue of the Turing jump in the structure of the

Turing degrees $\mathcal{D}_T$, a natural question here is if there are any jump inversion theorems such as Friedberg's?

In **Chapter 2.** of our work we introduce the basic notions, methods and facts that we need. We begin with the partial computable functions, c.e. sets and the structure of Turing degrees. We show some basic properties of the Turing reducibility and the notion of Turing jump. One of the important methods used in computable structure theory is the method of forcing, introduced first by Cohen. We demonstrate the forcing method and present the standard construction of 1-generic set. We prove some basic properties of the 1-generic sets and Friedberg's [Fri57] jump inversion theorem. The last theorem we generalize for structures in Chapter 3., using the forcing method. Next we consider the notion and the properties of enumeration reducibility. Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about one set is used to produce positive information about another set. The structure of enumeration degrees is an extension of the structure of Turing degrees. We introduce the notion of strong minimal cover and show some properties of the relativized variant of 1-genericity for the enumeration reducibility, since we use these properties in Chapter 7. The most common measure of the computational complexity of a structure is through degree spectra. We present the properties of the degree spectra and the enumeration degree spectra. Another way to characterize the complexity of a structure $\mathcal{A}$ is to analyze the definable sets in $\mathcal{A}$. This gives a finer measure as it may happen that two structures have the same degree spectra but greatly differ in their definability power and model theoretic properties. We present the normal form of relatively intrinsically Σ^0_α relations in a given countable structure for a computable ordinal α, by computable infinitary Σ^c_α formulas.

In **Chapter 3.** we answer to the following questions:

(1) How to define the jump of a structure as an analogue of the Turing jump in the degree structure $\mathcal{D}_T$ of Turing degrees? Are there any typical structural properties such as jump inversion theorems? Is the set of all jumps of the elements of the degree spectrum of a structure also a spectrum of another structure?

The idea of the *jump of a structure* is first considered by Soskov and his student Baleva [Bal06] in the context of s-reducibility between structures, a reducibility based on relative search computability of Moschovakis [Mos69]. We define the jump $\mathcal{A}'$ of the structure $\mathcal{A}$ by considering the Moschovakis extension of $\mathcal{A}$ together with a new predicate, an analogue of the Kleene's Halting set, which codes all the sets, definable by computable infinitary Σ^c_1 formulas with parameters. This changes the domain of the structure, but keeps the language finite, if the original is finite. Montalbán later, independently, gives another definition of the jump of a structure. Montalbán's approach [Mon09, Mon12, HM12] is to keep the domain of the structure the same and to add a complete set of relations definable by computable infinitary Π^c_1 formulas. In [Mon12, Mon] he changed the added complete set of relations by those, definable by computable infinitary Σ^c_1 formulas and received an equivalent notion as ours. Morozov [Mor04] and later Puzarenko [Puz09] also define the jump but for an admissible structure. Stukachev extends that definition to all structures in the terms of Σ-definability in hereditarily finite extension of the structure. Vatev [Vat13, Vat14, Vat15] extends the notion of jump of a structure to the α-th jump of a structure for arbitrary computable ordinal α.

We prove a jump inversion theorem, an analogue of the classical Friedberg's jump inversion theorem [Fri57]. We present a relativized version of the theorem to all structures. That is, if $\mathcal{A} \geq_w \mathcal{B}'$ then there is a structure $\mathcal{C} \geq_w \mathcal{B}$ such that $\mathcal{A} \equiv_w \mathcal{C}'$. In the proof we use Marker extensions. Actually, our proof is in the terms of degree spectra, i.e. if $DS(\mathcal{A}) \subseteq DS(\mathcal{B}')$, then there exists a structure $\mathcal{C}$ with the property $DS_1(\mathcal{C}) = DS(\mathcal{A})$ and $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$. This jump inversion theorem was proved later by Stukachev [Stu09, Stu10] for the notion of Σ-equivalence. Another way to formulate the jump inversion on structures is: for every structure $\mathcal{A}$, if $Y \subseteq \mathbb{N}$ computes a copy of the jump $\mathcal{A}'$, then there is $X \subseteq \mathbb{N}$ such that $X' \equiv_T Y$ and X computes a copy of $\mathcal{A}$. Montalán [Mon09, Mon12, Mon] call this the second jump inversion theorem. In other words: the jump spectrum of $\mathcal{A}$ is the spectrum of $\mathcal{A}'$, i.e. $DS_1(\mathcal{A}) = DS(\mathcal{A}')$. We prove this result with Soskov [SS07, SS09a] and independently later Montalbán [Mon09]. This result for any computable successor ordinal appears in some form in Goncharov, Harizanov, et.al. [GHK+05]. They proved the result above only as a tool to get other results to build a structure that is Δ_α-categorical but not relatively so. They do not mention the jump of a structure. Based on their method Vatev [Vat13, Vat14, Vat15] extends the jump inversion of a structure for arbitrary successor ordinal α. Soskov [Sos13] gives an example that the jump inversion theorem does not hold for a limit ordinal. We also present some applications of the jump inversion theorem which show that this a general method for lifting results from $n = 1$ to the arbitrary $n \in \mathbb{N}$.

Chapter 4. There is a more precise notion of jump inversion, but not all structures admit it. A structure $\mathcal{A}$ admits strong jump inversion whenever X' computes a copy of $\mathcal{A}'$ then X computes a copy of $\mathcal{A}$. The result of Downey and Jockusch [DJ94] shows that every Boolean algebra admits strong jump inversion. Lerman and Schmerl [LS79] prove that for every $\aleph_0$-categorical theory T, if $T \cap \Sigma_2$ is c.e., then every model of T admits strong jump inversion. Some equivalence structures and some abelian p-groups admits strong jump inversion. More recently, D. Marker and R. Miller [MM17] have shown that all countable models of the theory of differentially closed fields of characteristic

0 (DCF_0) admit strong jump inversion.

Not all structures admit strong jump inversion. Jockush and Soare [JS91] show that there are low linear orders without computable copies, and hence they do not admit a strong jump inversion. We are looking for model theoretic conditions which are sufficient for a structure to admit strong jump inversion. In Chapter 4. we answer to the following question.

(2) Are there any model theoretical conditions that are sufficient for a structure to admit strong jump inversion?

With Calvert, Frolov, et.al., [CFH⁺18], we establish a general result with sufficient conditions on a structure $\mathcal{A}$, which guarantee strong jump inversion of $\mathcal{A}$, expressed in terms of saturation and enumeration properties of sets of types having formulas of low arithmetic complexity, as computable enumeration R of the B_1-types, where these are made up of formulas that are Boolean combinations of existential formulas, effective type completion, and R-labeling of $\mathcal{A}$. When a structure $\mathcal{A}$ admits strong jump inversion, and $\mathcal{A}$ is low relative to an oracle X, we also consider the complexity of the isomorphisms between $\mathcal{A}$ and its X-computable copies.

Our general result applies to structures from some familiar classes, including certain classes of linear orderings, and trees. While we do not get the result of Downey and Jockusch for arbitrary Boolean algebras, we do get a result for Boolean algebras with no 1-atom, with some extra information on the complexity of the isomorphism. Such an isomorphism can be chosen to be Δ^0_3 relative to X. This is interesting because Knight and Stob established in 2000 that any low Boolean algebra has a computable copy and a corresponding Δ^0_4 isomorphism, and this bound has been proven to be sharp. We apply also our general conditions on the models of elementary first order theory T such that $T \cap \Sigma_2$ is computably enumerable and for each tuple of variables $\overline{x}$, there are only finitely many B_1-types in variables $\overline{x}$ consistent with T. Our general result includes the result of Marker and Miller. As a side result, we get that the saturated model of DCF_0 has a decidable copy.

Chapter 5. In many branches of mathematics, there is work classifying a collection of objects, up to isomorphism or other important equivalence, in terms of nice invariants. In descriptive set theory, there is a body of work using the notion of "Borel embedding" to compare the classification problems for various classes of structures (fields, graphs, groups, etc.). A Borel embedding of one class $\mathcal{K}$ into another class $\mathcal{K}'$ is a Borel function from $\mathcal{K}$ to $\mathcal{K}'$ that preserves the isomorphism types. There are some familiar examples of classes of structures $\mathcal{K}, \mathcal{K}'$ with a Turing computable embedding Θ from $\mathcal{K}$ to $\mathcal{K}'$. The Turing operator Θ takes structures in $\mathcal{K}$ to structures in $\mathcal{K}'$ such that for $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{K}$, $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\Theta(\mathcal{A}_1) \cong \Theta(\mathcal{A}_2)$. If $\mathcal{B} = \Theta(\mathcal{A})$, then $\mathcal{A}$ is coded in some way in $\mathcal{B}$. The effective decoding is given by Medvedev reducibility ($\leq_s$), a uniform variant of Muchnik reducibility. $\mathcal{A} \leq_s \mathcal{B}$ if there is a Turing operator Φ, for a copy of $\mathcal{B}$ it gives a copy of $\mathcal{A}$. There is a more precise notion of decoding, introduced by Montalbán [Mon14, Mon], - *an effective interpretation*, using computably infantry Σ_1^c formulas, which preserves the isomorphic copies. Every countable structure could be effectively interpreted in a graph. Under Borel embeddings and Turing computable embeddings, the class of linear orderings is maximally complicated, just like the class of graphs. It is natural to ask whether for an arbitrary graph G, there is a linear ordering L such that every copy of G computes a copy of L, and G is effectively interpreted in L. A class $\mathcal{K}$ is said to be on top for Turing computability if any other class of structures Turing-computably embeds in $\mathcal{K}$. The class of 2-step nilpotent groups lies on the top of Borel and Turing computable embeddings. Maltsev [Mal60] gave a computable definition of fields, even rings, in this class, using parameters. The question is: if the class of fields is effectively interpreted in the class of 2-step nilpotent groups without parameters? And since the class of fields are on the top of effective interpretability, is the class of of 2-step nilpotent groups on the top as well?

In Chapter 5. we answer the following question.

(3) For the known effective codings of one class of structures into another class, is there an effective or more difficult decoding for some special classes as linear orderings and 2-step nilpotent groups, which are on the top of Turing computable embeddings?

Historically Borel reducibility is introduced by Friedman and Stanley [FS89]. The effective version is the Turing-computable reducibility [CCKM04, KMVB07], introduced by Julia Knight and her students. R. Miller proposed a notion of effective interpretability, based on computable functor—a pair of Turing operators, the first one gives the Medvedev reduction and the second the preserving the isomorphism, between copies. Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] prove that these the two notions of effective uniform interpretability coincide. Harrison-Trainor, R. Miller, and Montalbán [HTMM18] show similar result for Borel functors and interpretations by infantry $L_{\omega 1,\omega}$ formulas..

With Knight and Vatev [KAV19], we give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering. We observe that any graph can be coded in the second jump of a linear ordering, so we have a Medvedev reduction. For the known Turing computable embedding of graphs in linear orderings, due to Friedman and Stanley [FS89], we show that there is no uniform effective interpretation, defined even by $L_{\omega_1\omega}$ formulas. Our conjecture is that there is no effective uniform way for coding graphs in linear orders with uniform effective decoding, even with decoding defined by $L_{\omega_1\omega}$ formulas. In support of this Montalbán and Harrison-Trainor [HT] independently have proved recently that for the Friedman and Stanley's embedding there is no uniform decoding.

Our second result here is positive. With Alvir, Calvert, et.al. [ACG+20], we consider an effective uniform interpretation of fields in some 2-step nilpotent groups. We improve on and generalize a 1960 result of Maltsev. For a field F, we denote by $H(F)$ the Heisenberg group with entries in F. Maltsev [Mal60] showed that there is a copy of F defined in $H(F)$, using existential formulas with an arbitrary non-commuting pair (u, v) as parameters. We show that F is effectively interpreted in $H(F)$ using computable Σ_1-formulas with no parameters. We give two proofs. The first is an existence proof, relying on a result of Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] based on a computable functor. This proof allows the possibility that the elements of F are represented by tuples in $H(F)$ of no fixed arity. The second proof is direct, giving explicit finitary existential formulas that define the interpretation, with elements of F represented by triples in $H(F)$. Looking at what was used to arrive at this parameter-free interpretation of F in $H(F)$, we give some general conditions, sufficient to eliminate parameters from interpretations.

For an algebraically closed field C of characteristic 0, let $SL_2(C)$ be a special linear group of 2×2 matrices over C with determinant 1. Clearly, $SL_2(C)$ is defined in C without parameters. With Alvir, Knight, R. Miller, [AKMS] we define an interpretation of the field C in $SL_2(C)$ using finitary existential formulas with two parameters. There are old model theoretic results, due to Poizat [Poi01], that give uniform definability of a copy of C in $SL_2(C)$ using elementary first order formulas without parameters. So, we have, not necessarily an *effective* interpretation without parameters, but one that is defined by elementary first order formulas. We do not know the complexity of the formulas.

Chapter 6. We are interested also in effective variants of some model theoretic popular constructions, such as the ultraproducts and ultrapowers. Cohesive powers of computable structures, introduced by Dimitrov, [Dim09], can be viewed as effective ultrapowers over effectively indecomposable sets called cohesive sets, where cohesive sets play the role of ultrafilters. It is possible a computable structure to have copies which are not computable. For example the linear order on the natural numbers has a presentation in which the successor relation is not computable. So here the question is: given two isomorphic structures, are their cohesive powers elementary equivalent? Or more specific: for any two copies of a computable linear order do their cohesive powers have the same order type?

In Chapter 6. we answer the following question.

(4) For any two copies of a computable order type, do their cohesive powers has the same order type?

First Skolem constructed a countable nonstandard model of the true arithmetic using similar construction. Various countable nonstandard models of fragments of arithmetic have been later studied by Feferman, Scott, Tennenbaum, Hirschfeld, Wheeler, Lerman, McLaughlin and others (see [FST59]). An effective version of cohesive powers of computable structures, based on partial computable functions has been introduced by Dimitrov, [Dim09], in relation to the study of automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ of effective vector spaces.

With Dimitrov, Harizanov, Morozov, Shafer and Vatev [DHM⁺19] and [DHM⁺20] we consider some properties of cohesive powers of linear orders. We show that if $\mathcal{A}$ is a computable structure that is ultrahomogeneous in a uniformly computable way, then $\mathcal{A}$ is isomorphic to its cohesive powers. We investigate the isomorphism types of cohesive powers for familiar computable linear orders $\mathcal{L}$. From [Dim09] it follows that the cohesive power of a linear order is a linear order. If $\mathcal{L}$ is a computable copy of ω that is computably isomorphic to the standard presentation of ω, then every cohesive power of $\mathcal{L}$ has order-type $\omega + \zeta\eta$ (ζ-the order type of integers, η - the order type of rationals). There is a computable copy $\mathcal{L}$ of ω that is not computably isomorphic to the standard presentation of ω, but every cohesive power of $\mathcal{L}$ has order-type $\omega + \zeta\eta$. However, there are computable copies of ω, necessarily not computably isomorphic to ω, having cohesive powers of order-type $\omega + \eta$, i.e. not elementarily equivalent to $\omega + \zeta\eta$. Our general result is that if $X \subseteq \mathbb{N} \setminus \{0\}$ is either a Σ^0_2 set or a Π^0_2 set, thought of as a set of finite order-types, then there is a computable copy of ω with a cohesive power of order-type $\omega + \sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$, where $\sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$ denotes the shuffle sum of the order-types in X and the order-type $\omega + \zeta\eta + \omega^*$. Furthermore, if X is finite and non-empty, then there is a computable copy of ω with a cohesive power of order-type $\omega + \sigma(X)$.

Chapter 7. Enumeration reducibility, introduced by Friedberg and Rogers [FR59], is a positive reducibility. The structure of the Turing degrees $\mathcal{D}_T$, properly embeds into the structure of enumeration degrees $\mathcal{D}_e$, and forms an automorphism base for $\mathcal{D}_e$. The images of the Turing degrees are the total degrees. There are cases that $\mathcal{D}_e$ is more useful analyzing the complexity of objects studied in effective mathematics. An early example of this phenomenon was given by Richter [Ric81], who proved that we cannot associate a Turing degree to every countable linear ordering. In fact, the only countable linear orderings that have a Turing degree are the ones with computable presentations. In search for an answer to a similar question—"Does every continuous function on the unit interval have a name of least Turing degree?" —Miller [Mil04] introduced the continuous degrees to measure of the complexity of continuous functions, and, more generally, points in computable metric spaces. He proved that the Turing degrees properly embed into the continuous degrees, and that the continuous degrees, in turn, properly embed into the enumeration degrees. Recently, it was shown that the total degrees are definable in the structure $\mathcal{D}_e$ [CGL⁺16]. We are interested of the question are there more substructures of $\mathcal{D}_e$ with interesting properties.

In Chapter 7. we will answer the following question.

(5) Are there any substructures with interesting properties in the degree structure $\mathcal{D}_e$ of the enumeration degrees, other than the total and the continuous degrees?

With Andrews, Ganchev, et.al. [AGK⁺19] we investigate the properties of a substructure of the enumeration degrees: the cototal degrees. A set $A \subseteq \mathbb{N}$ is *cototal* if it is enumeration reducible to its complement, $\overline{A}$. The *skip* of A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality and related properties, using the skip operator as a tool in our investigation. We give many examples of classes of enumeration degrees that either guarantee or prohibit cototality. Our study of cototality is motivated by two examples of cototal sets that were pointed out to us by Jeandel [Jea15]. He shows that the set of non-identity words in a finitely generated simple group is cototal. Jeandel also gives an example from symbolic dynamics: The set of words that appear in a minimal subshift is cototal.

The complement of a graph of a total function is cototal and these degrees that contain such set we call graph-cototal. An enumeration degree is weakly-cototal if it contains a set A such that $\overline{A}$ has total enumeration degree. We have

$$\text{graph-cototal} \implies \text{cototal} \implies \text{weakly cototal.}$$

We show that these three properties are distinct. The harder separation is to construct a cototal degree that is not graph-cototal, where we use a priority method with an infinite-injury argument relative to $\mathbf{0}'$.

We show that every Σ^0_2-set is cototal, in fact, graph-cototal. We show also that the complement of a maximal independent subset of a computable graph is cototal, and that every cototal degree contains the complement of a maximal independent subset of $\omega^{<\omega}$. Ethan McCarthy [McC18] proves that the same is true of complements of maximal antichains in $\omega^{<\omega}$. We show that joins of nontrivial K-pairs are cototal. And that the natural embedding of the continuous degrees, introduced by Miller [Mil04], into the enumeration degrees maps into the cototal degrees. Finally, we note that Harris [Har10] proved that sets with a good approximation have cototal degree. Recently Miller and M. Soskova proved that the cototal enumeration degrees are exactly the enumeration degrees of sets with good approximations and that the cototal enumerations degrees are dense.

In some ways, the skip is analogous to the jump operator in the Turing degrees. For example, a standard diagonalization argument shows that $A^\Diamond \nleq_{\mathrm{e}} A$. We restate the well-known fact that $A \leq_{\mathrm{e}} B$ if and only if $A^\Diamond \leq_1 B^\Diamond$, mirroring the jump in the Turing degrees. We prove a skip inversion theorem, as analogues of Friedberg's jump inversion theorem. The biggest difference between the skip and the Turing jump is that it is not always the case that $A \leq_e A^\Diamond$ (because not all enumeration degrees are cototal). In fact, there is a set that is its own double skip. We investigate the properties of the skip operator for the class of enumeration degrees of 1-generic sets and skips of nontrivial K-pairs.

We have some open questions arising from this investigation. The main problem is: which cototality notions are first-order definable in the enumeration degrees? Is the skip first-order definable in the enumeration degrees? Kalimullin [Kal03] showed that the enumeration jump is first-order definable. Note that a positive answer to the second question would imply, that the cototal degrees are definable. Another open question: Is there a continuous enumeration degree that is not graph-cototal?

The answers to the questions (1) — (5) are the original contributions of the work. They are from ten papers: [Sos07, SS07, SS09a, SS09b, CFH+18, KAV19, ACG+20, DHM+19, DHM+20, AGK+19] in the references, eight are published, [ACG+20, DHM+20] are submitted for publication. The papers with IF (3,303) are [AGK+19] in *Transactions of the American Mathematical Society*, [KAV19] in the *Journal of Symbolic Logic*, [SS09a, CFH+18] in *Journal of Logic and Computation*. With SJR (0,720) in *Lecture Notes in Computer Science* are [Sos07, DHM+19] and in Proceedings of the Panhellenic Logic Symposium are [SS07, SS09b]. In all joint papers the authors have equal participation. The author has 78 citations (without auto citations), from them on the topics of the dissertation are 48, (29 with IF or SJR, 9 in monographs, 2 in dissertations, 2 without either IF or SJR and 6 are not published yet).

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Chapter 2

Preliminaries

The numbering of the definitions, the propositions, the theorems, etc. are from the dissertation.

2.1 Turing reducibility

The concept of Turing reducibility goes back to Turing [Tur37, Tur39]. Turing wanted to formally capture the notion of an algorithmically computable function as a computable by a mathematical abstract system - Turing machine.

In today's language, we would say that a function $f : \mathbb{N} \to \mathbb{N}$ is *partial computable* if there is a computer program that on input n halts and outputs $f(n)$, or does not halt, if f is not defined. A partial computable function is *computable* if it stops on every input, i.e. if is total. A set $A \subseteq \mathbb{N}$ is computably enumerable (c.e.) if A is a domain of a partial computable function.

In 1939 Turing extended his model of computability by Turing machine to allow for questions to an oracle, i.e. the Turing machine is allowed to use the function f as a primitive function during its computation; that is, the program can ask questions about the value of $f(n)$ for different n's and can use the answers to make decisions while the program is running. The function f is called the oracle of this computation. For a partial function $f : \mathbb{N} \to \mathbb{N}$ we define φ_e^f to be the function computed by the e-th Turing machine using as oracle the function f. We shall assume that if during a computation, the oracle f is called with an argument outside its domain, then the computation is unsuccessfully. For $B \subseteq \mathbb{N}$ we define φ_e^B to denote the function computed by the e-th Turing machine using as oracle the set B, and actually we mean $\varphi_e^{\chi_B}$, where χ_B is the characteristic function of B.

Definition 2.1.1. A partial function f is *Turing reducible to* a partial function g (denoted $f \leq_T g$) if $f = \varphi_e^g$ for some e. We say that a set of natural numbers A is *computable from or Turing reducible to* a set of natural numbers B (denoted $A \leq_T B$) if and only if the characteristic function of the set A is φ_e^B for some natural number e.

The relation $\leq_T$ is a preorder on the subsets of the natural numbers and induces an equivalence relation: $A \equiv_T B$ if and only if $A \leq_T B$ and $B \leq_T A$. The equivalence class of a set A under this relation is the Turing degree of A, denoted by $d_T(A)$. The Turing degrees are ordered by $d_T(A) \leq d_T(B)$ if and only if $A \leq_T B$. The least upper bound of two degrees $d_T(A) \vee d_T(B)$ is $d_T(A \oplus B)$, where $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is the disjoint union of A and B, also known as join of A and B. The set $\mathbf{0}$ of all computable sets is the smallest degree. Finally relativizing the halting problem (The Kleene set) $K = \{e \mid \varphi_e^A(e)\downarrow\}$,to any set A, we have $K^A = \{e \mid \varphi_e^A(e)\downarrow\}$, denoted by A'. The set K^A we call *the jump* A' of A and induces over degree structure *the jump operation* which maps a degree $\mathbf{a}$ to a degree $\mathbf{a}'$, such that $\mathbf{a} < \mathbf{a}'$ (see below).

Post and Kleene [KP54] established basic algebraic facts about the structure of the Turing degrees $\mathcal{D}_T$: it is an uncountable upper semi-lattice with least element and jump operation. They showed that every countable partial ordering can be embedded in the Turing degrees. Their successors, including Shoenfield, Spector, Sacks, Jockusch, Posner and many others, developed more sophisticated methods and showed further structural properties, for example the existence of minimal elements in the structure. The theory of the Turing degrees was revealed as mathematically non-trivial, rich in ideas and results. The next generation of researchers had sufficient tools to tackle problems related to first order definability in the structure. The general question is which interesting relations on $\mathcal{D}_T$ are actually definable in terms of relative computability alone. The most notable result in this direction is by Slaman and Shore [SS99]: they showed that the jump operation is first order definable in $\mathcal{D}_T$. Their solution relies on a methodology introduced by Slaman and Woodin [SW86] to analyze the automorphism group of $\mathcal{D}_T$.

All of the following properties could be found in [Rog67, Soa87, Odi99, Coo04].

A stronger reducibility is the many-one reducibility (m-reducibility), which gives a very natural way of comparing the computability of different — possibly incomputable — sets of natural numbers A and B. The set A is *many-one reducible* (m-reducible) to B ($A \leq_m B$) if there is a computable function h with the property $(\forall n)(n \in A \iff h(n) \in B)$. Let $A \leq_1 B \iff A \leq_m B$ by an one to one computable function.

It is clear that if B is computable (c.e) and $A \leq_m B$ then A is computable (c.e.). Moreover, A is c.e. iff $A \leq_m K$. We call such sets as K complete sets for the c.e. sets.

The set A is *computably enumerable (c.e.) in* B iff for some e $A = dom(\varphi_e^B) = W_e^B$.

One can easily proof from the definitions the following properties:

1. $A \leq_T B \Rightarrow A$ is c.e. in B.
2. A is c.e. in B and $B \leq_T C \Rightarrow A$ is c.e. in C.

Theorem 2.1.2 (Post)**.** $A \leq_T B \iff A$ is c.e. in B and $\overline{A}$ is c.e. in B.

The Turing jump $A' = K^A$ of a set A has the following properties.

Proposition 2.1.3.
1. K^A is c.e. in A.
2. If B is c.e. in A then $B \leq_m K^A$.
3. $A <_T A'$.

Proposition 2.1.4.
$A \leq_T B \iff A' \leq_m B' \iff A' \leq_1 B'$.

Corollary 2.1.5 (Monotonicity of the jump)**.**
$A \leq_T B \Rightarrow A' \leq_T B'$.

Definition 2.1.6. $(d_T(A))' = d_T(A')$.

Since $A <_T K^A$, then $d_T(A) < d_T(A')$.

The computably enumerable sets, and correspondingly degrees, appear in many other branches of mathematics. The solution to Hilbert's tenth problem by Davis, Putnam, Robinson and Matiyasevich [Mat93] essentially relies on the existence of a computably enumerable set that is not computable. Friedberg and Muchnik developed a powerful method used to construct c.e. degrees with specific properties, the priority method. we will use this method in Chapter 7.

A recent result of Sleman and Soskova [SS18] shows a relationship between the local structure $\mathcal{D}_T(\leq \mathbf{0}') = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'\}$ and first order arithmetic, similar to the one proved by Slaman and Woodin[SW05] for the global structure $\mathcal{D}_T$ and second order arithmetic.

The jump hierarchy, also known as the high/low hierarchy, was introduced independently by Cooper (see [Coo04]) and Soare [Soa74]. The jump classes are: $H_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \;\&\; \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}$ of *high*$_n$ degrees and $L_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \;\&\; \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}$ of *low*$_n$ degrees.

2.2 Genericity and forcing

In this section, we give an introduction to the forcing method in computable structure theory. We consider 1-generics, which have relatively low computational complexity. The notion of forcing was introduced by Cohen to prove that the continuum hypothesis does not follow from the ZFC axioms of set theory. Every finite mapping $\tau : [0; n-1] \longrightarrow \mathbb{N}$ we call *a finite part.* We denote by $|\tau| = n$ the length of the interval, where τ is defined.

Definition 2.2.2. The set G is *1-generic*, if for every c.e. set S of finite parts:

$$(\exists \sigma \subseteq G)\underbrace{(\sigma \in S \vee (\forall \rho \supseteq \sigma)(\rho \notin S))}_{\sigma \text{ decides } S}.$$

We call such sets *generic* sets for short. For n-generic sets the difference is that the set of finite parts S is Σ^0_n, not only c.e.(Σ^0_1).

A set of finite parts S is called *dense in* G, if $(\forall \sigma \subseteq G)(\exists \rho \in S)(\sigma \subseteq \rho)$. Equivalently, G is generic, if whenever S is dense in G, then G meets S, i.e. $(\exists \sigma \subseteq G)(\sigma \in S)$.

If G is generic then G is not c.e., and for every c.e. $V \subseteq \mathbb{N}$ if $V \leq_T G$, then V is computable. The only sets that are c.e. in all generic sets are the ones that are already c.e.

The set G *models* the formula $F_e(x)$:

$$G \models F_e(x) \iff \{e\}^G(x)\downarrow \iff x \in W_e^G.$$

The finite part σ *forces* formula $F_e(x)$:

$$\sigma \Vdash F_e(x) \iff \{e\}^\sigma(x)\downarrow.$$

We use these relations in Chapter 3. Here are some properties of these relations, following from the definitions.

1. $\sigma \subseteq G \,\&\, \sigma \Vdash F_e(x) \Rightarrow G \models F_e(x)$.
2. $\sigma \subseteq \rho \,\&\, \sigma \Vdash (\neg)F_e(x) \Rightarrow \rho \Vdash (\neg)F_e(x)$.
3. $G \models F_e(x) \Leftrightarrow (\exists \sigma \subseteq G)(\sigma \Vdash F_e(x))$.

Lemma 2.2.5. The set $\{(\sigma, e, x) \mid \sigma \Vdash F_e(x)\}$ is c.e.

$$G \models \neg F_e(x) \iff G \not\models F_e(x) \iff \neg\{e\}^G(x)\downarrow.$$
$$\sigma \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \sigma)(\rho \nVdash F_e(x)).$$

Theorem 2.2.6. Let G be a generic set. Then

$$G \models \neg F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash \neg F_e(x)).$$

Corollary 2.2.7 (Truth lemma)**.** If G is generic, then

$$G \models (\neg)F_e(x) \iff (\exists \sigma \subseteq G)(\sigma \Vdash (\neg)F_e(x)).$$

Notice that $\{(\sigma, e, x) \mid \sigma \Vdash \neg F_e(x)\} \leq_T \varnothing'$.

Corollary 2.2.8. For every generic G we have $G' \equiv_T G \oplus \varnothing'$.

In Chapter 3. we prove the Friedberg's jump inversion theorem for the structures. Here is the original theorem.

Theorem 2.2.9 (Friedberg's jump inversion theorem)**.** [Fri57] Let $\varnothing' \leq_T B$. There exists a generic G, such that $G' \equiv_T B$, and hence $B \equiv_T G' \equiv_T G \oplus \varnothing'$.

Corollary 2.2.10. There exists a generic $G \not\equiv_T \varnothing$ such that $G' \equiv_T \varnothing'$.

2.3 Enumeration reducibility

Enumeration reducibility was defined by Friedberg and Rogers [FR59] in the late 1950's to capture a notion of reducibility between sets in which only positive information about membership in the set is either used or computed. This notion turns out to be as natural as Turing reducibility in a number of settings, e.g., in group theory and computable model theory.

A set A is *enumeration reducible to* a set B if there is an effective uniform way, given by an *enumeration operator*, to obtain an enumeration of A given any enumeration of B. The enumeration operators are interesting in themselves, as they give the semantics of the type free λ-calculus in graph models, suggested by Plotkin [Plo72] in 1972. The interest in enumeration reducibility is also supported by the fact that the structure of the enumeration degrees contains the structure of the Turing degrees without being elementary equivalent to it. Contemporary defninability results [CGL+16, GS15, GS12, SS12] in the theory of the enumeration degrees show that the structure is useful for the study of the structure of Turing degrees.

Definition 2.3.1. Let A and B be sets of natural numbers. The set A is *enumeration reducible to* the set B, written $A \leq_e B$, if there is a c.e. set W_e, such that:

$$A = W(B) = \{x \mid (\exists D)[\langle x, D\rangle \in W_e \ \& \ D \subseteq B]\},$$

where D is a finite set coded in the standard way.

The definition above associates an effective operator on sets to every c.e. set W_e, the aforementioned enumeration operator. Let $\{\Gamma_e\}_{e\in\omega}$ be an effective list of all enumeration operators.

Just like Turing reducibility, enumeration reducibility is a pre-order on the natural numbers, it induces an equivalence relation $\equiv_e$ and a degree structure $\mathcal{D}_e$. The structure of the enumeration degrees is also an upper semi-lattice. The set $A \oplus B$ is a least upper bound of A and B with respect to $\leq_e$. Two sets A and B are *enumeration equivalent* ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$. The equivalence class of a set A under this relation is its *enumeration degree* $d_e(A)$. The set $\mathcal{D}_e$ consisting of all enumeration degrees, together with the naturally induced partial order and least upper bound operation is the *upper semi-lattice of the enumeration degrees*. It has a least element $\mathbf{0}_e$ consisting of all computably enumerable sets. For an introduction to the enumeration degrees the reader might consult Cooper [Coo90].

There is a strong relationship between the relations that we defined: $A \leq_T B$ if and only if $A \oplus \overline{A}$ is c.e. in B if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$. The set $A \oplus \overline{A}$ codes in a positive way the positive and negative information about a set A. This suggests a relationship between Turing reducibility, enumeration reducibility and the relation "c.e. in" formally expressed as follows.

Proposition 2.3.3. Let A and B be sets of natural numbers.

1. $A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.
2. A is c.e. in B if and only if $A \leq_e B \oplus \overline{B}$.

This gives the natural embedding ι of the Turing degrees into the enumeration degrees ([Med55, Myh61]):

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}).$$

A set A is called *total* if and only if $A \equiv_e A \oplus \overline{A}$. Examples of total sets are the graphs of total functions. An enumeration degree is *total* if it contains a total set. The enumeration degrees in the range of ι coincide with the total enumeration degrees.

The following theorem by Selman shows that the total enumeration degrees play an important role in the structure: an enumeration degree can be characterized by the set of total degrees above it.

Theorem 2.3.4. [Sel71] For any $A, B \subseteq \mathbb{N}$ the following are equivalent:

1. $A \leq_e B$;
2. $\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}$;
3. $\{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x} \text{ is total } \& \ d_e(B) \leq \mathbf{x}\} \subseteq \{\mathbf{x} \in \mathcal{D}_e \mid \mathbf{x} \text{ is total } \& \ d_e(A) \leq \mathbf{x}\}$.

Finally, we give the definition of a jump operator for the enumeration degrees, originally due to Cooper and studied by McEvoy [Coo84, McE85].

Definition 2.3.5. Let $K_A = \{\langle e, x\rangle \mid x \in \Gamma_e(A)\}$. The set

$$A'_e = K_A \oplus \overline{K_A}$$

is called the enumeration jump of A and $d_e(A)' = d_e(A'_e)$.

Note that $K_A = \oplus_{e \in \omega}\Gamma_e(A) = \{\langle e, x\rangle \mid x \in \Gamma_e(A)\}$. It is clear that $K_A \equiv_e A$. Denote by $A^+ = A \oplus \overline{A}$. The enumeration jump is monotone and agrees with the Turing jump in the following sense: $(A')^+ \equiv_e (A^+)'_e$, and $A' \equiv_T (A^+)'_e$ [Coo84, McE85].

We will use Soskov's jump inversion theorem for the enumeration jump:

Theorem 2.3.6. [Sos00] For every enumeration degree $\mathbf{a}$ there exists a total enumeration degree $\mathbf{b}$, such that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$.

The pioneering work on the enumeration degrees dates back to Case [Cas71] and Medvedev [Med55]. In particular, Case shows that $\mathcal{D}_e$ is not a lattice as a consequence of the exact pair theorem and Medvedev proves the existence of quasi-minimal degrees: a degree is *quasi-minimal* if it bounds no nonzero total enumeration degree. Cooper laid the foundations of the study of the enumeration degrees in his survey paper [Coo84] from 1990. He established many important algebraic properties of the global and local structure, such as the lack of minimal elements, which shows that the theory of the enumeration degrees is different from the theory of the Turing degrees. McEvoy [McE85], a student of Cooper, defined the enumeration jump operation, which maps an enumeration degree $\mathbf{a}$ to a total enumeration degree $\mathbf{a}'$, such that $\mathbf{a} <_e \mathbf{a}'$. McEvoy then showed that the embedding ι preserves the jump operation. Kalimullin obtained a definable class of pairs of enumeration degrees which came to be known as Kalimullin pairs, or $\mathcal{K}$-pairs. Kalimullin [Kal03] showed that the enumeration jump is definable in $\mathcal{D}_e$. Ganchev and M. Soskova [GS15] give an alternative proof of the definability of the enumeration jump. Their proof is an instance of a more general phenomenon: they introduce the notion of a maximal $\mathcal{K}$-pair and conjecture that a nonzero enumeration degree is total if and only if it is the least upper bound of the elements of a maximal $\mathcal{K}$-pair. They show that if this conjecture is true than this would imply the first order definability the image (under the embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$) of the relation on Turing degrees "c.e. in". In [GS12] they show that the first order theory of true arithmetic can be interpreted in $\mathcal{D}_e(\leq \mathbf{0'}_e)$, using coding methods based on $\mathcal{K}$-pairs, settling an open problem from Cooper's 1990 survey paper. In [GS15] they show further that the class of low enumeration degrees is first order definable. More importantly, they show that their conjecture for the first order definability of the total Σ^0_2 degrees in $\mathcal{D}_e(\leq \mathbf{0'}_e)$ using maximal $\mathcal{K}$-pairs is true for the local structure $\mathcal{D}_e(\leq \mathbf{0'}_e)$, thus settling the local version of Rogers' 1967 question. The full answer to Rogers' 1967 question is finally obtained through the collaboration of Cai, Ganchev, Lempp, Miller and M. Soskova, confirming Ganchev and Soskova's conjecture.

Theorem 2.3.7 (Cai, Ganchev, Lempp, Miller, M. Soskova)**.** [CGL+16] The total enumeration degrees are first order definable in $\mathcal{D}_e$. A nonzero enumeration degree is total if and only if it is the least upper bound of the members of a maximal Kalimullin pair.

Recent work [GS18] of Ganchev and M. Soskova shows that all classes of high enumeration degrees $H_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'_e \;\&\; \mathbf{a}^{(n)} = \mathbf{0}_e^{(n+1)}\}$ and low enumeration degrees $L_n = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}'_e \;\&\; \mathbf{a}^{(n)} = \mathbf{0}_e^{(n)}\}$ are definable in $\mathcal{D}_e$, for each $n \geq 1$.

The relationship between enumeration degrees and abstract models of computability inspires a new direction in the field of computable structure theory. You could see more in our expository paper with M. Soskova [SS17].

In the last chapter we show our latest results on a subclass of the enumeration degrees — the cototal degrees. Call a set $A \subseteq \mathbb{N}$ *cototal* if $A \leq_e \overline{A}$ and call an enumeration degree *cototal* if it contains a cototal set. We introduce an analog of jump operation - *the skip operator*. We investigate the skip for the class of enumeration degrees of 1-generic sets, studied by Copestake [Cop88]. We define a relativized form of 1-genericity, suitable for the context of the enumeration degrees. We use the notation "relative to $\langle X \rangle$" to denote "relative to the enumeration degree of X" (not of $X \oplus \overline{X}$ as in Turing degrees).

Definition 2.3.8. Let G and X be sets of natural numbers. G is 1-*generic relative to* $\langle X \rangle$ if and only if for every set of finite parts S such that $S \leq_e X$:

$$(\exists \sigma \subseteq G)(\sigma \in S \vee (\forall \tau \supseteq \sigma)[\tau \notin S]).$$

If $X = \emptyset$, then we call G simply 1-*generic* and if $X = \overline{K}$, then G is 2-*generic*.

Note that G is 1-generic relative to X in the usual sense if and only if G is 1-generic relative to $\langle X \oplus \overline{X} \rangle$ in the sense of the definition above.

Definition 2.3.9. An enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total enumeration degree bounded by $\mathbf{a}$ is $\mathbf{0}_e$.

McEvoy [McE85] proved that the enumeration jump restricted to the quasiminimal degrees has the same range as the unrestricted jump operator. Relativizing the notion of quasiminimality, we get the following two notions:

Definition 2.3.10. An enumeration degree $\mathbf{a}$ is a *quasiminimal cover* of an enumeration degree $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and there is no total enumeration degree $\mathbf{x}$ such that $\mathbf{b} < \mathbf{x} \leq \mathbf{a}$. The degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and every total enumeration degree $\mathbf{x}$ bounded by $\mathbf{a}$ is below $\mathbf{b}$.

The next proposition exhibits two important properties of generic enumeration degrees.

Proposition 2.3.11. Let G be 1-generic relative to $\langle X \rangle$.

1. $d_e(G \oplus X)$ is a strong quasiminimal cover of $d_e(X)$.
2. $\overline{G}$ is 1-generic relative to $\langle X \rangle$.

2.4 Degree spectra

The Turing degree spectrum of a countable structure $\mathcal{A}$ provides a natural measure of the complexity of the isomorphism type of that structure. The spectrum of $\mathcal{A}$ is introduced by Richter [Ric81], as the set of those Turing degrees $\mathbf{a}$ such that for some copy $\mathcal{B}$ of $\mathcal{A}$ (that is, for some $\mathcal{B} \simeq \mathcal{A}$ with domain $\mathbb{N}$), the atomic diagram of $\mathcal{B}$ has Turing degree $\mathbf{a}$.

Let $\mathcal{A} = (A, R_1, \ldots, R_k)$ be a countable relational structure. If in the language of the structure there are some functions symbols we represent them by their graphs. An enumeration of $\mathcal{A}$ is a total surjective mapping of $\mathbb{N}$ onto $|\mathcal{A}|$. Given an enumeration f of $\mathcal{A}$ and a subset of B of $|\mathcal{A}|^a$, let

$$f^{-1}(B) = \{\langle x_1, \dots, x_a\rangle \mid (f(x_1), \dots, f(x_a)) \in B\}.$$

Denote by $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=)$. By $D(\mathcal{A})$ we denote the atomic diagram of $\mathcal{A}$.

Definition 2.4.1. The *degree spectrum of* $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{d_T(f^{-1}(\mathcal{A})) \mid f \text{ is an enumeration of } \mathcal{A}\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathcal{A})$, then $\mathbf{a}$ is called *the degree of* $\mathcal{A}$.

We use the following two simple properties of the degree spectra. They are proved by Soskov in [Sos04] for enumeration degree spectra. Suppose that $\mathcal{A}$ is infinite and the domain of $\mathcal{A}$ is the set of the natural numbers.

Proposition 2.4.2. Let f be an arbitrary enumeration of $\mathcal{A}$. Then there exists an injective enumeration g of $\mathcal{A}$ such that $g^{-1}(\mathcal{A}) \leq_{\mathrm{T}} f^{-1}(\mathcal{A})$.

One noticeable difference with the standard definition of Turing degree spectra is that in the definition of the degree spectra, we use the surjective enumerations, instead of bijective enumerations. Proposition 2.4.2 shows that from the point of view of the existence of a degree of a structure this difference does not matter. But the advantage is that the spectrum is always upwards closed (see Proposition 2.4.3). Knight proved in [Kni86], that the spectrum with injective enumerations is closed upwards only in nontrivial structures. In a trivial structure there is a finite tuple such that every permutation of the domain fixing that tuple is an automorphism of $\mathcal{A}$.

Proposition 2.4.3. For every structure $\mathcal{A}$ the degree spectrum $\mathrm{DS}(\mathcal{A})$ is upwards closed.

For every computable ordinal α, following Knight [Kni86] we define *the* α*-th jump spectrum* $DS_\alpha(\mathcal{A})$ of a structure $\mathcal{A}$ to be the set of all αth jumps of the elements of the degree spectrum of $\mathcal{A}$. If $\mathbf{a}$ is the least element of $DS_\alpha(\mathcal{A})$, then $\mathbf{a}$ is called the α*-th jump degree* of $\mathcal{A}$. We show in Chapter 3. that the first jump spectrum is always upwards closed. Richter's [Ric81] proved, as we mention in the introduction that the Turing degree spectrum $DS(\mathcal{A})$ of a linear ordering has a degree then it is computable, i.e. this degree should be $\mathbf{0}$-the set of all computable sets. Knight [Kni86] extended Richter's result to show that the only possible first jump Turing degree of a linear ordering is $\mathbf{0}'$, so not every linear ordering has a first jump degree. Downey and Knight [DK92] proved next that for every computable ordinal α there exists a linear order $\mathcal{A}$ such that $\mathcal{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$ there is no βth jump degree of $\mathcal{A}$. Slaman [Sla98] and independently Wehner

[Weh98] gave an example of a structure $\mathcal{A}$ whose Turing degree spectrum consists of all nonzero Turing degrees, $DS(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{0} < \mathbf{a}\}$. We give very simple proofs of the last two results in Chapter 3. as an application of the jump inversion theorem for structures.

The enumeration degree spectrum $DS_e(\mathcal{A})$ of a countable structure $\mathcal{A}$ is introduced by Soskov [Sos04] as the set of all enumeration degrees generated by the presentations (homomorphic copies in $\mathbb{N}$) of $\mathcal{A}$. It is also closed upwards with respect to total degrees, i.e. if $\mathbf{a} \in DS_e(\mathcal{A})$, $\mathbf{b}$ is a total e-degree and $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{b} \in DS_e(\mathcal{A})$. The minimal degree of $DS_e(\mathcal{A})$, if it exists, is called *e-degree* of $\mathcal{A}$.

Just like Turing reducibility can be expressed in terms of enumeration reducibility, the Turing degree spectrum of a structure $\mathcal{A}$ corresponds to the enumeration degree spectrum of a structure, denoted by $\mathcal{A}^+$, which codes in a positive way both the positive and negative facts about the predicates in $\mathcal{A}$. If $\mathcal{A} = (A, R_1, \dots, R_k)$ then let $\mathcal{A}^+ = (A, R_1, \dots, R_k, \neg R_1, \dots, \neg R_k)$. The image of the Turing degree spectrum of $\mathcal{A}$ under the natural embedding is exactly $DS_e(\mathcal{A}^+)$.

Co-spectrum $CS(\mathcal{A})$ of a structure $\mathcal{A}$ is the set of all lower bounds of the enumeration degree spectrum of the structure $\mathcal{A}$. If $CS(\mathcal{A})$ has a greatest element, then it is the *co-degree* of $\mathcal{A}$. For every computable ordinal α we denote by $CS_\alpha(\mathcal{A})$ the co-spectrum of $DS_\alpha(\mathcal{A})$.

An application of Selman's theorem shows that the co-spectrum of $\mathcal{A}$ depends only on the total elements of the spectrum of $\mathcal{A}$. Soskov [Sos04] proved that for every computable ordinal α and $\mathbf{b} \in DS_\alpha(\mathcal{A})$ there exist total e-degrees $\mathbf{f_0}$ and $\mathbf{f_1}$ such that : $\mathbf{f_0}^{(\alpha)} \leq \mathbf{b}$ and $\mathbf{f_1}^{(\alpha)} \leq \mathbf{b}$, and $\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)} \notin CS_\beta(\mathcal{A})$ for $\beta < \alpha$, and $\{\mathbf{x} \mid \mathbf{x} \in \mathcal{D}_e \ \&\ \mathbf{x} \leq \mathbf{f_0}^{(\beta)} \ \&\ \mathbf{x} \leq \mathbf{f_1}^{(\beta)}\} = CS_\beta(\mathcal{A})$ for every $\beta + 1 < \alpha$. He shows that there exist quasi minimal enumeration degrees for the degree spectrum, i.e. an e-degree $\mathbf{q} \notin CS(\mathcal{A})$, and every total $\mathbf{x} \leq \mathbf{q} \to \mathbf{x} \in DS(\mathcal{A})$, and every total $\mathbf{x} \geq \mathbf{q} \to \mathbf{x} \in CS(\mathcal{A})$. This is an analogue of a quasi minimal degree in $\mathcal{D}_e$.

Kalimullin [Kal09b], building on Wehner's result, transfers these ideas to enumeration degree spectra: There is a structure $\mathcal{A}$ such that $DS_e(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_e \ \&\ \mathbf{a} > \mathbf{0}_e\}$.

The co-degree and e-degree of a structure are closely related to what Knight [Kni98] and Montalbán [Mon] call the "enumeration degree of a structure". A set $X \subseteq \mathbb{N}$ is the "enumeration degree" of a structure $\mathcal{A}$ if every enumeration of X computes a copy of $\mathcal{A}$, and every copy of $\mathcal{A}$ computes an enumeration of X. Thus by Selman's theorem the enumeration degree of X is the co-degree of the structure $\mathcal{A}^+$. This co-degree, however has an additional property: $DS(\mathcal{A}^+)$ is exactly the set of total enumeration degrees above $d_e(X)$.

Soskov [Sos04], building on results of Downey and Jockusch [DJ94], and Coles, Downey and Slaman [CDS00] proved that for a torsion free abelian group $\mathcal{G}$ the enumeration degree of the characteristic of the group $S(\mathcal{G})$ is a co-degree of $\mathcal{G}$. Moreover he showed that the first jump degree (the smallest one in $DS_1(\mathcal{G})$) is the enumeration jump of this co-degree. Another consequence of this is that every principal ideal of enumeration degrees is a co-spectrum of a structure, namely the co-spectrum of some torsion free abelian group of rank one. Further Soskov proved [Sos04] that every countable ideal of enumeration degrees is the co-spectrum of a structure.

Understanding which subsets of the Turing degrees can be realized as degree spectra is an important open problem in the area. A natural question here: is every set of degrees that is upwards closed with respect to total elements the enumeration spectrum of a structure? The answer is, of course, 'No'. One way to see this is via the notion of a *base* and its relationship to the existence of a degree.

A subset $\mathcal{B} \subseteq \mathcal{C}$ of a set of enumeration degrees $\mathcal{C}$ is *a base of* $\mathcal{C}$ if $(\forall \mathbf{a} \in \mathcal{C})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a})$. Using generic enumerations and an argument much like that used in Selman's theorem we can show the following.

Theorem 2.4.6. [Sos04] A structure $\mathcal{A}$ has an e-degree if and only if $DS_e(\mathcal{A})$ has a countable base.

In particular the union of two cones above incomparable degrees (and even countable cones) cannot be the enumeration degree spectrum of a structure (just like it cannot be the Turing degree spectrum of a structure). Nevertheless, degree spectra play well with co-spectra and behave structurally with respect to their elements just like the cone of total degrees above a fixed enumeration degree.

2.5 Definability in a structure

Another way to characterize the complexity of a structure $\mathcal{A}$ is to analyze the definable sets in $\mathcal{A}$. This gives a finer measure as it may happen that two structures have the same degree spectra but greatly differ in their definability power and model theoretic properties.

2.5.1 Relatively intrinsically Σ^0_α relations

Let $\mathcal{A} = (A, R_1, R_2, \dots, R_k)$ be a countable structure. For simplicity we suppose that $A = \mathbb{N}$.

Definition 2.5.6. A relation R on $\mathcal{A}$ is relatively intrinsically Σ^0_α in a structure $\mathcal{A}$ if for each $(\mathcal{B}, P) \simeq (\mathcal{A}, R)$ the relation P is Σ^0_α in the atomic diagram $D(\mathcal{B})$, which in our terms means that for every enumeration f of $\mathcal{A}$, $f^{-1}(R) \in \Sigma^0_\alpha(f^{-1}(\mathcal{A}))$.

For example, consider a linear ordering $\mathcal{A} = (A, <)$, and S-successor relation. S is relatively intrinsically Π^0_1 in $\mathcal{A}$, since $\neg S(x,y) \iff x \nless$

$y \vee \exists z(x < z \ \&\ z < y)$ is relatively intrinsically Σ^0_1 in $\mathcal{A}$. The "block" relation $B(x,y) \iff$ there are finitely many elements $z_1,\dots,z_n$ such that $S(x,z_1)$,$S(z_1,z_2),\dots,S(z_n,y)$ is relatively intrinsically Σ^0_2 in $\mathcal{A}$, and there is no Σ^0_2 first order formula, which defines B. But B can be defined by a computable infinite disjunction of such formulas as we shall see in the next subsection.

2.5.2 Computable infinitary formulas

Let L be a fixed computable language. Some mathematical properties, such as the Archimedean property (true of subfields of the ordered field of reals), are expressed in a natural way by an infinitely long formula. We consider formulas of $L_{\omega_1,\omega}$ (see Keisler [Kei71]). Here ω_1 indicates that the disjunctions and conjunctions are over only countable sets, and ω indicates that there is only finite nesting of quantifiers. For example, in the language of ordered fields, there is a sentence, which adding it to the axioms of ordered fields, the models are exactly the Archimedean ordered fields $(\forall x)\bigvee_n(x < \tau_n)$, where $\tau_n = \underbrace{1+1+\dots+1}_{n}$.

The computable infinitary Σ_α and Π_α formulas, denoted by Σ^c_α and Π^c_α, (see [AK00]) with free variables among $x_1,\dots,x_l$, are defined by transfinite induction on α as follows.

The Σ^c_0 and Π^c_0 formulas are quantifier free formulas on $x_1,\dots,x_l$.

For $\alpha > 0$, a Σ^c_α formula is the disjunction of a c.e. set of formulas of the form $\exists y_1\dots\exists y_m\Psi(x_1,\ \dots,x_l,y_1,\dots,y_m)$, where Ψ is a Π^c_β formula, for some $\beta < \alpha$, with free variables among $x_1,\dots,x_l,\ y_1,\dots,y_m$.

A Π^c_α formula is the conjunction of a c.e. set of formulas of the form $\forall y_1\dots\forall y_m\Psi(x_1,\dots,x_l,y_1,\dots,y_m)$, where Ψ is a Σ^c_β formula, for some $\beta < \alpha$, with free variables among $x_1,\dots,x_l,\ y_1,\dots,y_m$.

Definition 2.5.7. A relation $R \subseteq |\mathcal{A}|^l$ is *definable* in a structure $\mathcal{A}$ by a Σ^c_α formula $\Phi(x_1,\dots,x_l,w_1,\dots,w_r)$, if there are parameters $t_1,\dots,t_r \in |\mathcal{A}|$ such that for every $a_1,\dots,a_l \in |\mathcal{A}|$ the following equivalence holds:

$$(a_1,\dots,a_l) \in R \iff \mathcal{A} \models \Phi(x_1/a_1,\dots,x_l/a_l,w_1/t_1,\dots,w_r/t_r).$$

Ash, Knight, Manasse and Slaman [AKMS89] and independently Chisholm [Chi90] prove that the relatively intrinsically Σ^0_α relations in the structure $\mathcal{A}$ are the definable ones by a Σ^c_α formula with finitely many parameters in $\mathcal{A}$.

Theorem 2.5.8. Let R be a relation on the structure $\mathcal{A}$. The following are equivalent:

1. R is relatively intrinsically Σ^0_α in a structure $\mathcal{A}$.

2. R is definable by a Σ^c_α formula with finitely many parameters in $\mathcal{A}$.

Antonio Montalbán extends in his book [Mon] this result for $\alpha = 1$ not only for relations with a fixed number of arguments but also of those $R \subseteq |A|^{<\omega}$. For example over a Q-vector space V, the relation $LD \subseteq V^{<\omega}$ of linear dependence is always c.e. in V. To enumerate LD in a $D(V)$-computable way, go through all the possible non-trivial Q-linear combinations $q_0 v_0 + \cdots + q_k v_k$ of all possible tuples of vectors $\langle v_0, \dots, v_k \rangle \in V^{<\omega}$, and if you find one that is equal to $\vec{0}$, enumerate $\langle v_0, \dots, v_k \rangle$ into LD. It is clear that we could write a Σ_1^c formula that define this relation but the free variable will not be fixed.

In the definition of effective interpretation 5.1.10 the interpretation is defined by formulas that have no specific arity. So, we use generalized Σ_1^c-definition of a relation. Here, the arity of a formula is the number of its free variables.

Definition 2.5.9 (Generalized Σ_1^c-definition)**.** Let $R \subseteq |\mathcal{A}|^{<\omega}$, and let $\varphi_n(\overline{x}_n)_{n\in\omega}$ be a computable sequence of Σ_1^c formulas, where $\varphi_n(\overline{x}_n)$ has arity n. If for each n, $\varphi_n(\overline{x}_n)$ defines $R \cap \mathcal{A}^n$, then we say that $\bigvee_n \varphi_n(\overline{x}_n)$ is a *generalized* Σ_1^c *definition* of R.

Montalbán [Mon] proved that the result of Theorem 2.5.8 holds for such relations $R \subseteq |\mathcal{A}|^{<\omega}$, i.e. R is relatively intrinsically Σ_1^0 in a structure $\mathcal{A}$ if and only if R is definable by generalized Σ_1^c formulas in $\mathcal{A}$ with parameters. He called these relations relatively intrinsically c.e. (r.i.c.e.).We use this theorem in Chapter 5.

Chapter 3

Jump of a structure

The notion of jump of a structure is an analogue of the jump operation in the degree structures. It contains information about the sets definable by computable infinitary Σ_1^c formulas. This notion has been independently defined various times in the last few years, as we explained in Chapter 1.

In [Sos07, SS07, SS09a], with Soskov, we prove two jump inversion theorems. The first one is an analogue of the Friedberg's Jump inversion theorem. Stukachev [Stu09, Stu10] shows a similar result for the Σ definability. The second theorem shows that every jump degree spectrum $DS_1(\mathcal{A})$ is a degree spectrum of the structure - the jump of $\mathcal{A}$. Independently, later Montalbán [Mon09] proved similar result.

I want to mention that Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon [GHK⁺05] give an idea how the jump inversion - Friedberg's style could be generalized for a computable successor ordinal. They only do it for graphs, but we know [HKSS02] any degree spectrum can be realized as the degree spectrum of a graph. They proved the result above only as a tool to get get other results about and relative intrinsically relations. Vatev [Vat14] uses this idea and proves the jump-inversion theorem for any computable successor ordinal. Soskov proves in [Sos13] that such theorem is not true for computable limit ordinals.

3.1 Jump of a structure

Let $\mathcal{A} = (A; R_1, \dots, R_s)$ be a countable structure and let equality be among the predicates $R_1, \dots, R_s$. We suppose that the domain A of $\mathcal{A}$ is infinite.

Following Moschkovakis [Mos69] the least acceptable extension of the structure $\mathcal{A}$ is defined as follows.

Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let A^* be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π.

We associate an element n^* of A^* with each natural number $n \in \mathbb{N}$ by induction:

$$\begin{array}{ll} 0^* & = 0; \\ (n+1)^* & = \Pi(0, n^*). \end{array}$$

The set of all elements n^* defined above will be denoted by $\mathbb{N}^*$.

Let L and R be the functions on A^* satisfying the following conditions:

$$\begin{array}{l} L(0) = R(0) = 0; \\ (\forall t \in A)(L(t) = R(t) = 1^*); \\ (\forall s, t \in A^*)(L(\Pi(s,t)) = s \ \& \ R(\Pi(s,t)) = t). \end{array}$$

The pairing function allows us to code finite sequences of elements: let $\Pi_1(t_1) = t_1$, $\Pi_{n+1}(t_1, t_2, \dots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \dots, t_{n+1}))$ for every $t_1, t_2, \dots, t_{n+1} \in A^*$.

For each predicate R_i of the structure $\mathcal{A}$ define the respective predicate R_i^* on A^* by

$$R_i^*(t) \iff (\exists a_1 \in A) \dots (\exists a_{r_i} \in A)(t = \Pi_{r_i}(a_1, \dots, a_{r_i}) \ \& \ R_i(a_1, \dots, a_{r_i})).$$

Definition 3.1.1. *Moschovakis' extension of* $\mathcal{A}$ is the structure

$$\mathcal{A}^* = (A^*; A_0, R_1^*, \dots, R_s^*, G_\Pi, G_L, G_R, =),$$

where G_Π, $G_L$and G_R are the graphs of Π, L and R respectively.

Lemma 3.1.2. Let f be an enumeration of $\mathcal{A}$. There exists an enumeration f^* of $\mathcal{A}^*$ such that $(f^*)^{-1}(\mathcal{A}^*) \equiv_T f^{-1}(\mathcal{A})$.

Proposition 3.1.3. $DS(\mathcal{A}) = DS(\mathcal{A}^*)$.

Let f be an enumeration of $\mathcal{A}$. Given natural numbers e and x let

$$f \models F_e(x) \iff x \in W_e^{f^{-1}(\mathcal{A})}$$

and let

$$f \models \neg F_e(x) \iff f \not\models F_e(x).$$

Given a finite part δ and $R \subseteq A^n$, let $\delta^{-1}(R)$ be the finite function on the natural numbers taking values in $\{0, 1\}$ such that

$$\begin{array}{l} \delta^{-1}(R)(u) \simeq 1 \iff (\exists x_1, \dots x_n \in dom(\delta))(u = \langle x_1, \dots, x_n \rangle \ \& \\ (\delta(x_1), \dots, \delta(x_n)) \in R) \text{ and} \\ \delta^{-1}(R)(u) \simeq 0 \iff (\exists x_1, \dots x_n \in dom(\delta))(u = \langle x_1, \dots, x_n \rangle \ \& \\ (\delta(x_1), \dots, \delta(x_n)) \notin R). \end{array} \tag{3.1.1}$$

By $\delta^{-1}(\mathcal{A})$ we shall denote the finite function $\delta^{-1}(R_1) \oplus \dots \oplus \delta^{-1}(R_s)$.

Definition 3.1.4. For any $e, x \in \mathbb{N}$ and for every finite part δ, define *the forcing relations* $\delta \Vdash F_e(x)$ and $\delta \Vdash \neg F_e(x)$ as follows:

$$\delta \Vdash F_e(x) \iff x \in W_e^{\delta^{-1}(\mathcal{A})}$$

$$\delta \Vdash \neg F_e(x) \iff (\forall \tau \supseteq \delta)(\tau \nVdash F_e(x)).$$

The following two properties of the forcing relation are obvious:

(F1) $\delta \Vdash (\neg)F_e(x)$ & $\delta \subseteq \tau \Rightarrow \tau \Vdash (\neg)F_e(x)$.

(F2) For every enumeration f of $\mathcal{A}$,

$$f \vDash F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash F_e(x)).$$

Definition 3.1.5. An enumeration f of $\mathcal{A}$ is *generic* if for every $e, x \in \mathbb{N}$:

$$(\exists \tau \subseteq f)(\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)).$$

Note, that this is equivalent to Definition 2.2.2 for a 1-generic set, only take $G = f^{-1}(\mathcal{A})$ and S to be the set of finite parts $\{\tau \mid \tau \Vdash F_e(x)\}$. It is clear that S is c.e.

We know from Theorem 2.2.6 that for every generic enumeration f of $\mathcal{A}$ for all $e, x \in \mathbb{N}$,

$$f \vDash \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash \neg F_e(x)).$$

With each finite part $\tau \neq \varnothing$ such that $\mathrm{dom}(\tau) = \{x_1, \ldots, x_n\}$ and $\tau(x_1) = s_1, \ldots, \tau(x_n) = s_n$, we associate the element $\tau^* = \Pi_n(\Pi(x_1^*, s_1), \ldots, \Pi(x_n^*, s_n))$ of A^*. Let $\tau^* = 0$ if $\tau = \varnothing$.

Define $K_{\mathcal{A}} = \{\Pi_3(\delta^*, e^*, x^*) \mid (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))$ & $e^*, x^* \in \mathbb{N}^*\}$.

Definition 3.1.6. *The jump of the structure* $\mathcal{A}$ is the following structure:

$$\mathcal{A}' = (A^*; A_0, R_1^*, \ldots, R_s^*, G_\Pi, G_L, G_R, =, K_{\mathcal{A}}).$$

The following proposition follows directly from Lemma 3.1.2.

Proposition 3.1.7. Let f be an enumeration of $\mathcal{A}$. Then

$$(f^*)^{-1}(\mathcal{A}') \equiv_T f^{-1}(\mathcal{A}) \oplus (f^*)^{-1}(K_{\mathcal{A}}).$$

3.2 Every Jump Spectrum is Spectrum

Theorem 3.2.1. For every structure $\mathcal{A}$ there exists a structure $\mathcal{B}$ such that $DS_1(\mathcal{A}) = DS(\mathcal{B})$.

Let $\mathcal{B} = \mathcal{A}'$ defined above. We shall prove that $DS_1(\mathcal{A}) = DS(\mathcal{B})$. We divide the proof into two parts.

Proposition 3.2.2. $DS_1(\mathcal{A}) \subseteq DS(\mathcal{B})$.

Let g be an enumeration of $\mathcal{A}$ such that $g^{-1}(\mathcal{A})' \in DS_1(\mathcal{A})$. By Proposition 2.4.2, there exists an injective enumeration f of $\mathcal{A}$ such that $f^{-1}(\mathcal{A}) \leq_T g^{-1}(\mathcal{A})$. Since $f^{-1}(\mathcal{A})' \leq_T g^{-1}(\mathcal{A})'$ and $DS(\mathcal{B})$ is closed upwards, it is sufficient to show that $d_T(f^{-1}(\mathcal{A}))' \in DS(\mathcal{B})$. For we show that $(f^*)^{-1}(\mathcal{B}) \leq_T f^{-1}(\mathcal{A})'$ and use once more the fact that $DS(\mathcal{B})$ is closed upwards. We use the following idea. Since the natural numbers are represented in $\mathcal{A}^*$ we computably transfer all the natural numbers and finite parts in $\mathbb{N}^*$. We prove that $(f^*)^{-1}(K_\mathcal{A})$ is c.e. in $f^{-1}(\mathcal{A})$. From here it follows that $(f^*)^{-1}(K_\mathcal{A}) \leq_T f^{-1}(\mathcal{A})'$. Therefore, by Proposition 3.1.7, $(f^*)^{-1}(\mathcal{B}) \leq_T f^{-1}(\mathcal{A})'$.

Now we turn to the proof of the reverse inclusion. We shall need the following property of the jump spectrum:

Lemma 3.2.3. Every jump spectrum is closed upwards.

Proposition 3.2.4. $DS(\mathcal{B}) \subseteq DS_1(\mathcal{A})$.

Let $\mathbf{a} \in DS(\mathcal{B})$ and m be an enumeration of $\mathcal{B}$ such that $m^{-1}(\mathcal{B}) \in \mathbf{a}$. By Proposition 2.4.2, there exists an injective enumeration f of $\mathcal{B}$ such that $f^{-1}(\mathcal{B}) \leq_T m^{-1}(\mathcal{B})$. We construct an enumeration g of the structure $\mathcal{A}$ such that $g^{-1}(\mathcal{A})' \leq_\mathrm{T} f^{-1}(\mathcal{B})$. Then, by Lemma 3.2.3, $\mathbf{a} \in DS_1(\mathcal{A})$. Using the above idea we computably transfer all the natural numbers and finite parts in $\mathbb{N}^*$. We construct the enumeration g of $\mathcal{A}$ as 1-generic enumeration using the forcing method and such that the transfer $g^\#$ of g in $\mathbb{N}^*$ is computable in $f^{-1}(\mathcal{B})$.

3.3 Jump Inversion Theorem

Naturally, once we have a jump of a structure, the question of jump inversion arises: Given a structure $\mathcal{A}$ with $DS(\mathcal{A})$ consisting of total degree above $\mathbf{0}'$, is there a structure $\mathcal{C}$ such that $DS_1(\mathcal{C}) = DS(\mathcal{A})$. We prove an even more general Friedberg's style Jump inversion theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$ (so, all elements of $DS(\mathcal{A})$ are above $\mathbf{0}'$). Then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_1(\mathcal{C}) = DS(\mathcal{A})$.

The proof of this theorem uses the method of Marker extensions, which will be discussed in detail in the next subsection.

3.3.1 Marker's Extensions

Marker [Mar89] presented a method of constructing for any $n \geq 1$ an $\aleph_0$-categorical almost strongly minimal theory which is not Σ_n-axiomatizable. Further Goncharov and Khoussainov [GK02] adapted the construction to the general case in order to find for any $n \geq 1$ examples of $\aleph_1$-categorical computable models as well as $\aleph_0$-categorical computable models whose theories are Turing equivalent to $\varnothing^{(n)}$. We shall give the definition of Marker's $\exists$ and $\forall$ extensions following [GK02].

Let $\mathcal{A} = (A; R_1, \ldots, R_s, =)$ be a countable structure such that each predicate R_i has arity r_i.

Marker's $\exists$-extension of R_i, denoted by $R_i^{\exists}$, is defined as follows. Consider a set X_i with new elements such that $X_i = \{x^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid R_i(a_1, \ldots, a_{r_i})\}$. We shall call the set X_i an $\exists$-fellow for R_i. We suppose that all sets A, X_1,..., X_s are pairwise disjoint.

The predicate $R_i^{\exists}$ is a predicate of arity $r_i + 1$ such that

$$R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff a_1, \ldots, a_{r_i} \in A \ \&\ x \in X_i \ \&\ x = x^i_{\langle a_1,\ldots,a_{r_i}\rangle}.$$

The property of $R_i^{\exists}$ is that for every $a_1, \ldots, a_{r_i} \in A$

$$(\exists x \in X_i) R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff R_i(a_1, \ldots, a_{r_i}). \tag{3.3.1}$$

Definition 3.3.1. The structure $\mathcal{A}^{\exists}$ is defined as follows:

$$(A \cup \bigcup_{i=1}^{s} X_i; R_1^{\exists}, \ldots, R_s^{\exists}, X_1, \ldots, X_s, =),$$

where each $R_i^{\exists}$ is the Marker's $\exists$-extension of R_i with the $\exists$-fellow X_i.

Further, Marker's $\forall$-extension of $R_i^{\exists}$, denoted by $R_i^{\exists\forall}$, is defined as follows. Consider an infinite set Y_i of new elements such that

$$Y_i = \{y^i_{\langle a_1,\ldots,a_{r_i},x\rangle} : \neg R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \ \&\ a_1, \ldots, a_{r_i} \in A, \ \&\ x \in X_i\}.$$

We shall call the set Y_i a $\forall$-fellow for $R_i^{\exists}$. We suppose that all sets A, X_1,..., X_s and Y_1,..., Y_s are pairwise disjoint.

The predicate $R_i^{\exists\forall}$ is a predicate of arity $r_i + 2$ such that

1. If $R_i^{\exists\forall}(a_1, \ldots, a_{r_i}, x, y)$ then $a_1, \ldots, a_{r_i} \in A$, $x \in X_i$ and $y \in Y_i$;
2. If $a_1, \ldots, a_{r_i} \in A$, $\&\ x \in X_i \ \&\ y \in Y_i$ then

$$\neg R_i^{\exists\forall}(a_1, \ldots, a_{r_i}, x, y) \iff y = y^i_{\langle a_1,\ldots,a_{r_i},x\rangle} \ .$$

From the definition of $R_i^{\exists\forall}$ it follows that if $a_1, \ldots, a_{r_i} \in A$ and $x \in X_i$ then

$$(\forall y \in Y_i) R_i^{\exists\forall}(a_1, \ldots, a_{r_i}, x, y) \iff R_i^{\exists}(a_1, \ldots, a_{r_i}, x). \tag{3.3.2}$$

Definition 3.3.2. The structure $\mathcal{A}^{\exists\forall}$ is defined as follows

$$(A \cup \bigcup_{i=1}^{s} X_i \cup \bigcup_{i=1}^{s} Y_i; R_1^{\exists\forall}, \dots, R_s^{\exists\forall}, X_1, \dots, X_s, Y_1, \dots, Y_s, =),$$

where X_i is the $\exists$-fellow for R_i and Y_i is the $\forall$-fellow for $R_i^{\exists}$.

The structure $\mathcal{A}^{\exists\forall}$ has the following properties:

Proposition 3.3.3. 1. Let $a_1, \dots, a_{r_i} \in A$. Then:

(a) $R_i(a_1, \dots, a_{r_i}) \iff (\exists x \in X_i)(\forall y \in Y_i) R_i^{\exists\forall}(a_1, \dots, a_{r_i}, x, y)$;

(b) If $R_i(a_1, \dots, a_{r_i})$ then there exists a unique $x \in X_i$ such that $(\forall y \in Y_i) R_i^{\exists\forall}(a_1, \dots, a_{r_i}, x, y)$;

2. For each sequence $a_1, \dots, a_{r_i} \in A$ and $x \in X_i$ there exists at most one $y \in Y_i$ such that $\neg R_i^{\exists\forall}(a_1, \dots, a_{r_i}, x, y)$;

3. For each $y \in Y_i$ there exists a unique sequence $a_1, \dots, a_{r_i} \in A$ and $x \in X_i$ such that $\neg R_i^{\exists\forall}(a_1, \dots, a_{r_i}, x, y)$;

4. For each $x \in X_i$ there exists a unique sequence $a_1, \dots, a_{r_i} \in A$ such that for all $y \in Y_i$ the predicate $R_i^{\exists\forall}(a_1, \dots, a_{r_i}, x, y)$ is true.

Let $\mathcal{A} = (A; R_1, \dots, R_s, =)$ and $\mathcal{B} = (B; P_1, \dots, P_t, =)$ be countable structures in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Suppose that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{=\}$ and $A \cap B = \emptyset$. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{A, B\}$, where A and B are unary predicates.

Definition 3.3.4. *The join of the structures* $\mathcal{A}$ *and* $\mathcal{B}$ is the structure $\mathcal{A} \oplus \mathcal{B} = (A \cup B; R_1, \dots, R_s, P_1, \dots, P_t, A, B, =)$ in the language $\mathcal{L}$, where

(a) the predicate A is true only over the elements of A and similarly B is true only over the elements of B;

(b) each predicate R_i is defined on the elements of A as in the structure $\mathcal{A}$ and false if some of the arguments of R_i are not in A and similarly each predicate P_j is defined as in the structure $\mathcal{B}$ over the elements of B and false if some of the arguments of P_j are not in B.

Lemma 3.3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures and $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$. Then $DS(\mathcal{C}) \subseteq DS(\mathcal{A})$ and $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$.

3.3.2 Representation of $\Sigma_2^0(D)$ Sets

Let $D \subseteq \mathbb{N}$. A set $M \subseteq \mathbb{N}$ is in $\Sigma_2^0(D)$ if there exists a computable in D predicate Q such that

$$n \in M \iff \exists a \forall b Q(n, a, b) \ .$$

Definition 3.3.6. [GK02] If $M \in \Sigma_2^0(D)$ then M is *one-to-one representable* if there exists a computable in D predicate Q with the following properties:

1. $n \in M \iff \exists a \forall b Q(n,a,b)$;
2. $n \in M \iff$ there exists a unique a such that $\forall b Q(n,a,b)$;
3. for every pair $\langle n,a\rangle$ there is at most one b such that $\neg Q(n,a,b)$;
4. for every b there is a unique pair $\langle n,a\rangle$ such that $\neg Q(n,a,b)$;
5. for every a there exists a unique n such that $\forall b Q(n,a,b)$.

The predicate Q from the above definition is called *an one-to-one representation of M*. Goncharov and Khoussainov [GK02] proved the following lemma:

Lemma 3.3.7. If M is a co-infinite $\Sigma^0_2(D)$ subset of $\mathbb{N}$ and there is an infinite computable in D subset S of M such that $M \setminus S$ is infinite, then M has an one-to-one representation.

Let $\mathcal{A} = (A; R_1, \dots, R_s, =)$ be a countable structure. Recall that the set A is infinite. We can easily find a structure $\mathcal{A}^\#$ with the same degree spectrum as $\mathcal{A}$ and such that for every injective enumeration $f^\#$ of $\mathcal{A}^\#$ and for each predicate R of $\mathcal{A}^\#$ the set ${f^\#}^{-1}(R)$ is co-infinite and there is a computable infinite subset S of ${f^\#}^{-1}(R)$ such that ${f^\#}^{-1}(R) \setminus S$ is infinite.

Lemma 3.3.8. There is a structure $\mathcal{A}^\#$, such that $DS(\mathcal{A}) = DS(\mathcal{A}^\#)$ and for every injective enumeration $f^\#$ of $\mathcal{A}^\#$ and each nontrivial predicate ${R_i}^\#$ the set ${f^\#}^{-1}({R_i}^\#)$ is co-infinite and there is a computable infinite set $S \subseteq {f^\#}^{-1}({R_i}^\#)$ such that ${f^\#}^{-1}({R_i}^\#) \setminus S$ is infinite.

3.3.3 The Jump Inversion Theorem

Theorem 3.3.9. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_1(\mathcal{B})$. Then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_1(\mathcal{C}) = DS(\mathcal{A})$.

Let $\mathcal{A} = (A; R_1, \dots, R_s, =)$. For every predicate R_i consider a new predicate R_i^c which is equal to the negation of R_i.

By Lemma 3.3.8 we may suppose that for every injective enumeration f of $\mathcal{A}$ and each nontrivial predicate R_i the sets $f^{-1}(R_i)$ and $f^{-1}(R_i^c)$ are co-infinite and there are computable infinite sets $S \subseteq f^{-1}(R_i)$ and $P \subseteq f^{-1}(R_i^c)$ such that $f^{-1}(R_i) \setminus S$ and $f^{-1}(R_i^c) \setminus P$ are infinite.

We extend the structure $\mathcal{A}$ including the negations of the predicates as follows:

$$\overline{\mathcal{A}} = (A; R_1, R_1^c, \dots, R_s, R_s^c, =).$$

It is clear that $DS(\mathcal{A}) = DS(\overline{\mathcal{A}})$ since for each enumeration f of $\mathcal{A}$ we have that $f^{-1}(\mathcal{A}) \equiv_T f^{-1}(\overline{\mathcal{A}})$.

Consider now the structure $\overline{\mathcal{A}}^{\exists\forall}$. Let X_j be the $\exists$-fellow of $\overline{R}_j$ and Y_j be the $\forall$-fellow of $\overline{R}_j^{\exists}$, $j = 1,\dots,2s$.

Without loss of generality we may assume that the structures $\mathcal{B} = (B; P_1,\dots,P_t,=)$ and $\overline{\mathcal{A}}^{\exists\forall}$ are disjoint.

Let $\mathcal{C} = \mathcal{B} \oplus \overline{\mathcal{A}}^{\exists\forall}$. By Lemma 3.3.5, $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$. We prove that $DS_1(\mathcal{C}) = DS(\overline{\mathcal{A}})$ using Proposition 3.3.3 and Lemma 3.3.7.

3.4 Some Applications

Let $\mathcal{A}^{(n)}$ be the n-th jump of structure $\mathcal{A}$ defined inductively:

$$\mathcal{A}^{(0)} = \mathcal{A}; \qquad \mathcal{A}^{(n+1)} = (\mathcal{A}^n)'.$$

Clearly $DS_0(\mathcal{A}) = DS(\mathcal{A})$ and $DS_{n+1}(\mathcal{A}) = \{\mathbf{a}' : \mathbf{a} \in DS_n(\mathcal{A})\}$. Using this and Theorem 3.2.1, one can easily see by induction on n that for every n there exists a structure $\mathcal{A}^{(n)}$ such that $DS_n(\mathcal{A}) = DS(\mathcal{A}^{(n)})$.

Theorem 3.4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be structures such that $DS(\mathcal{A}) \subseteq DS_n(\mathcal{B})$. Then there exists a structure $\mathcal{C}$ such that $DS(\mathcal{C}) \subseteq DS(\mathcal{B})$ and $DS_n(\mathcal{C}) = DS(\mathcal{A})$.

The definitions above of the jump spectrum can be naturally generalized for all computable ordinals α. In [DK92] Downey and Knight proved with a fairly complicated construction that for every computable ordinal α there exists a linear order $\mathcal{A}$ such that $\mathcal{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathcal{A}$. Now we can obtain this theorem for the finite ordinals as an application of Theorem 3.3.9. Consider a structure $\mathcal{B}$ such that $DS(\mathcal{B})$ consists of all degrees above $\mathbf{0}^{(n)}$ and has no least element, and such that $\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathcal{B})$. Let $\mathcal{A}$ be any total computable structure, e.g. $\mathcal{A} = (\mathbb{N}; =)$ Clearly $DS(\mathcal{B}) \subseteq DS_n(\mathcal{A})$. By Theorem 3.4.2 there exists a structure $\mathcal{C}$ such that $DS_n(\mathcal{C}) = DS(\mathcal{B})$. Therefore $\mathcal{C}$ does not have a n-th jump degree and so no k-th jump degree for $k \leq n$. On the other hand $DS_{n+1}(\mathcal{C}) = DS_1(\mathcal{B})$ and hence the $(n+1)$-th jump degree of $\mathcal{C}$ is $\mathbf{0}^{(n+1)}$. Such a structure $\mathcal{B}$ is constructed as a torsion free abelian group with a characteristic a quasi-minimal relative to $\varnothing^n$ set S, such that $\varnothing^{(n+1)} \equiv_T S'_e$. The set could be constructed relativizing the Jump inversion theorem of McEvoy [McE85].

An easy application of Theorem 3.2.1 is the main property of the jump of a structure. Consider a relation $R \subseteq A^n$. The relation R is relatively intrinsically Σ^0_2 on $\mathcal{A}$ if and only if R is relatively intrinsically Σ^0_1 on $\mathcal{A}'$.

Our next application is a generalization of results of Slaman [Sla98] and Wehner [Weh98]. They give an example of a structure with degree spectrum consisting of all noncomputable Turing degrees.

Theorem 3.4.3. [Weh98] There is a family of finite sets, which has no c.e. enumeration, i.e. c.e. universal set, and for each noncomputable set X there is a enumeration computable in X.

First we relativize this theorem.

Theorem 3.4.4. Let $B \subseteq \mathbb{N}$. There is a family $\mathcal{F}$ of sets, which has no c.e. in B enumeration, and for each set $X >_T B$ there is a enumeration of the family $\mathcal{F}$, computable in X.

Following an idea of Kalimullin [Kal09b] we consider the following family of sets

$$\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F \text{ finite set, } F \neq W_n^B\}.$$

Proposition 3.4.5. Let $X \subseteq \mathbb{N}$. If a universal for $\mathcal{F}$ set U is c.e. in X then $X >_T B$.

Proposition 3.4.6. Let $B <_T X$. There exists a universal set U for the family $\mathcal{F}$, such that $U \leq_T X$.

Theorem 3.4.7. (Wehner, Slaman)[Weh98][Sla98] There is a structure $\mathcal{C}$, for which $DS(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{0}\}$.

The relativized result is the following:

Theorem 3.4.8. For each $n \in \mathbb{N}$ and every Turing degree $\mathbf{b} \geq \mathbf{0}^{(n)}$ there exists $\mathcal{C}$, for which $DS_n(\mathcal{C}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{b}\}$.

We construct the structure $\mathcal{A}$, such that $DS(\mathcal{A}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{b}\}$, using the family $\mathcal{F}$ in the same way as is done in [Weh98]. Let $\mathcal{B} = (\mathbb{N}; =)$. It is clear that $\mathbf{b} \in DS_n(\mathcal{B})$ for each $\mathbf{b} \geq \mathbf{0}^{(n)}$. Thus $DS(\mathcal{A}) \subseteq DS_n(\mathcal{B})$. By the jump inversion Theorem 3.4.2 there exists a structure $\mathcal{C}$, such that $DS_n(\mathcal{C}) = DS(\mathcal{A})$.

In conclusion would like to point out that the Jump inversion theorem gives a method to lift some interesting results for degree spectra to the nth jump spectra.

Chapter 4

Strong jump inversion

We present a general result with sufficient conditions for a countable structure to admit strong ump inversion. We show several classes of structures where these conditions apply, such as some classes of linear orderings, Boolean algebras, trees, models of theories with few types and differentially closed fields. These investigations are join with Wesley Calvert, Andrey Frolov, Valentina Harizanov, Julia Knight, Charles McCoy, Stefan Vatev, and started when most of them visited Sofia in 2013 and are published in the paper [CFH+18].

4.1 Canonical jump and strong jump inversion

We are interested in the following notion of jump inversion.

Definition 4.1.1. A structure $\mathcal{A}$ *admits strong jump inversion* provided that for all sets X, if X' computes $D(\mathcal{C})'$ for some $\mathcal{C} \cong \mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$.

Remark 4.1.2. The structure $\mathcal{A}$ admits strong jump inversion iff for all X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X. Here when we say that $\mathcal{C}$ is low over X, we mean that $D(\mathcal{C})' \leq_T X'$.

The definition of strong jump inversion was motivated by the following result of Downey and Jockusch [DJ94].

Theorem 4.1.3 (Downey-Jockusch)**.** All Boolean algebras admit strong jump inversion.

Here are some further examples of structures that admit strong jump inversion.

Example 4.1.4 (*Equivalence structures*)**.** Each equivalence structure is characterized up to isomorphism by the number of equivalence classes of various sizes. We consider equivalence structures with infinitely many infinite classes. It is well-known, and easy to prove, that such an equivalence structure has an X-computable copy iff the set of pairs (n,k) such that there are at least k classes of size n is Σ^0_2 relative to X. (See [AK00] for a complete characterization of the equivalence structures with computable copies.)

Proposition 4.1.5. Let $\mathcal{A}$ be an equivalence structure with infinitely many infinite classes. Then $\mathcal{A}$ admits strong jump inversion.

Example 4.1.6 (*Abelian p-groups of length* ω)**.** By Ulm's Theorem, a countable Abelian p-group is characterized up to isomorphism by the Ulm sequence and the dimension of the divisible part. For an account of this, see [Kap69]. An Abelian p-group of length ω can be expressed as a direct sum of copies of $Z_{p^{n+1}}$, for finite n, and the Prüfer group Z_{p^∞}. Then the Ulm sequence is $(u_n(G))_{n\in\omega}$, where $u_n(G)$ is the number of direct summands of form $Z_{p^{n+1}}$. The dimension of the divisible part is the number of direct summands of form Z_{p^∞}. It is well-known [AK00], and easy to prove, that if G is an Abelian p-group of length ω with a divisible part of infinite dimension, then G has an X-computable copy iff the set $\{(n,k) : u_n(G) \geq k\}$ is Σ^0_2 relative to X.

Proposition 4.1.7. Let G be an Abelian p-group of length ω such that the divisible part has infinite dimension. Then G admits strong jump inversion.

Not all countable structures admit strong jump inversion.

Example 4.1.8. Jockusch and Soare [JS91] showed that there are low linear orderings with no computable copy.

Example 4.1.9. Let T be a low completion of PA. There is a model $\mathcal{A}$ such that the atomic diagram $D(\mathcal{A})$, and even the complete diagram $D^c(\mathcal{A})$, are computable in T. Then $D(\mathcal{A})'$ is Δ^0_2. By a well-known result of Tennenbaum, since $\mathcal{A}$ is necessarily non-standard, there is no computable copy.

We used in previous chapter our definition of the jump of structure when we proved the jump inversions theorems. Here we will use the Montalbán's definition from [Mon09] but modified in [Mon12], in order to show that it is equivalent to our notion.

Definition 4.1.10 (Canonical jump)**.** For a structure $\mathcal{A}$, the *canonical jump* is a structure $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$, where $(R_i)_{i\in\omega}$ are relations from which we can uniformly compute all r.i.c.e. relations on $\mathcal{A}$, and from the index i of the relation R_i, we can compute the arity of R_i and a computable Σ^c_1 formula (without parameters) that defines it in $\mathcal{A}$.

Remark 4.1.11. The set $\varnothing'$ is included in the canonical jump. We may give it by a family of relations $R_{f(e)}$, for a computable function f, where $R_{f(e)}$ is always true if $e \in \varnothing'$ and always false otherwise. We may define $R_{f(e)}$ by the computable Σ^c_1 formula $\bigvee_s \tau_{e,s}$, where $\tau_{e,s}$ is $\top$ if e has entered $\varnothing'$ by step s and $\bot$ otherwise.

The proposition below shows that we can express strong jump inversion in terms of copies of the canonical jump structure $\mathcal{A}'$, as opposed to the Turing jump of the atomic diagram for various copies $\mathcal{B}$.

Proposition 4.1.14. For any structure $\mathcal{A}$, the following are equivalent:

(1) $\mathcal{A}$ admits strong jump inversion.

(2) For all sets X, if X' computes a copy of the canonical jump $\mathcal{A}'$ of $\mathcal{A}$, then X computes a copy of $\mathcal{A}$.

(3) For all sets X and Y, if $X' \equiv_T Y'$ and Y computes a copy of $\mathcal{A}$ then so does X.

4.2 General result

In this section, we give a result with conditions sufficient to guarantee that a structure admits strong jump inversion. The result is not difficult to prove. However, there are a number of examples where it applies. To state the result, we need some definitions.

Definition 4.2.1. Let S be a countable family of sets. An *enumeration* of S is a set R of pairs (i,k) such that S is the family of sets $R_i = \{k : (i,k) \in R\}$. If $A = R_i$, we say that i is an *R-index* for A.

Definition 4.2.2.

1. A *B_n-formula* is a finite Boolean combination of ordinary finite elementary Σ_n-formulas.

2. A *B_n-type* is the set of B_n-formulas in the complete type of some tuple in some structure for the language.

Definition 4.2.3. Fix a structure $\mathcal{A}$. Let S be a set of B_1-types including all those realized in $\mathcal{A}$. Let R be an enumeration of S. An *R-labeling of $\mathcal{A}$* is a function taking each tuple $\overline{a}$ in $\mathcal{A}$ to an R-index for the B_1-type of $\overline{a}$.

We are interested in structures $\mathcal{A}$ with the following property.

Definition 4.2.4 (Effective type completion)**.** The structure $\mathcal{A}$ satisfies *effective type completion* if there is a uniform effective procedure that, given a B_1-type $p(\overline{u})$ realized in $\mathcal{A}$ and an existential formula $\varphi(\overline{u},x)$ such that $(\exists x)\varphi(\overline{u},x) \in p(\overline{u})$, yields a B_1-type $q(\overline{u},x)$ with $\varphi(\overline{u},x) \in q(\overline{u},x)$, such that if $\overline{a}$ in $\mathcal{A}$ realizes $p(\overline{u})$, then some b in $\mathcal{A}$ realizes $q(\overline{a},x)$.

Here is our general result.

Theorem 4.2.5. A structure $\mathcal{A}$ admits strong jump inversion if it satisfies the following conditions:

(1) There is a computable enumeration R of a set of B_1-types including all those realized by tuples in $\mathcal{A}$.

(2) $\mathcal{A}$ satisfies effective type completion.

(3) For all sets X, if X' computes the jump of some copy of $\mathcal{A}$, then X' computes a copy of $\mathcal{A}$ with an R-labeling.

Moreover, if $\mathcal{C}$ is a copy of $\mathcal{A}$ with an X'-computable R-labeling, then we get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an X'-computable isomorphism from $\mathcal{B}$ to $\mathcal{C}$.

Remark 4.2.6. For some structures $\mathcal{A}$, Condition (3) is satisfied in a strong way. For any $\mathcal{C} \cong \mathcal{A}$, $D(\mathcal{C})'$ computes an R-labeling of $\mathcal{C}$. Hence, if $\mathcal{A}$ is low, there is a Δ^0_2 isomorphism from $\mathcal{A}$ to a computable copy.

In several examples, $\mathcal{A}$ has effective type completion because it satisfies a property that we call *weak* 1*-saturation*. To describe this property, we need a preliminary definition.

Definition 4.2.7. Suppose $p(\overline{u})$ and $q(\overline{u}, x)$ are B_1-types. We say that $q(\overline{u}, x)$ is *generated by the formulas of* $p(\overline{u})$ *and existential formulas* provided that $q(\overline{u}, x) \supseteq p(\overline{u})$, and for any universal formula $\psi(\overline{u}, x)$ (in the indicated variables), writing $neg(\psi)$ for the natural existential formula logically equivalent to $\neg\psi$, we have $\psi(\overline{u}, x) \in q(\overline{u}, x)$ iff there is a finite conjunction $\chi(\overline{u}, x)$ of existential formulas in $q(\overline{u}, x)$ such that $(\exists x)[\chi(\overline{u}, x) \mathbin{\&} neg(\psi(\overline{u}, x))]$ is not in $p(\overline{u})$.

Definition 4.2.8. The structure $\mathcal{A}$ is *weakly* 1*-saturated* provided that if $p(\overline{u})$ is the B_1-type of a tuple $\overline{a}$, and $q(\overline{u}, x)$ is a B_1-type generated by formulas of $p(\overline{u})$ and existential formulas, then $q(\overline{a}, x)$ is realized in $\mathcal{A}$.

Lemma 4.2.9. Let $p(\overline{u})$ be a B_1-type. Suppose $q(\overline{u}, x)$ is a B_1-type that is generated by formulas of $p(\overline{u})$ and existential formulas. Then $q(\overline{u}, x)$ is consistent with all extensions of $p(\overline{u})$ to a complete type in variables $\overline{u}$.

Proposition 4.2.10. If $\mathcal{A}$ is weakly 1-saturated, then it satisfies effective type completion.

4.3 Examples

4.3.1 Linear orderings

Frolov proved strong jump inversion for two special classes of linear orderings, with further results on complexity of isomorphisms. The results are given in [Fro06], [Fro10], [Fro12]. Here we prove these results using Theorem 4.2.5.

First, we describe the possible B_1 types in linear orderings. Every B_1-type $p(\overline{u})$ is determined uniquely by the sizes of the intervals to the left of the first element, between successive elements, and to the right of the last element. Thus, we can define a computable enumeration R of all B_1-types realized in linear orderings so that from the index i of the B_1-type R_i, we can effectively obtain the sizes of the intervals.

Proposition 4.3.1. Let $\mathcal{A}$ be a linear ordering such that every infinite interval can be split into two infinite parts. Then $\mathcal{A}$ is weakly 1-saturated.

Here is the simpler of the two results on linear orderings.

Theorem 4.3.2. Let $\mathcal{A}$ be a linear ordering such that each element lies on a maximal discrete set that is finite. Suppose there is a finite bound on the sizes of these sets. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy with an isomorphism that is Δ^0_2 relative to X.

The next result, Theorem 4.3.3, is more complicated. Before we state the result, we review some well-known, basic concepts about linear orderings. Recall the *block equivalence relation* $\sim$ on a linear ordering $\mathcal{A}$, where $a \sim b$ iff $[a,b]$ is finite. For any linear ordering $\mathcal{A}$, each equivalence class under this relation is an interval that is either finite or of order type ω, ω^*, or $\zeta = \omega^* + \omega$. Furthermore, the quotient structure $\mathcal{A}/_\sim$ is itself a linear ordering, where each distinct point represents an equivalence class under $\sim$.

In Theorem 4.3.3, for a given $\mathcal{A}$ that is low over X, it is not clear that $\mathcal{A}$ itself has an R-labeling that is Δ^0_2 relative to X. However, we can build a copy $\mathcal{B}$ with such an R-labeling. We write η for the order type of the rationals.

Theorem 4.3.3. Let $\mathcal{A}$ be a linear ordering for which the quotient $\mathcal{A}/_\sim$ has order type η. Suppose also that in $\mathcal{A}$, every infinite interval has arbitrarily large finite successor chains. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 over X from $\mathcal{A}$ to $\mathcal{B}$.

Lemma 4.3.4. Suppose $\mathcal{A}$ is low over X. There is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 over X. Moreover, there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that f is Δ^0_3 relative to X.

Assuming the lemma, we complete the proof of Theorem 4.3.3 as follows. Given $\mathcal{A}$, low over X, the lemma gives a copy $\mathcal{B}$ with an R-labeling that is Δ^0_2 relative to X, and an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_3 relative to X. By Theorem 4.2.5, there is an X-computable copy $\mathcal{C}$ with an isomorphism g from $\mathcal{C}$ to $\mathcal{B}$ that is Δ^0_2 relative to X. Then $f \circ g$ is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ that is Δ^0_3 relative to X.

4.3.2 Boolean algebras

As we mentioned in the introduction, Downey and Jockusch [DJ94] showed that every low Boolean algebra has a computable copy. In [KS00], it is shown that for a low Boolean algebra $\mathcal{A}$, there is a computable copy $\mathcal{B}$ with a Δ^0_4 isomorphism. In unpublished work, Knight and Stob proved that this is best possible, in the sense that there is a low Boolean algebra with no Δ^0_3 isomorphism taking $\mathcal{A}$ to a computable copy $\mathcal{B}$.

For every element a in the Boolean algebra $\mathcal{B}$, we say that *a has size n* if it is the join of n atoms of $\mathcal{B}$. If a is not the join of finitely many atoms of $\mathcal{B}$, then we say that a has *infinite* size. Here we consider Boolean algebras with no 1-atoms, which means that every infinite element splits into two infinite elements.

Lemma 4.3.5. If $\mathcal{A}$ is a Boolean algebra with no 1-atoms, then $\mathcal{A}$ is weakly 1-saturated.

Lemma 4.3.6. Let $\mathcal{A}$ be Boolean algebra with no 1-atom. If $\mathcal{A}$ is low over X, then X' computes a copy $\mathcal{B}$ with an R-labeling. Moreover, there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ that is Δ^0_3 relative to X.

Proposition 4.3.7. Suppose $\mathcal{A}$ is an infinite Boolean algebra with no 1-atoms. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 relative to X.

4.3.3 Trees

We consider some special classes of subtrees of $\omega^{<\omega}$. Our trees grow downward. The top node is $\varnothing$. For the language of trees, we use the predecessor function, where $\varnothing$—the root—is its own predecessor. We consider two special classes of trees. The first is very simple.

Proposition 4.3.8. Suppose $\mathcal{A}$ is a tree such that the top node is infinite (i.e., it has infinitely many successors), and each infinite node has only finitely many successors that are terminal, with the rest all infinite. Then $\mathcal{A}$ admits strong jump inversion.

The second class of trees is a bit more complicated. We use some definitions and notation. If T is a sub-tree of $\omega^{<\omega}$, and $a \in T$, we write T_a for the tree consisting of a and all nodes below.

Definition 4.3.9. For nodes a in a fixed tree T,

(1) we say that a is *finite* if T_a is finite,

(2) we say that a is *infinite* if T_a is infinite. (For the trees we consider below, if a is infinite, we will require not only that T_a is infinite, but also that a has infinitely many successors, so we will have agreement with the definition we used in Proposition 4.3.8.)

Notation. Let a be finite, with T_a the subtree below a. Let T_a^1 be a possible re-labeling of the nodes in T_a in which the nodes in a subtree are labeled ∞. We write $(T_a^1)^*$ for the infinite tree that results from extending the labeled tree T_a^1 so that all new nodes in $(T_a^1)^*$ are labeled ∞, and each node labeled ∞ has infinitely many successors labeled ∞. (No finite node in T_a^1 acquires successors in $(T_a^1)^*$.)

Here is the result for the second class of trees.

Proposition 4.3.10. Suppose T is a subtree of $\omega^{<\omega}$ such that the top node is infinite, and for any infinite node a, there are only finitely many finite successors. Suppose also that for any infinite node a, for any finite successor b, if T_b^1 is a possible re-labeling of T_b making all nodes in a certain subtree infinite, then there are infinitely many successors b_n of a such that $T_{b_n} \cong (T_b^1)^*$. Then T admits strong jump inversion.

We prove that $\mathcal{A}$ is weakly 1-saturated. We have a computable enumeration of the possible finite labeled subtrees, and, hence, of the B_1-types realized in trees of this kind. Let R be this computable enumeration of B_1-types. To apply Theorem 4.2.5, we need the following.

Lemma 4.3.11. There is a copy $\mathcal{B}$ of T with a Δ_2^0 R-labeling.

Applying Theorem 4.2.5, we get a computable copy of T.

4.3.4 Models of a theory with few B_1-types

Lerman and Schmerl [LS79] gave conditions under which an $\aleph_0$-categorical theory T has a computable model. They assumed that the theory is arithmetical and $T \cap \Sigma_{n+1}$ is Σ_n^0 for each n. In [Kni94], the assumption that T is arithmetical is dropped, and, instead, it is assumed that $T \cap \Sigma_{n+1}$ is Σ_n^0 uniformly in n. The proof in [LS79] gives the following.

Theorem 4.3.12 (Lerman-Schmerl)**.** Let T be an $\aleph_0$-categorical theory that is Δ_N^0 and suppose that for all $1 \leq n < N$, $T \cap \Sigma_{n+1}$ is Σ_n^0. Then T has a computable model.

To prove this, Lerman and Schmerl showed the following.

Lemma 4.3.13. For any $n < N$, if $\mathcal{A}$ is a model whose B_{n+1}-diagram is computable in X', and $T \cap \Sigma_{n+2}$ is Σ_1^0 in X, then there is a model $\mathcal{B}$ whose B_n-diagram is computable in X.

Let T be as in the Lerman-Schmerl Theorem. Let $\mathcal{A}$ be a model of T that is low over X. Then the Σ_1 diagram of $\mathcal{A}$ is computable in X'. Of course, $T \cap \Sigma_2$ is Σ_1^0, so it is Σ_1^0 relative to X. The lemma implies that $\mathcal{A}$ has an X-computable copy. In fact, we get the following.

Theorem 4.3.14. Let T be an elementary first order theory, in a computable language, such that $T \cap \Sigma_2$ is Σ_1^0. Suppose that for each tuple of variables $\overline{x}$, there are only finitely many B_1-types in variables $\overline{x}$ consistent with T. Then every model $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ_2^0 relative to X.

First, we show that there is a computable enumeration R of all the B_1-types. Next, we show that $\mathcal{A}$ is weakly 1-saturated. And we prove the next Lemma.

Lemma 4.3.15. If $\mathcal{A}$ is low over X, then there is an R-labeling of $\mathcal{A}$ that is Δ_2^0 relative to X.

Finally, we apply Theorem 4.2.5 to get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ that is Δ_2^0 relative to X.
Note: There are non-$\aleph_0$-categorical theories satisfying the conditions of Theorem 4.3.14.

4.3.5 Differentially closed fields

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A *differential field* is a field with one or more derivations satisfying the following familiar rules:

1. $\delta(u + v) = \delta(u) + \delta(v)$, and
2. $\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v$.

We consider differential fields of characteristic 0, and with a single derivation δ.

Trivially, $\mathbb{Q}$ is a differential field, under the derivation that takes all elements to 0. If a is an element of a differential field K, then *a generates* a differential field $F \subseteq K$, where the elements of F are gotten from a by closing under addition, multiplication, subtraction, division, and derivation.

DCF_0

Roughly speaking, a *differentially closed field* is a differential field in which differential polynomials have roots, where a differential polynomial is a polynomial $p(x)$ in x and its various derivatives. We write DCF_0 for the theory of differentially closed fields (of characteristic 0, with a single derivation). A.

Robinson showed that the theory DCF_0 admits elimination of quantifiers. L. Blum, in her thesis, gave a nice computable set of axioms, showing that the theory is decidable. Thus, the elimination of quantifiers is effective. Blum also showed that DCF_0 is ω-stable. Then general model-theoretic results imply the existence and uniqueness of prime models over an arbitrary set. The existence and uniqueness of differential closures were not proved by algebraic methods—they really used the model theoretic results. For a discussion of differentially closed fields, emphasizing Blum's results, see Sacks [Sac10].

Differential polynomials

We consider *differential polynomials* $p(x)$ in a single variable x. A differential polynomial $p(x)$, over a differential field K, may be thought of as an algebraic polynomial in $K[x, \delta(x), \delta^{(2)}(x), \ldots, \delta^{(n)}(x)]$, for some n. We write $K\langle x\rangle$ for the set of differential polynomials over K. Initially, we let K be $\mathbb{Q}$, where $\delta(q) = 0$ for all $q \in \mathbb{Q}$. Later, K will be a finitely generated extension of $\mathbb{Q}$. Differential fields satisfy the quotient rule—this is easy to prove from the product rule. From this, it follows that if a is an element of a differential field extending K, and F is the differential subfield generated over K by a, then each element of F can be expressed in the form $\frac{p(a)}{q(a)}$, where $p(x), q(x) \in K\langle x\rangle$.

Definition 4.3.16 (Order)**.** For $p(x) \in K\langle x\rangle$, the *order* is the greatest n such that $\delta^{(n)}(x)$ appears non-trivially in $p(x)$. There are some special cases. An algebraic polynomial in x (with no derivatives) has order 0. The 0 polynomial has order ∞.

Definition 4.3.17 (degree, rank, order of ranks)**.** For $p(x) \in K\langle x\rangle$ of finite order n, the *degree* of $p(x)$ is the highest power k of $\delta^{(n)}(x)$ that appears. The *rank* of $p(x)$ is the ordered pair (n, k), where n is the order and k is the degree. We order the possible ranks of differential polynomials lexicographically.

Definition 4.3.18. A differential polynomial $p(x) \in K\langle x\rangle$ of order n is said to be *irreducible* if it is irreducible when considered as an algebraic polynomial in $K[x, \delta(x), \ldots, \delta^{(n)}(x)]$ (think of x and its derivatives as indeterminates). We count the 0 polynomial as irreducible.

Blum's axioms for DCF_0

Blum's axioms say that a differentially closed field (of characteristic 0 and with a single derivation), is a differential field K such that

(1) for any pair of differential polynomials $p(x), q(x) \in K\langle x\rangle$ such that the order of $q(x)$ is less than that of $p(x)$, there is some x satisfying $p(x) = 0$ and $q(x) \neq 0$,

(2) if $p(x)$ has order 0, then $p(x)$ has a root.

The axioms of form (2) say that K is algebraically closed.

Types

We want to understand the types, in any number of variables, realized in models of DCF_0. For a single variable x, each type over $\varnothing$ is determined by an irreducible differential polynomial $p(x) \in \mathbb{Q}\langle x\rangle$. If $p(x) \in \mathbb{Q}\langle x\rangle$ is irreducible of order n, then the corresponding type consists of formulas provable from the axioms of DCF_0, the formula $p(x) = 0$ and further formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$ of order less than n. The formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$ of order less than n, say that $x, \delta(x), \delta^{(2)}(x), \ldots, \delta^{(n-1)}(x)$ are algebraically independent over $\mathbb{Q}$. We allow the case where $p(x)$ is the 0 polynomial, which has order ∞. In this case, the corresponding type λ_p consists of the formulas provable from the axioms of DCF_0 and the formulas $q(x) \neq 0$ for $q(x)$ of all finite orders.

Similarly, for a differential field K, each type over K (to be realized in some extension of K to a model of DCF_0) is determined by an irreducible differential polynomial $p(x) \in K\langle x\rangle$. If $p(x)$ is irreducible of order n, the corresponding type $\lambda_{K,p}$ consists of formulas provable from the axioms of DCF_0, the atomic diagram of K, the formula $p(x) = 0$, and further formulas $q(x) \neq 0$, for $q(x)$ of order less than n. The formulas $q(x) \neq 0$, taken together, say that $x, \delta(x), \ldots, \delta^{(n-1)}(x)$ are algebraically independent over K.

A proof of the following result can be found in Sacks [Sac10], pp. 297-298.

Proposition 4.3.19.

1. If $p(x) \in \mathbb{Q}\langle x\rangle$ is irreducible, the corresponding type λ_p is complete over $\varnothing$. Moreover, all types over $\varnothing$ (in the variable x) have this form.
2. For a differential field K, if $p(x) \in K\langle x\rangle$ is irreducible, then $\lambda_{K,p}$ is a complete type over K, and all types over K (in the variable x) have this form.

Among the types in one variable (over $\varnothing$, or over K), there is a unique type, obtained from the 0 polynomial, that is *differential transcendental*. The other types, obtained from differential polynomials of finite rank, are *differential algebraic*.

Types in several variables

In general, we can determine a type in variables $(x_1, \ldots, x_n)$ by giving the type of x_1 (over $\varnothing$), the type of x_2 over x_1, the type of x_3 over (x_1, x_2), and so on. To describe a type in variables $(x_1, \ldots, x_n)$, we imagine a large differentially closed field M and we consider various elements and differential subfields. The type of x_1 is λ_{p_1} for some irreducible $p_1 \in \mathbb{Q}\langle x_1\rangle$. Let K_1 be the differential subfield of M generated by x_1 over $\mathbb{Q}$, where x_1 satisfies λ_{p_1} in M. The type of x_2 over K_1 is λ_{K_1,p_2} for some irreducible $p_2 \in K_1\langle x_2\rangle$. Let K_2 be the differential field generated by x_2 over K_1. In general, given

K_i generated by $x_1, \dots, x_i$, the type of x_{i+1} over K_i is $\lambda_{K_i, p_{i+1}}$ for some irreducible $p_{i+1} \in K_i\langle x_{i+1}\rangle$, and then K_{i+1} is the differential subfield of M generated by x_{i+1} over K_i.

Toward strong jump inversion

Marker and R. Miller [MM17] showed that all models of DCF_0 admit strong jump inversion. Our goal in this subsection is to obtain this result using our Theorem 4.2.5. In the earlier applications of Theorem 4.2.5, the structures satisfied the condition of effective type completion because they were weakly 1-saturated. Among the countable models of DCF_0, only the saturated one is weakly 1-saturated. There are $2^{\aleph_0}$ non-isomorphic countable models. (In fact, Marker and Miller gave a method for coding an arbitrary countable graph in a model of DCF_0.) We will need to show effective type-completion in some other way. There is a lemma in [MM17] that does exactly this. Since we have effective quantifier elimination, we can work with quantifier-free types. Most of our effort goes into producing a computable enumeration R of the quantifier-free types realized in models of DCF_0. Once we have this, we can show easily that for any model $\mathcal{A}$, $D(\mathcal{A})'$ computes an R-labeling of $\mathcal{A}$. This puts us in position to apply Theorem 4.2.5.

Computable enumeration of types

It may at first seem that it should be easy to produce a computable enumeration of types. After all, the theory DCF_0 is decidable and all types are computable. However, T. Millar [Mil78] gave an example of a decidable theory T, with all types computable, such that there is no computable enumeration of all types. So, we have some work to do.

By quantifier elimination, we can pass effectively from a quantifier-free type $\lambda(\overline{x})$ to the complete type generated by $DCF_0 \cup \lambda(\overline{x})$. In what follows, we will enumerate quantifier-free types. We will consider realizations of the quantifier-free types in differential fields K that are not differentially closed, bearing in mind that a tuple realizing $\lambda(\overline{x})$ in K will realize the corresponding complete type generated by $DCF_0 \cup \lambda(\overline{x})$ in any extension of K to a model of DCF_0.

We eventually give a uniform procedure that, for a given tuple of variables $\overline{x}$, yields an enumeration of the types in $\overline{x}$. But first, we give a procedure for a single variable x in order to elucidate the relevant issues before proceeding to the full procedure. We determine a type $\lambda(x)$ corresponding to each differential polynomial $p(x) \in \mathbb{Q}\langle x\rangle$, irreducible or not. Let $(\varphi_s)_{s\in\omega}$ be a computable list of the atomic formulas in variable x, in order of Gödel number. At each stage, we put into $\lambda(x)$ finitely many formulas, always checking consistency with DCF_0.

At stage 0, we put into the type $\lambda(x)$ just the formula $p(x) = 0$, assuming that this is consistent. We also determine the order of $p(x)$—we can do this just by inspection. At stage s, we will decide φ_s, putting it or its negation into $\lambda(x)$. If $p(x)$ is irreducible, there will be a proof of exactly one of φ_s, $\neg\varphi_s$ from DCF_0, $p(x) = 0$, and the formulas $q(x) \neq 0$, for $q(x) \in \mathbb{Q}\langle x\rangle$ of order less than that of $p(x)$. So, we search for a proof. Being reducible is c.e., and if $p(x)$ is reducible, we will eventually see this.

At stage s, we search until we either find a proof of $\pm\varphi_s$ or discover that $p(x)$ is reducible. If we find a proof of φ_s (or $\neg\varphi_s$), then we add this formula to our type, provided that it is consistent to do so. If we find that $p(x)$ is reducible, then we just decide φ_s so as to maintain consistency with DCF_0. The procedure we have just described gives a type λ corresponding to each $p \in \mathbb{Q}\langle x\rangle$. If p is irreducible, then $\lambda = \lambda_p$. Thus, by considering all $p \in \mathbb{Q}\langle x\rangle$, we get all types in the variable x.

A type in one variable corresponded to a differential polynomial $p(x)$ over $\mathbb{Q}$. Intuitively, we'd like to enumerate types in n variables using all n-tuple of polynomials, according to the pattern described above in types in several variables. Unfortunately, since the fields themselves depend on the polynomials in the tuple, it is not even clear if a potential polynomial would make sense; one of its coefficients might actually be undefined. Therefore, our enumeration construction takes these obstacles into account with a more formal approach. A type in n variables will correspond to an n-tuple of formal differential polynomials $p_1(x_1), \dots, p_n(x_n)$. Here $p_1(x_1)$ is an actual differential polynomial with coefficients in $\mathbb{Q}$. For $i \geq 1$, $p_{i+1}(x_{i+1})$ looks like a differential polynomial, but the coefficients come from a set K_i^F of formal names for possible elements of a differential field generated by elements $x_1, \dots, x_i$. We say more about these formal names below. We define the sets K_i^F and $K_i^F\langle x_{i+1}\rangle$ by induction on i.

The many lemmas below allow us to prove Proposition 4.3.32, the computable enumeration of types, from the basic definitions and results in [Sac10].

Definition 4.3.20.

1. $K_0^F = \mathbb{Q}$, and $K_0^F\langle x_1\rangle = \mathbb{Q}\langle x_1\rangle$,
2. $K_i^F\langle x_{i+1}\rangle$ is the set of formal expressions that look like differential polynomials in the variable x_{i+1} but have coefficients in K_i^F as opposed to a well-defined differential field,
3. K_{i+1}^F consists of the expressions $\frac{r(x_{i+1})}{s(x_{i+1})}$, where $r, s \in K_i^F\langle x_{i+1}\rangle$.

Lemma 4.3.21. Uniformly in n, we can enumerate the n-tuples $p_1(x_1), \dots, p_n(x_n)$, where $p_{i+1}(x_{i+1}) \in K_i^F\langle x_{i+1}\rangle$.

Given an n-tuple of formal differential polynomials $p_1, \dots, p_n$ as above, we will obtain a type $\lambda(x_1, \dots, x_n)$ by producing a sequence of differential fields

$K_0, \ldots, K_n$, where $K_0 = \mathbb{Q}$, and K_{i+1} is generated over K_i by an element x_{i+1} satisfying a chosen type λ_{i+1} that depends on p_{i+1}. In the end, K_n will be generated by $x_1, \ldots, x_n$, and $\lambda(x_1, \ldots, x_n)$ will be the type realized by $x_1, \ldots, x_n$ that generates K_n. We give several lemmas.

Lemma 4.3.22. There is a uniform effective procedure that, given a differential field K and a type $\lambda(x)$ over K, yields a differential field $K' \supseteq K$ that is generated over K by an element x realizing λ.

Given an actual differential field K_i, generated by elements $x_1, \ldots, x_i$, some names from K_i^F have a definite value in K_i, while others do not. Recall that the names are quotients. We do not get a value if the denominator is 0.

Lemma 4.3.23. There is a uniform effective procedure that, given a differential field K_i generated by elements $x_1, \ldots, x_i$, and an element $f \in K_i^F$, determines whether f makes sense, and if so, assigns to f a definite value in K_i.

Lemma 4.3.24. There is a uniform effective procedure that, given $p \in K_i^F\langle x_{i+1}\rangle$ and a differential field K_i generated by elements $x_1, \ldots, x_i$, determines whether p makes sense (i.e., whether the coefficients all have value in K_i), and if so, identifies p with an element of $K_i\langle x_{i+1}\rangle$.

Lemma 4.3.25. There is a uniform effective procedure that, given a differential field K and a differential polynomial $p(x)$ over K, enumerates the differential polynomials $q(x)$ of order lower than that of $p(x)$.

Lemma 4.3.26. There is a uniform effective procedure that, given a differential field K and a differential polynomial $p(x)$ over K, enumerates the proofs of formulas $\varphi(x)$ (with parameters in K) from DCF_0, $D(K)$, $p(x) = 0$, and $q(x) \neq 0$, for q of lower order.

In Lemma 4.3.26, we did not assume that $p(x)$ is irreducible. So, the set of axioms may not generate a consistent, complete type over K.

Lemma 4.3.27. There is a uniform effective procedure that, given a differential field K, enumerates the reducible differential polynomials $p(x)$ over K.

Lemma 4.3.28. Let K be a differential field. For any tuple $\overline{k}$ in K, DCF_0 together with the quantifier-free type of $\overline{k}$ generates a complete type that would be realized by $\overline{k}$ in any extension of K to a model of DCF_0.

Lemma 4.3.29. There is a uniform effective procedure for determining, for a differential field K and a formula $\varphi(\overline{k}, x)$ (with parameters $\overline{k}$ in K), whether $\varphi(\overline{k}, x)$ is consistent with $DCF_0 \cup D(K)$.

Lemma 4.3.30. There is a uniform effective procedure that, given a differential field K and $p(x) \in K\langle x\rangle$, enumerates a type $\lambda(x)$ for x over K. Moreover, if $p(x)$ is irreducible, then $\lambda(x) = \lambda_{K,p}$.

Proposition 4.3.31. Uniformly in n, we can enumerate the types in n variables.

As planned, we combine the enumerations of types in variables $x_1, \ldots, x_n$, for various n.

Proposition 4.3.32. There is a computable enumeration R of all complete types realized in models of DCF_0.

Now, we can prove the result of Marker and Miller, using our Theorem 4.2.5.

Proposition 4.3.33. Every countable model of DCF_0 admits strong jump inversion.

By Proposition 4.3.32, there is a computable enumeration R of the complete types realized in models of DCF_0, and thus, of the B_1 types. Thus, Condition (1) of Theorem 4.2.5 holds. The following lemma shows that Condition (3) holds in the strong way.

Lemma 4.3.34. Let X be a subset of ω, and let $\mathcal{A}$ be a model of DCF_0 that is low over X. Then X' computes an R-labeling of $\mathcal{A}$.

We need to establish Condition (2), effective type completion. There is a uniform effective procedure for computing, from a type $p(\overline{u})$ and a formula $\varphi(\overline{u}, x)$, consistent with $p(\overline{u})$, a type $q(\overline{u}, x)$ such that if $\overline{c}$ satisfies $p(\overline{u})$, then some a satisfies $q(\overline{c}, x)$. Marker and Miller [MM17] needed this for the same reason we do. It is Lemma 4.3 in their paper. (The type $q(\overline{c}, x)$ will be realized in the differential closure of $\overline{c}$.) The conditions for Theorem 4.2.5 are all satisfied. Therefore, $\mathcal{A}$ admits strong jump inversion.

Decidable saturated model of DCF_0

In general, a structure $\mathcal{A}$ is computable if its atomic diagram is computable, and $\mathcal{A}$ is decidable if the complete diagram is computable. By elimination of quantifiers, a model of DCF_0 is decidable iff it is computable. Using Proposition 4.3.32, we can show that the countable saturated model of DCF_0 has a decidable copy. We need the following result from Morley [Mor76].

Theorem 4.3.35. Let T be a countable complete elementary first order theory for a computable language. Then the following are equivalent:

1. T has a decidable saturated model,

2. there is a computable enumeration of all types realized in models of T.

Using Theorem 4.3.35 and Proposition 4.3.32, we get the following.

Corollary 4.3.36. The saturated model of DCF_0 has a decidable copy.

Chapter 5

Effective embeddings and interpretations

There are different notions that describe the coding (and decoding) of a structure $\mathcal{A}$ in another structure $\mathcal{B}$. The main idea is to see which classes of structures have more expressive power. We are interested in cases where there is a uniform effective procedure for coding and decoding, and in cases where there is no such procedure. We give one negative and one positive result.There is a body of work in mathematical logic dealing with comparing the complexity of the classification problem for various classes of structures.

In Model Theory they are looking at *the cardinality of the set of isomorphism types*, we know that the classification problem for the class of countable linear orderings ($2^{\aleph_0}$ many isomorphism types) must be more complicated than the classification problem for the class of Q-vector spaces ($\aleph_0$ many isomorphism types) In Descriptive Set Theory they are using *Borel embeddings* and the $\leq_B$ partial ordering induced by the embeddings, we can make distinctions among classes with $2^{\aleph_0}$ many isomorphism types. For instance, it is known that the class of Abelian p-groups of length ω lies strictly below the class of countable linear orderings in the $\leq_B$ partial ordering.

Friedman and Stanley [FS89] considered a Borel embedding of directed graphs in linear orderings. In [CCKM04], the authors relaxed the convention that the structures have universe $\mathbb{N}$, to allow finite structures. They introduced an effective version of Borel embedding. A Turing computable embedding Φ of class of structures K into another class of structures K' gives a uniform effective procedure for coding each structure from K in a structure from K', which preserves the back-and-forth structure [KMVB07] and isomorphisms. It is based on the Turing operator. Similar notion, based on the enumeration operator is introduced in [CCKM04]. Since the enumeration operator is monotone, it preserves the substructures. Recently, the interest of this notion is growing [GKV18, BGV19].

The decoding may or may not be effective. Some of the known examples of Turing computable embeddings involve uniformly defined effective interpretations. In particular, this is true of the standard codings (due to Lavrov, Nies, and Marker) of directed graphs, or structures from an arbitrary computable language, in undirected graphs. One step of decoding gives us the Medvedev reducibility. Recall that a structure $\mathcal{A}$ is Medvedev reducible to a structure $\mathcal{B}$ if there is a Turing operator Φ, that takes a copy of $\mathcal{B}$ to a copy of $\mathcal{A}$. Let Θ be a Turing computable embedding of directed graphs $\mathcal{A}$ in undirected graphs (see [Mar02]). There is a fixed tuple of existential formulas that give a *uniform* effective interpretation; i.e., for all directed graphs $\mathcal{A}$, these formulas interpret $\mathcal{A}$ in $\Theta(\mathcal{A})$. So, these existential formulas gives us the decoding. It follows that $\mathcal{A}$ is Medvedev reducible to $\Theta(\mathcal{A})$ uniformly; i.e., $\mathcal{A} \leq_s \Theta(\mathcal{A})$ with a fixed Turing operator Φ that serves for all $\mathcal{A}$.

Hirschfeldt, Khoussainov, Shore, and Slinko [HKSS02] give conditions for completeness of a class of structures. The idea is that the structures in such class capture all the theoretical-model and structural properties which a computable structure posses. A class of structures $\mathcal{K}$ is complete with respect to degree spectra, effective dimensions, expansion by constants, and degree spectra of relations if for every structure $\mathcal{B}$ (in a computable language), there is a structure $\mathcal{A} \in \mathcal{K}$ with the following properties $\mathcal{A}$ and $\mathcal{B}$ share many properties—having the same spectrum, the same computable dimension (if $\mathcal{B}$ has a computable copy), which is preserved under expansion by constants, and If $S \subseteq \mathcal{B}$, there exists $U \subseteq \mathcal{A}$, such that S and U have the same spectra of relations. In [HKSS02] is shown that the class of undirected graphs, partial orderings, lattices, the class of rings (with zero- divisors), integral domains of arbitrary characteristic, commutative semigroups, and the class of 2-step nilpotent groups are complete.

A more general notion is considered by Montalbàn [Mon14] - the notion of effective bi-interpretability. Two structures are effectively-bi-interpretable if there are effective-interpretations of each structure in the other and the composition of the isomorphisms interpreting one structure inside the other and then interpreting the other back into the first one to be effective. He shows that the effective bi-interpretability preserves the most computability theoretic properties. A more recent result of R. Miller, Poonen, Schoutens, and Shlapentokh [MPSS18] shows that undirected graphs can be effectively interpreted in fields and fields are on top for effective-bi-interpretability.

In the next section we present our joint results with Julia Knight and Stefan Vatev [KAV19] for coding and decoding graphs in linear orderings. In the second section of this chapter we present an effective interpretation of fields in 2-step nilpotent groups — Heisenberg groups [ACG+20]. The last section is devoted to an interpretation of an algebraic closed field $\mathcal{C}$ with characteristic 0 in a special linear group $SL_2(\mathcal{C})$.

5.1 Coding and decoding of graphs in linear orderings

The class of undirected graphs and and the class of linear orderings both lie on top under Turing computable embeddings. The standard Turing computable embeddings of directed graphs (or structures for an arbitrary computable relational language) in undirected graphs come with uniform effective interpretations. We give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering. Any graph can be Medvedev reducible to the second jump of a linear ordering. For the known Turing computable embedding of graphs in linear orderings, due to Friedman and Stanley [FS89], we show that there is no uniform interpretation defined by $L_{\omega_1\omega}$ formulas; that is, no fixed tuple of $L_{\omega_1\omega}$ formulas can interpret every graph in its Friedman-Stanley ordering.

We assume that the language of each structure is computable.We may assume that the languages are relational. We restrict our attention to structures with universe equal to $\mathbb{N}$. Let $Mod(L)$ be the class of L-structures with this universe. We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. We may identify this, via Gödel numbering, with a set of natural numbers, or with an element of 2^ω. Thus, we think of $Mod(L)$ as a subclass of 2^ω. For a class of structures $\mathcal{K} \subseteq Mod(L)$, we suppose that $\mathcal{K}$ is axiomatized by an $L_{\omega_1\omega}$ sentence. By a result of López-Escobar [LE65], this is the same as assuming that $\mathcal{K}$ is a Borel subclass of $Mod(L)$ closed under isomorphism.

5.1.1 Borel embeddings

The following definition is from [FS89] in order to investigate a classification of classes of structures.

Definition 5.1.1. We say that a class $\mathcal{K}$ is *Borel embeddable* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_B \mathcal{K}'$, if there is a Borel function $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Borel embedding of $\mathcal{K}$ into $\mathcal{K}'$ represents a uniform procedure for coding structures from $\mathcal{K}$ in structures from $\mathcal{K}'$.

Theorem 5.1.2. The following classes lie on top under $\leq_B$, i.e. every structure could be Borel embedded in this class.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

Friedman and Stanley [FS89] defined an embedding of graphs in fields of any fixed characteristic. They also defined an embedding of graphs in linear orderings. For the other classes listed above, Friedman and Stanley credit earlier sources. Lavrov [Lav63] defined an embedding of $Mod(L)$ (structures with a domain $\mathbb{N}$ in the language L) in undirected graphs, for any language L. There are similar constructions due to Nies [Nie96] and Marker [Mar02]. Mekler [Mek81] defined an embedding of graphs in 2-step nilpotent groups. Alternatively, we get an embedding of graphs in 2-step nilpotent groups by composing the embedding of graphs in fields with an earlier embedding by Maltsev [Mal60] of fields in 2-step nilpotent groups.

5.1.2 Turing computable embeddings

Knight and her students consider effective embeddings [CCKM04], [KMVB07].

Definition 5.1.4. We say that a class $\mathcal{K}$ is *Turing computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{tc} \mathcal{K}'$, if there is a Turing operator $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Turing computable embedding represents an effective coding procedure. In [CCKM04] is proven that the same classes from Theorem 5.1.2 are on the top of Turing computable embedding. The reason for this is that the Borel embeddings of Friedman-Stanley, Lavrov, Nies, Marker, Mekler, and Maltsev are all, in fact, Turing computable.

5.1.3 Medvedev reductions

A *problem* is a subset of 2^ω or $\mathbb{N}^\omega$. Problem P is Medvedev reducible to problem Q if there is a Turing operator Φ that takes elements of Q to elements of P. The problems that interest us ask for copies of particular structures, where each copy is identified with an element of 2^ω.

Definition 5.1.6. We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$ if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

Supposing that $\mathcal{A}$ is coded in $\mathcal{B}$, a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

In a number of familiar examples where $\mathcal{A} \leq_s \mathcal{B}$, the structure $\mathcal{A}$ is defined or interpreted in $\mathcal{B}$ using formulas that let us recover a copy of $\mathcal{A}$ from each copy of $\mathcal{B}$.

The notion of Medvedev reducibility captures part of the idea of effective recovery (decoding) of a copy of $\mathcal{A}$ from a copy of $\mathcal{B}$.

5.1.4 Sample embedding

Below, we describe Marker's Turing computable embedding of directed graphs in undirected graphs.

1. For each point a in the directed graph $\mathcal{A}$, the undirected graph $\mathcal{B}$ has a point b_a connected to a triangle.

2. For each ordered pair of points (a, a') from $\mathcal{A}$, $\mathcal{B}$ has a point $p_{(a,a')}$ that is connected directly to b_a and with one stop to $b_{a'}$. The point $p_{(a,a')}$ is connected to a square if there is an arrow from a to a', and to a pentagon otherwise.

For structures $\mathcal{A}$ with more relations, the same idea works—we use more special points and more n-gons.
Fact: For Marker's embedding Φ of directed graphs in undirected graphs, there are finitary existential formulas that, for all inputs $\mathcal{A}$, define the following.

1. the set D of b_a connected to a triangle,

2. the set of ordered pairs $(b_a, b_{a'})$ such that the special point $p_{(a,a')}$ is connected to a square,

3. the set of ordered pairs $(b_a, b_{a'})$ such that the special point $p_{(a,a')}$ is connected to a pentagon.

This guarantees that any copy of $\Phi(\mathcal{A})$ computes a copy of $\mathcal{A}$.

5.1.5 Effective interpretations and computable functors

Informally, a structure $\mathcal{A}$ is effectively interpretable in a structure $\mathcal{B}$ if there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ (as in Model theory [Mar02]), but the domain of the interpretation is allowed to be a subset of $B^{<\omega}$, while in the classical definition it is required to be a subset of B^n for some n), and where all sets in the interpretation are required to be computable within the structure (while in the classical definition they should be first-order definable). The formulas defining the interpretation are *generalized computable infitary* Σ_1^c as we defined in Chapter 2. Definition 2.5.9. A version with parameters of the effective interpretability is introduced by Ershov [Ers85] — the Σ-definability over $\mathbb{HF}(\mathcal{B})$, the structure of hereditarily finite sets over $\mathcal{B}$. It uses the first-order logic over $\mathbb{HF}(\mathcal{B})$, and is studied in Russia over the last twenty years [EPS11, Puz09, MK08, Stu13, Kal09a]. Antonio Montalbán in [Mon, Mon12] shows that Σ-definability over $\mathbb{HF}(\mathcal{B})$ corresponds to effective interpretability in $\mathcal{B}$ with parameters.

Antonio Montalbán defined in [Mon14] a very general kind of interpretation of $\mathcal{A}$ in $\mathcal{B}$ guaranteeing that $\mathcal{A} \leq_s \mathcal{B}$. The tuples in $\mathcal{B}$ that represent elements of $\mathcal{A}$ have no fixed arity.

As we know by a result from [AKMS89], [Chi90], Theorem 2.5.8, for a relation R and a structure $\mathcal{A}$, R is relatively intrinsically c.e. (or Σ_α^0) on $\mathcal{A}$ iff it is defined in $\mathcal{A}$ by a computable Σ_1^c (or computable Σ_α^c) formula, with a finite tuple $\bar{c}$ of parameters in $\mathcal{A}$. Actually, as Montalbán proved in [Mon12], a relation $R \subset A^{<\omega}$ is relatively intrinsically c.e. on $\mathcal{A}$ if it is defined by a generalized computable Σ_1^c formula with no parameters but with with infinitely many free variables.

Example 5.1.9. The dependence relation on tuples in a $\mathbb{Q}$-vector space is a familiar relation with no fixed arity. It is defined by a Σ_1^c formula $\bigvee_n \varphi_n(\overline{x}_n)$ of the kind that we use for effective interpretations. We let $\varphi_n(\overline{x}_n) = \bigvee_\lambda \lambda(\overline{x}_n) = 0$, where λ ranges over the non-trivial rational linear combinations of $\overline{x}_n = (x_1, \ldots, x_n)$.

Definition 5.1.10. A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there is a set $D \subseteq \mathcal{B}^{<\omega}$, Σ_1^c-definable over $\varnothing$, and there are relations $\sim$ and R_i^* on D, computable Δ_1-definable over $\varnothing$, such that $(D, R_i^*)/_\sim \cong \mathcal{A}$.

Above, we described Marker's Turing computable embedding of directed graphs in undirected graphs, and we saw there are uniform finitary existential formulas that in the output directed graph a set D and relations $\pm R^*$ such that (D, R^*) is isomorphic to the input undirected graph. A recent embedding of graphs in fields, due to R. Miller, Poonen, Schoutens, and Shlapentokh [MPSS18], gives a uniform effective interpretation.

Harrison-Trainor, Melnikov, R. Miller, and Montalbán [HTMMM17] defined a second notion witch gives an equivalent definition.

Definition 5.1.11. [Computable functor][HTMMM17]
A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators, Φ and Ψ, with the following features:

(1) For each $\mathcal{C} \cong \mathcal{B}$, we have $\Phi(\mathcal{C}) \cong \mathcal{A}$,

(2) For any $\mathcal{B}_1, \mathcal{B}_2 \cong \mathcal{B}$ and any isomorphism f from $\mathcal{B}_1$ onto $\mathcal{B}_2$, $\Psi(\mathcal{B}_1, \mathcal{B}_2, f)$ is an isomorphism from $\Phi(\mathcal{B}_1)$ onto $\Phi(\mathcal{B}_2)$. The operator Ψ is required to satisfy some natural properties.

   (a) If $\mathcal{B}_1 = \mathcal{B}_2 \cong \mathcal{B}$ and f is the identity function, then $\Psi(\mathcal{B}_1, \mathcal{B}_2, f)$ is the identity on $\Phi(\mathcal{B}_1)$.

   (b) For $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \cong \mathcal{B}$, and isomorphisms f from $\mathcal{B}_1$ to $\mathcal{B}_2$ and g from $\mathcal{B}_2$ to $\mathcal{B}_3$, $\Psi(\mathcal{B}_1, \mathcal{B}_3, g \circ f) = \Psi(\mathcal{B}_2, \mathcal{B}_3, g) \circ \Psi(\mathcal{B}_1, \mathcal{B}_2, f)$.

The main result from [HTMMM17] gives the equivalence of the two notions.

Theorem 5.1.12. For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Corollary 5.1.13. If $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, then $\mathcal{A} \leq_s \mathcal{B}$.

Kalimullin [Kal12] showed that the converse of the corollary fails. We can have a Turing operator Φ taking copies of $\mathcal{B}$ to copies of $\mathcal{A}$ without having a Turing operator Ψ taking triples $(\mathcal{B}_1, \mathcal{B}_2, f)$ to g, where $\mathcal{B}_1, \mathcal{B}_2$ are copies of $\mathcal{B}$ and $\mathcal{B}_1 \cong_f \mathcal{B}_2$ and $\Phi(\mathcal{B}_1) \cong_g \Phi(\mathcal{B}_2)$.

In the proof of Theorem 5.1.12, it is important that the set D in the interpretation consist of tuples from $\mathcal{B}$ of arbitrary arity. The same is true in the proof of the following.

Proposition 5.1.14. If $\mathcal{A}$ is computable, then $\mathcal{A}$ is effectively interpreted in all structures $\mathcal{B}$.

It is also natural to ask whether, when $\mathcal{A}$ is effectively interpreted in $(\mathcal{B}, \bar{b})$ with parameters $\bar{b}$, it must be effectively interpreted in $\mathcal{B}$ without parameters. Kalimullin [Kal12] gave examples providing negative answers to both questions.

Maltsev's embedding Φ of fields in 2-step nilpotent groups involves interpreting F in $\Phi(F)$ using formulas with parameters. Recently, we show that there is a uniform computable functor from $\Phi(F)$ to F. Hence, there is a uniform effective interpretation of F in $\Phi(F)$ in which the formulas do not have parameters. We will prove this in Section 5.3.

5.1.6 Interpretations by more general formulas

We may consider interpretations of $\mathcal{A}$ in $\mathcal{B}$, where D, $\pm \sim$, and $\pm R_i^*$ are defined in $\mathcal{B}$ by Σ_2^c formulas, and we have $(D, (R_i^*)_{i\in\mathbb{N}})/_\sim \cong \mathcal{A}$.

Harrison-Trainor, R. Miller, and Montalbán [HTMM18] proved the analogue of the result from [HTMMM17] in which the interpretations are defined by formulas of $L_{\omega_1\omega}$, and the functors are Borel. Again for an interpretation of $\mathcal{A}$ in $\mathcal{B}$, the set of tuples in $\mathcal{B}$ that represent elements of $\mathcal{A}$ may have arbitrary arity.

Theorem 5.1.15. [HTMM18]
A structure $\mathcal{A}$ is interpreted in $\mathcal{B}$ using $L_{\omega_1\omega}$-formulas iff there is a Borel functor (Φ, Ψ) from $\mathcal{B}$ to $\mathcal{A}$.

5.2 Interpreting graphs in linear orderings

As we have seen, any structure can be effectively interpreted in a graph. Linear orderings do not have so much interpreting power. To show this, we use the following result of Linda Jean Richter [Ric81].

Proposition 5.2.1 (Richter)**.** For a linear ordering L, the only sets computable in all copies of L are the computable sets.

Proposition 5.2.2. There is a graph G such that for all linear orderings L, $G \nleq_s L$.

The following result, from [Kni86], is a lifting of Proposition 5.2.1.

Proposition 5.2.3 (Knight)**.** For a linear ordering L, the only sets computable in all copies of L' (or in the jumps of all copies of L) are the Δ^0_2 sets.

This yields a lifting of Proposition 5.2.2.

Proposition 5.2.4. There is a graph G such that for all linear orderings L, $G \nleq_s L'$.

The pattern above does not continue. The following is well-known (see Theorem 9.12 [AK00]).

Proposition 5.2.5. For any set S, there is a linear ordering L such that for all copies of L, the second jump computes S.

For a set A, the ordering $\sigma(A \cup \{\omega\})$ (the "shuffle sum" of orderings of type n for $n \in A$ and of type ω) consists of densely many copies of each of these orderings. The degrees of copies of $\sigma(A \cup \{\omega\})$ are the degrees of sets X such that A is c.e. relative to $X^{(2)}$. Let $A = S \oplus S^c$, where S^c is the complement of S. Consider the linear ordering $L = \sigma(A \cup \{\omega\})$. Then we have a pair of finitary Σ_3 formulas saying that $n \in S$ iff L has a maximal discrete set of size $2n$ and $n \notin S$ iff L has a maximal discrete set of size $2n+1$. It follows that any copy of $L^{(2)}$ uniformly computes the set S.

Using Proposition 5.2.5, we get the following.

Proposition 5.2.6. For any graph G, there is a linear ordering L such that $G \leq_s L^{(2)}$,

5.2.1 Turing computable embedding of graphs in linear orderings

The class of linear orderings, like the class of graphs, lies on top under Turing computable embeddings. We describe the Turing computable embedding L, given in [FS89], of directed graphs in linear orderings.

Friedman-Stanley embedding. First, let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets. Let $(t_n)_{1\leq n<\omega}$ be a list of the atomic types in the language of directed graphs. We let t_1 be the type of $\emptyset$, we put the types for single elements next, then the types for distinct pairs, then the types for distinct triples, etc. For a graph G, the ordering $L(G)$ is a sub-ordering of $\mathbb{Q}^{<\omega}$, with the lexicographic ordering. The elements of $L(G)$ are the finite sequences $r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k \in \mathbb{Q}^{<\omega}$ such that

1. for $i < n$, $r_i \in A_0$, and $r_n \in A_1$,
2. there is a special tuple in G, of length n, satisfying the atomic type t_m, and k is a natural number less than m,
3. if $n \geq 1$ and the special tuple is $a_1, \ldots, a_n$, then for all i with $1 \leq i \leq n$, $q_i \in A_{a_i}$.

In talks, Knight has claimed, without any proof, that this embedding does not represent an interpretation. Our goal in the rest of the section is to prove the following theorem.

Theorem 5.2.7 (Main Theorem)**.** There do not exist $L_{\omega_1\omega}$-formulas that, for all graphs G, interpret G in $L(G)$.

We begin with some definitions and simple lemmas about $L(G)$.

Definition 5.2.8. Let $b = r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k \in L(G)$. We say that b *mentions* $\overline{a}$ if $\overline{a}$ is the special tuple in G of length n, such that for $1 \leq i \leq n$, $q_i \in A_{a_i}$.

Lemma 5.2.9. Suppose $b \in L(G)$ mentions $\overline{a}$. Then b lies in a maximal discrete interval of some finite size $m \geq 1$. The number m tells us the atomic type of $\overline{a}$; in particular, it tells us the length of $\overline{a}$.

The structure of the linear ordering $L(G)$ does not directly tell us the lengths of the elements b (as elements of $\mathbb{Q}^{<\omega}$). However, if b mentions $\overline{a}$ of length n, then b has length $2n + 2$.

Lemma 5.2.10. If $b \in L(G)$ has length $2n + 2$, then there is an infinite interval around b that consists entirely of elements of length at least $2n + 2$.

Lemma 5.2.11. Let $b, b' \in L(G)$, where $b < b'$, and let d be an element of $[b, b']$ of minimum length. If d mentions $\overline{c}$, then all elements of $[b, b']$ mention extensions of $\overline{c}$.

Let $\overline{b}$ be a tuple in $L(G)$. For each b_i in $\overline{b}$, let $\overline{a}_i$ be the tuple in G mentioned by b_i. The formulas true of $\overline{b}$ in $L(G)$ are determined by the formulas true in G of the various $\overline{a}_i$, together with the "shape" of $\overline{b}$.

Definition 5.2.12. For a tuple $\overline{b} = (b_1, \ldots, b_n)$ in $L(G)$, with $b_1 < b_2 < \ldots < b_n$, the *shape* encodes the following information:

1. the order type of $\overline{b}$—for simplicity, we suppose that $b_1 < b_2 < \ldots < b_n$,
2. the size of each interval (b_i, b_{i+1})—we note that the interval is infinite unless b_i, b_{i+1} belong to the same finite discrete set in $L(G)$, which means that they agree on all but the last term,

3. the location of each b_i in the finite discrete interval to which it belongs,
4. the length of each b_i,
5. for $i < n$, the number k_i such that $2k_i + 2$ is the length of a shortest element d in the interval $[b_i, b_{i+1}]$—d mentions a tuple $\overline{c}$ of length k_i, and all elements of $[b_i, b_{i+1}]$ mention tuples that extend $\overline{c}$.

Proposition 5.2.13. For each n-tuple $\overline{b}$, there exist Π^c_4, and Σ^c_4 formulas in the language of linear orderings saying, in $L(G)$ for any G, that the n-tuple $\overline{x}$ has the same shape as some fixed tuple $\overline{b}$.

Remarks on elements of length 2: Suppose d has length 2. Then $\varnothing$ is the tuple mentioned by d and the atomic type of $\varnothing$ is t_1, so d has the form $r_0 0$, where $r_0 \in A_1$. Note that d is the only element of $L(G)$ that starts with r_0. If $b < d < b'$, then b has first term r and b' has first term r', where $r < r_0 < r'$. Since all A_i are dense in $\mathbb{Q}$, essentially everything happens in the intervals (b, d) and (d, b').

Lemma 5.2.14. Suppose $c < c^* < c'$ in $L(G)$, where c^* has length 2.

(1) For any $\overline{e}$ in (c, ∞), there is an automorphism of (c, ∞) taking $\overline{e}$ to some $\overline{e}'$ in the interval (c, c^*).

(2) For any $\overline{e}$ in $(-\infty, c')$, there is an automorphism of $(-\infty, c')$ taking $\overline{e}$ to some $\overline{e}'$ in the interval (c^*, c').

If $a < b$ in the ordering $L(G)$, we may say that a lies *to the left of* b, or that b lies *to the right of* a.

Lemma 5.2.15. Let $\overline{b}$ be a finite tuple in $L(G)$, and let c be an element of $L(G)$.

(1) There is an automorphism of $L(G)$ taking $\overline{b}$ to a tuple $\overline{b}'$ entirely to the right of c, with elements of length 2 in between.

(2) There is also an automorphism taking $\overline{b}$ to a tuple $\overline{b}''$ entirely to the left of c, with elements of length 2 in between.

5.2.2 The relations $\sim^\gamma$

Below, we recall a family of equivalence relations, defined for pairs of tuples, from the same structure, or from two different structures.

Definition 5.2.16. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for a fixed finite relational language. Let $\overline{a}$ and $\overline{b}$ be tuples of the same length, where $\overline{a}$ is in $\mathcal{A}$ and $\overline{b}$ is in $\mathcal{B}$.

(1) $(\mathcal{A},\overline{a}) \sim^0 (\mathcal{B},\overline{b})$ if the tuples $\overline{a}$ and $\overline{b}$ satisfy the same atomic formulas in their respective structures.

(2) For $\gamma > 0$, $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$ if for all $\beta < \gamma$,

   (a) for all $\overline{c} \in \mathcal{A}$, there exists $\overline{d} \in \mathcal{B}$ such that $(\mathcal{A},\overline{a},\overline{c}) \sim^\beta (\mathcal{B},\overline{b},\overline{d})$,

   (b) for all $\overline{d} \in \mathcal{B}$, there exists $\overline{c} \in \mathcal{A}$ such that $(\mathcal{A},\overline{a},\overline{c}) \sim^\beta (\mathcal{B},\overline{b},\overline{d})$.

Note: We write $\mathcal{A} \sim^\gamma \mathcal{B}$ to indicate that $(\mathcal{A},\varnothing) \sim^\gamma (\mathcal{B},\varnothing)$.

Lemma 5.2.17. Let $\mathcal{A}$ be a computable structure for a finite relational language. For any $\gamma < \omega_1^{CK}$ and for any tuple $\overline{a}$ in $\mathcal{A}$, we can effectively find a $\Pi^c_{2\gamma}$-formula $\varphi^\gamma_{\overline{a}}(\overline{x})$ such that $\mathcal{A} \vDash \varphi^\gamma_{\overline{a}}(\overline{b})$ iff $\overline{a} \sim^\gamma \overline{b}$.

Lemma 5.2.18. Let L be a fixed finite relational language. For any computable ordinal γ, and any tuples of variables $\overline{x}$, $\overline{y}$, of the same length, we can effectively find a computable $\Pi_{2\gamma}$-formula $\varphi^\gamma(\overline{x},\overline{y})$ such that for any L-structure $\mathcal{A}$, and any tuples $\overline{a}$ and $\overline{b}$ from $\mathcal{A}$, $\mathcal{A} \vDash \varphi^\gamma(\overline{a},\overline{b})$ iff $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{A},\overline{b})$.

The next lemma is well-known, and the proof is straightforward.

Lemma 5.2.19. Let $\mathcal{A}$ and $\mathcal{B}$ be structures for the same countable language, and let $\overline{a}$ and $\overline{b}$ be tuples of the same length, in $\mathcal{A}$ and $\mathcal{B}$,respectively. Then for any countable ordinal γ, if $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$, then the Σ^c_γ formulas true of $\overline{a}$ in $\mathcal{A}$ are the same as the those true of $\overline{b}$ in $\mathcal{B}$.

5.2.3 $\sim^\gamma$-equivalence in linear orderings

In a linear orderings, the $\sim^\gamma$-classes of a tuple $\overline{a}$ are determined by the $\sim^\gamma$-classes of the intervals with endpoints in $\overline{a}$. Let $\mathcal{A}$ and $\mathcal{B}$ be linear orderings. Let $\overline{a} = a_1 < \ldots < a_n$ be a tuple in $\mathcal{A}$, and let $\overline{b} = b_1 < \ldots < b_n$ be a tuple in $\mathcal{B}$. Let $I_0,\ldots,I_n$ and $J_0,\ldots,J_n$ be the intervals in $\mathcal{A}$ and $\mathcal{B}$ determined by $\overline{a}$ and $\overline{b}$; i.e., I_0 is the interval $(-\infty,a_1)$ in $\mathcal{A}$, J_0 is the interval $(-\infty,b_1)$ in $\mathcal{B}$, for $i < n$, I_i is the interval (a_i,a_{i+1}) in $\mathcal{A}$, J_i is the interval (b_i,b_{i+1}) in $\mathcal{B}$, I_n is the interval (a_n,∞) in $\mathcal{A}$, and J_n is the interval (b_n,∞) in $\mathcal{B}$. The next lemma is well-known, and the proof is straightforward.

Lemma 5.2.20. $(\mathcal{A},\overline{a}) \sim^\gamma (\mathcal{B},\overline{b})$ iff for $i \le n$, $I_i \sim^\gamma J_i$.

5.2.4 More on the orderings $L(G)$

We return to the orderings of form $L(G)$. In the next subsection, we will prove that there do not exist $L_{\omega_1\omega}$ formulas that, for all G, interpret G in $L(G)$. Roughly speaking, the outline is as follows. We assume that there are such formulas. The formulas are Σ_α, for some countable ordinal α. Moreover, they are X-computable Σ_α for some X such that $\alpha < \omega_1^X$. Taking G to be the ordering ω_1^X, we will produce tuples $\overline{b},\overline{c},\overline{b}'$ in $L(G)$ representing elements a,e,a' of G such that $\overline{b},\overline{c} \sim^\gamma \overline{c},\overline{b}'$, although in G, we have $a < e$ and $a' < e$. This is a contradiction. The current subsection gives several lemmas about the relations $\sim^\gamma$ on tuples in $L(G)$, and about automorphisms of $L(G)$. These lemmas are what we need to produce the tuples $\overline{b},\overline{c},\overline{b}'$.

To start off, we note that if $a_1,a_2 \sim^1 b_1,b_2$, then the sizes of the intervals (a_1,a_2) and (b_1,b_2) match. Moreover, if $a \sim^2 b$, then a and b belong to maximal discrete intervals of the same size.

Lemma 5.2.21. Let $I = (b,b')$, where $b < b'$, and let $J = (c,c')$, where $c < c'$. Suppose $b \sim^\gamma c$ and $b' \sim^\gamma c'$, where some $b^* \in I$ and some $c^* \in J$ each have length 2. Then $I \sim^\gamma J$.

Lemma 5.2.22. Let $\overline{b}_1,\overline{b}_2$, $\overline{c}_1,\overline{c}_2$ be increasing sequences in $L(G)$, where $\overline{b}_1 \sim^\gamma \overline{c}_1$ and $\overline{b}_2 \sim^\gamma \overline{c}_2$. Suppose further that there is an element of length 2 between the last element of $\overline{b}_1$ and the first element of $\overline{b}_2$, and there is an element of length 2 between the last element of $\overline{c}_1$ and the first element of $\overline{c}_2$. Then $\overline{b}_1,\overline{b}_2 \sim^\gamma \overline{c}_1,\overline{c}_2$.

Lemma 5.2.23. Suppose $\overline{b}$, $\overline{b}'$ are tuples in $L(G)$ of the same shape. Let $\overline{a}$, $\overline{a}'$ be the full tuples from G mentioned by the b_i's, or the b_i''s. If $\overline{a} \sim^\gamma \overline{a}'$, then $\overline{b} \sim^\gamma \overline{b}'$.

Definition 5.2.24. We say that $\mathcal{A}$ is a computable infinitary substructure of $\mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$ and for all computable infinitary formulas $\varphi(\overline{x})$ and all $\overline{a}$ in $\mathcal{A}$, $\mathcal{B} \models \varphi(\overline{a})$ iff $\mathcal{A} \models \varphi(\overline{a})$. (The definition is the same as elementary substructure except that the formulas are not elementary (finitary) first order.)

Lemma 5.2.25. Let G_1 and G_2 be directed graphs such that G_1 is a computable infinitary substructure of G_2. Suppose also that G_2 is computable, so $L(G_2)$ is computable. Then $L(G_1)$ is a computable infinitary substructure of $L(G_2)$.

5.2.5 Proof of Theorem 5.2.7

Theorem 5.2.7 says that there are no $L_{\omega_1\omega}$-formulas that, for all directed graphs G, define an interpretation of G in $L(G)$. We introduce the ideas of the proof in Proposition 5.2.27. Among the directed graphs are the linear orderings. The Harrison ordering H [Har68] has order type $\omega_1^{CK}(1+\eta)$. While ω_1^{CK} has no computable copy, H does have a computable copy. It is well known that H and ω_1^{CK} satisfy the same computable infinitary sentences. In fact, they satisfy the same Π_α sentences of $L_{\omega_1\omega}$ for all computable ordinals α.

Let I be the initial segment of H of order type ω_1^{CK}. Thinking of H as a directed graph, we can form the linear orderings $L(H)$ and $L(I)$. By

Proposition 5.1.14, just because H has a computable copy, it is effectively interpreted in every structure $\mathcal{B}$. Our result will say that there are no computable infinitary formulas that define an interpretation of H in $L(H)$ and also define an interpretation of I in $L(I)$.

We apply Lemma 5.2.25 to conclude that $L(I)$ is a computable infinitary substructure of H.

Proposition 5.2.26. $L(I)$ is a computable infinitary substructure of $L(H)$.

Proposition 5.2.27. There do not exist computable infinitary formulas that define an interpretation of H in $L(H)$ and also define an interpretation of I in $L(I)$.

In order to prove Proposition 5.2.27 suppose there are computable infinitary formulas that define an interpretation of H in $L(H)$, and also define an interpretation of I in $L(I)$. Say D, $\sim$, and $\circledless$ are the sets of tuples defined by these formulas in $L(H)$.

For each computable ordinal α, we have a formula $\varphi_\alpha(x)$ saying of an element x in H that $pred(x) = \{y : y < x\}$ has order type α. Let $\psi_\alpha(\overline{x})$ be the translation formula saying of a tuple $\overline{x}$ that it is in D and the set of predecessors of the equivalence class of $\overline{x}$ has order type α. For each computable ordinal α, there is a tuple in D satisfying $\psi_\alpha(\overline{x})$ (for an appropriate $\overline{x}$). Since $L(I)$ is a computable infinitary substructure of $L(H)$, some tuple from D in $L(I)$ also satisfies $\psi_\alpha(\overline{x})$. Moreover, each tuple from D in $L(I)$ satisfies one of the formulas ψ_α. Recall that the ordering H is computable, and so is $L(H)$. We define equivalence relations $\equiv^\gamma$ on D.

Definition 5.2.28. For tuples $\overline{a}$ and $\overline{b}$ in D, let $\overline{a} \equiv^\gamma \overline{b}$ iff

1. $\overline{a}$ and $\overline{b}$ have the same shape and
2. $\overline{a} \sim^\gamma \overline{b}$.

Fact: For each computable ordinal γ and each $\overline{a}$ in D, the $\equiv^\gamma$-class of $\overline{a}$ is defined by a computable infinitary formula.

Lemma 5.2.29. For each computable ordinal γ, there is a $\equiv^\gamma$-class C such that there are arbitrarily large computable ordinals α for which some $\overline{b}$ in C satisfies ψ_α.

Suppose that the formulas defining D, $\circledless$, and $\sim$ are all Σ^c_γ. Since D may have no fixed arity, we mean that there is a computable sequence of Σ^c_γ formulas defining the sets of n-tuples in D, and similarly for $\circledless$ and $\sim$. By Lemma 5.2.29, there is a set $C \subseteq D$ in which all tuples have the same shape and are in the same $\sim^\gamma$-class—in particular, the tuples in C all have the same arity. We choose tuples $\overline{b}$ and $\overline{c}$ in $L(I)$, both belonging to C, such that $\overline{b}$ satisfies ψ_α and $\overline{c}$ satisfies ψ_β, where $\alpha < \beta$.

By Lemma 5.2.15, we may suppose that all elements of the tuple $\overline{b}$ lie to the left of the $<$-first element of $\overline{c}$, and the interval between the $<$-greatest element of $\overline{b}$ and the $<$-first element of $\overline{c}$ contains an element of length 2. Also, by the same lemma, we have a tuple $\overline{b}'$, automorphic to $\overline{b}$, such that all elements of $\overline{b}'$ lie to the right of the $<$-greatest element of $\overline{c}$, and the interval between the $<$-greatest element of $\overline{c}$ and the $<$-first element of $\overline{b}'$ contains an element of length 2. Since $\overline{b}$ satisfies ψ_α and $\overline{c}$ satisfies ψ_β, we should have $L(I) \models \overline{b} \mathbin{\textcircled{<}} \overline{c}$. Since $\overline{b}'$ is automorphic to $\overline{b}$, it should also satisfy ψ_α, so we should have $L(I) \models \overline{b}' \mathbin{\textcircled{<}} \overline{c}$. Applying Lemma 5.2.22, we get the fact that $\overline{b}, \overline{c} \sim^\gamma \overline{c}, \overline{b}'$. Therefore, since $L(I) \models \overline{b} \mathbin{\textcircled{<}} \overline{c}$, and $\textcircled{<}$ is defined by a Σ^c_γ-formula, we have $L(I) \models \overline{c} \mathbin{\textcircled{<}} \overline{b}'$. This is the contradiction that we were expecting when we set out to prove Proposition 5.2.27.

Proposition 5.2.30. There is no interpretation of ω_1^{CK} in $L(\omega_1^{CK})$ defined by computable infinitary formulas.

Suppose we have an interpretation of ω_1^{CK} in $L(\omega_1^{CK})$, defined by computable infinitary formulas. Say that the formulas that define the appropriate D, $\textcircled{<}$, and $\sim$ are Σ^c_γ. Our assumption gives the fact that for a Harrison ordering with well-ordered initial segment I, these formulas interpret I in $L(I)$. However, the assumption does not say that they also interpret H in $L(H)$. Thus, we are not in a position to use the important Lemma 5.2.29.

Lemma 5.2.31. Let $\mathcal{A}$ be a computable structure. If $\mathcal{B}$ satisfies the computable infinitary sentences true in $\mathcal{A}$, then the formulas $\varphi^\gamma_{\overline{d}}$ that define the $\sim^\gamma$-equivalence classes of all tuples in $\mathcal{A}$ also define the $\sim^\gamma$-equivalence classes of all tuples in $\mathcal{B}$. Moreover, if $\mathcal{B} \models \varphi^\gamma_{\overline{d}}(\overline{b})$, then the Σ^c_γ-formulas true of $\overline{b}$ in $\mathcal{B}$ are the same as those true of $\overline{d}$ in $\mathcal{A}$.

The next lemma gives the conclusion of Lemma 5.2.29. The proof involves locating ω_1^{CK} inside a larger ordering similar to the Harrison ordering.

Lemma 5.2.32. In $L(\omega_1^{CK})$, there are tuples $\overline{d}_\alpha$, corresponding to arbitrarily large computable ordinals α, such that all $\overline{d}_\alpha$ are in D, all have the same length and shape, all are $\sim^\gamma$-equivalent, and $\overline{d}_\alpha$ satisfies ψ_α.

We use Barwise-Kreisel Compactness. Let Γ be a Π^1_1 set of computable infinitary sentences describing a structure

$$\mathcal{U} = (U_1 \cup U_2, U_1, <_1, U_2, <_2, F, c)$$

such that

1. U_1 and U_2 are disjoint sets,
2. $(U_1, <_1)$ is a linear ordering that satisfies the computable infinitary sentences true in ω_1^{CK} and H—since H is computable, this is Π^1_1,

3. $(U_2, <_2)$ satisfies the computable infinitary sentences true in $L(\omega_1^{CK})$—this is Π_1^1 since $L(H)$ is computable and $L(I)$ is a computable infinitary substructure of $L(H)$,

4. F is a function from D^{U_2} to U_1 that induces an isomorphism between $(D^{U_2}, \circledless)/_{\sim^{U2}}$ and $(U_1, <_1)$,

5. c is a constant in U_1 such that $c >_1 \alpha$ for all computable ordinals α; i.e., there is a proper initial segment of $<_1$-$pred(c)$ of type α.

Every Δ_1^1 subset of Γ is satisfied by taking copies of ω_1^{CK}, $L(\omega_1^{CK})$, with an appropriate function F, and letting c be a sufficiently large computable ordinal. Therefore, the whole set Γ has a model. Let $\bar{b}$ be an element of D^{U_2} such that $F(\bar{b}) = c$. Let C be the set of tuples of U_2 having the shape of $\bar{b}$ and $\sim^\gamma$-equivalent to $\bar{b}$. Since $(U_2, <_2)$ satisfies the same computable infinitary sentences true in the computable structure $L(H)$, by the lemma above, the $\sim^\gamma$-equivalence class of $\bar{b}$ is defined in $(U_2, <_2)$ by a computable infinitary formula. For each computable ordinal α, we have a computable infinitary sentence χ_α saying that some tuple in C does not satisfy ψ_β for any $\beta < \alpha$. The sentence χ_α is true in our model of Γ, witnessed by $\bar{b}$ such that $F(\bar{b}) = c$. Therefore, the sentence χ_α is true also in $L(\omega_1^{CK})$, witnessed by some $\bar{b}'$. Since our formulas define an interpretation of ω_1^{CK} in $L(\omega_1^{CK})$, the witness $\bar{b}'$ for χ_α in $L(\omega_1^{CK})$ must satisfy ψ_γ for some $\gamma \geq \alpha$.

Now, we can proceed as in the proof of Proposition 5.2.27. We are working in $L(\omega_1^{CK})$. We choose $\bar{b}, \bar{c}$, from the sequence of $\bar{d}_\alpha$'s in the lemma, such that $\bar{b} \sim^\gamma \bar{c}$, where $\bar{b}$ satisfies ψ_α and $\bar{c}$ satisfies ψ_β, for $\alpha < \beta$. By Lemma 5.2.15, we may suppose that the elements of $\bar{b}$ all lie to the left of the $<$-first element of $\bar{c}$, and the interval between the $<$-greatest element of $\bar{b}$ and the $<$-first element of $\bar{c}$ contains an element of length 2. Since $\alpha < \beta$, we should have $L(\omega_1^{CK}) \models \bar{b} \circledless \bar{c}$. We can take $\bar{b}'$ automorphic to $\bar{b}$ such that all elements of $\bar{b}'$ lie to the right of the $<$-greatest element of $\bar{c}$, and the interval between the $<$-greatest element of $\bar{c}$ and the $<$-first element of $\bar{b}'$ contains an element of length 2. Clearly, $L(\omega_1^{CK}) \models \bar{b}' \circledless \bar{c}$ since $\bar{b}'$ satisfies $\psi_\alpha(\bar{x})$. Applying Lemma 5.2.22 we get the fact that $\bar{b}, \bar{c} \sim^\gamma \bar{c}, \bar{b}'$. It follows that $L(\omega_1^{CK}) \models \bar{c} \circledless \bar{b}'$, which is a contradiction.

We are ready to complete the proof of Theorem 5.2.7, saying that there is no tuple of $L_{\omega_1\omega}$-formulas that, for all directed graphs G, interprets G in $L(G)$.

Suppose that we have such formulas. For some X, the formulas are X-computable infinitary. Let G be a linear ordering of type ω_1^X. Relativizing Proposition 5.2.30, we have the fact that G is not interpreted in $L(G)$ by any X-computable formulas.

The Friedman-Stanley embedding represents a uniform effective encoding of directed graphs in linear orderings. We have seen that there is no uniform interpretation of the input graph in the output linear ordering.

Conjecture 1. Let Φ be a Turing computable embedding of directed graphs in linear orderings. There do not exist $L_{\omega_1\omega}$ formulas that, for all directed graphs G, define an interpretation of G in $\Phi(G)$.

5.3 Interpreting a field into the Heisenberg group

The Heisenberg group of a field F is the upper-triangular subgroup of $GL_3(F)$ in which all matrices have 1's along the diagonal and 0's below it. Maltsev [Mal60] showed that there are existential formulas with parameters, which, for every field F, define F in its Heisenberg group $H(F)$. In this section we will show that there are existential formulas without parameters, which, for every field F, interpret F in $H(F)$. Observing what is used to obtain this result, we will then formulate a general result on removing parameters from an interpretation.

Here is a uniform definition of the effective interpretation (see Definition 5.1.10), and a uniform definition of computable functor (see Definition 5.1.11).

Suppose $K \leq_{tc} K'$ via Θ.

(1) We say that the structures in K are *uniformly effectively interpreted* in their Θ-images if there is a fixed collection of generalized computable Σ_1^c formulas (without parameters) (see Definition 2.5.9) such that, for all $\mathcal{A} \in K$, define an interpretation of $\mathcal{A}$ in $\Theta(\mathcal{A})$.

(2) We say that Φ and Ψ form a *uniform computable functor* from the structures $\Theta(\mathcal{A})$ to $\mathcal{A}$ if these Turing operators serve for all $\mathcal{A} \in K$.

There is a uniform version of Theorem 5.1.12.

Theorem 5.3.3. For classes K, K' with $K \leq_{tc} K'$ via Θ, the following are equivalent:

1. there are computable Σ_1^c formulas (without parameters) which, for all $\mathcal{A} \in K$, effectively interpret $\mathcal{A}$ in $\Theta(\mathcal{A})$,
2. there are uniform Turing operators Φ, Ψ that, for all $\mathcal{A} \in K$, form a computable functor from $\Theta(\mathcal{A})$ to $\mathcal{A}$.

Maltsev defined a Turing computable embedding of fields in 2-step nilpotent groups. The embedding takes each field F to its *Heisenberg group* $H(F)$. To show that the embedding preserves isomorphism, Maltsev gave uniform existential formulas defining a copy of F in $H(F)$. The definitions involved a pair of parameters, whose orbit is defined by an existential (in fact, quantifier-free) formula. In Section 5.3.1, we recall Maltsev's definitions. In Section 5.3.2, we describe a uniform computable functor that, for all F, takes copies of $H(F)$, with their isomorphisms, to copies of F, with corresponding isomorphisms. By Theorem 5.3.3, it follows that there is a uniform effective interpretation of F in $H(F)$ with no parameters. In Section 5.3.3, we give explicit finitary existential formulas that define such an interpretation. In Section 5.3.4, we note that although F is effectively interpretable in $H(F)$

and $H(F)$ is effectively interpretable in F, we do not, in general, have effective bi-interpretability. In Section 5.3.5, we generalize what we did in passing from Maltsev's definition, with parameters, to the uniform effective interpretation, with no parameters. This is a joint work [ACG+20] with Alvir, Calvert, Goodman, Harizanov, Knight, Morozov, Miller, and Weisshaar.

5.3.1 Defining F in $H(F)$

In this section, we recall Maltsev's embedding of fields in 2-step nilpotent groups, and his formulas that define a copy of the field in the group. Recall that for a field F, the Heisenberg group $H(F)$ is the set of matrices of the form

$$h(a,b,c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with entries in F. Note that $h(0,0,0)$ is the identity matrix. We are interested in non-commuting pairs in $H(F)$. One such pair is $(h(1,0,0), h(0,1,0))$. For $u = h(u_1,u_2,u_3)$ and $v = h(v_1,v_2,v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.$$

For a group G, we write $Z(G)$ for the center. For group elements x,y, the *commutator* is $[x,y] = x^{-1}y^{-1}xy$. The following technical lemma provides much of the information we need to show that F is defined, with parameters, in $H(F)$.

Lemma 5.3.4.
1. (a) For u and v, the commutator, $[u,v]$, is $h(0,0,\Delta_{(u,v)})$, and
   (b) $[u,v] = 1$ iff $\Delta_{(u,v)} = 0$.
2. Let $u = h(u_1,u_2,u_3)$, and let $v = h(v_1,v_2,v_3)$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $u \in Z(H(F))$. If $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $[u,v] = 1$ iff there exists α such that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
3. $Z(H(F))$ consists of the elements of the form $h(0,0,c)$.
4. If $[u,v] \neq 1$, then $x \in Z(H(F))$ iff $[x,u] = [x,v] = 1$.

Corollary 5.3.5. If $x \in H(F)$ is fixed by all automorphisms of $H(F)$, then $x = 1$.

The next lemma tells us how, for any non-commuting pair u, v in the group $(H(F), *)$, we can define operations $+$ and $\cdot$, and an isomorphism f from F to $(Z(H(F)), +, \cdot)$.

Lemma 5.3.6. Let $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$ be a non-commuting pair. Assume that $\alpha, \beta, \gamma \in F$. Let $x = h(0, 0, \alpha \cdot \Delta_{(u,v)})$, $y = h(0, 0, \beta \cdot \Delta_{(u,v)})$, and $z = h(0, 0, \gamma \cdot \Delta_{(u,v)})$. Then

1. $\alpha + \beta = \gamma$ iff $x * y = z$, where $*$ is the matrix multiplication.
2. $\alpha \cdot \beta = \gamma$ iff there exist x' and y' such that $[x', u] = [y', v] = 1$, $[u, y'] = y$, $[x', v] = x$, and $z = [x', y']$.

The main result of the section follows directly from Lemmas 5.3.4 and 5.3.6.

Theorem 5.3.7. For an arbitrary non-commuting pair (u, v) in $H(F)$, we get $F_{(u,v)} = (Z(H(F)), \oplus, \otimes_{(u,v)})$ where

1. $x \in Z(H(F))$ iff $[x, u] = [x, v] = 1$,
2. $\oplus$ is the group operation from $H(F)$,
3. $\otimes_{(u,v)}$ is the set of triples (x, y, z) such that there exist x', y' with $[x', u] = [y', v] = 1$, $[x', v] = x$, $[u, y'] = y$, and $[x', y'] = z$,
4. the function $g_{(u,v)}$ taking $\alpha \in F$ to $h(0, 0, \alpha \cdot \Delta_{(u,v)}) \in H(F)$ is an isomorphism between F and $F_{(u,v)}$.

Note: From Part 4, it is clear that $h(0, 0, \Delta_{(u,v)})$ is the multiplicative identity in $F_{(u,v)}$—we may write $1_{(u,v)}$ for this element.

Proposition 5.3.8. There is a uniform Medvedev reduction Φ of F to $H(F)$.

Given $G \cong H(F)$, we search for a non-commuting pair (u, v) in G, and then use Maltsev's definitions to get a copy of F computable from G.

It turns out that the Medvedev reduction Φ is half of a computable functor. In the next subsection, we explain how to get the other half.

5.3.2 The computable functor

In the previous subsection, we saw that for any field F and any non-commuting pair (u, v) in $H(F)$, there is an isomorphic copy $F_{(u,v)}$ of F defined in $H(F)$ by finitary existential formulas with parameters (u, v). The defining formulas are the same for all F. Hence, there is a uniform Turing operator Φ that, for all fields F, takes copies of $H(F)$ to copies of F. In this subsection, we describe a companion operator Ψ so that Φ and Ψ together form a uniform computable functor. For any field F, and any triple (G_1, p, G_2) such that G_1 and G_2 are copies of $H(F)$ and p is an isomorphism from G_1 onto G_2, the function $\Psi(G_1, p, G_2)$ must be an isomorphism from $\Phi(G_1)$ onto $\Phi(G_2)$, and, moreover, the isomorphisms given by Ψ must preserve identity and composition. We saw in the previous subsection that for any field F, and any non-commuting pair (u, v) in $H(F)$, the function $g_{(u,v)}$ taking α to $h(0, 0, \alpha \cdot \Delta_{(u,v)})$ is an isomorphism from F onto $F_{(u,v)}$. We use this $g_{(u,v)}$ below.

Lemma 5.3.9. For any F and any non-commuting pairs (u, v), (u', v') in $H(F)$, there is a natural isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ onto $F_{(u',v')}$. Moreover, the family of isomorphisms $f_{(u,v),(u',v')}$ is functorial; i.e.,

1. for any non-commuting pair (u, v), the function $f_{(u,v),(u,v)}$ is the identity,
2. for any three non-commuting pairs (u, v), (u', v'), and (u'', v''),

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')}.$$

The next lemma says that there is a uniform existential definition of the family of isomorphisms $f_{(u,v),(u',v')}$.

Lemma 5.3.10. There is a finitary existential formula $\psi(u, v, u', v', x, y)$ that, for any two non-commuting pairs (u, v) and (u', v'), defines the isomorphism $f_{(u,v),(u',v')}$ taking $x \in F_{(u,v)}$ to $y \in F_{(u',v')}$.

We use Lemmas 5.3.9 and 5.3.10 to prove the following.

Proposition 5.3.11. There is a uniform computable functor that, for all fields F, takes $H(F)$ to F.

Corollary 5.3.12. There is a uniform effective interpretation of F in $H(F)$.

The result from [HTMMM17] gives a uniform interpretation of F in $H(F)$, valid for all countable fields F, using computable Σ_1^c formulas with no parameters. The tuples from $H(F)$ that represent elements of F may have arbitrary arity. In the next subsection, we will do better.

We note here that the uniform interpretation of F in $H(F)$ given in this subsection allows one to transfer the computable-model-theoretic properties of any graph G to a 2-step-nilpotent group, without introducing any constants. This is not a new result: in [Mek81], Mekler gave a related coding of graphs into 2-step-nilpotent groups, which, in concert with the completeness of graphs for such properties (see [HKSS02]), appears to yield the same fact, although Mekler's coding had different goals than completeness. Then, in [HKSS02], Hirschfeldt, Khoussainov, Shore, and Slinko used Maltsev's interpretation of an integral domain in its Heisenberg group with two parameters, along with the completeness of integral domains, to re-establish it. More recently, [MPSS18] demonstrated the completeness of fields, by coding graphs into fields, From that result, along with Corollary 5.3.12 and the usual definition of $H(F)$ as a matrix group given by a set of triples from F, we achieve a coding of graphs into fields, different from Mekler's coding, with no constants required.

5.3.3 Defining the interpretation directly

Our goal in this section is to give explicit existential formulas defining a uniform effective interpretation of a field in its Heisenberg group. We discovered the formulas for this interpretation by examining the infinitary formulas used in the interpretation in Corollary 5.3.12 and trimming them down to their essence, which turned out to be finitary.

Theorem 5.3.13. There are finitary existential formulas that, uniformly for every field F, define an effective interpretation of F in $H(F)$, with elements of F represented by triples of elements from $H(F)$.

We offer intuition before giving the formal proof. The domain D of the interpretation consists of those triples (u, v, x) from $H(F)$ with $uv \neq vu$ and x in the center: for each single (u, v), we apply Maltsev's definitions, with u, v as parameters, to get $F_{(u,v)} \cong F$. We view the triples arranged as follows:

$F_{(u,v)}$	$F_{(u',v')}$	$F_{(u'',v'')}$	$\cdots$
(u,v,x_0)	(u',v',x_0)	(u'',v'',x_0)	
(u,v,x_1)	(u',v',x_1)	(u'',v'',x_1)	
(u,v,x_2)	(u',v',x_2)	(u'',v'',x_2)	
(u,v,x_3)	(u',v',x_3)	(u'',v'',x_3)	
$\vdots$	$\vdots$	$\vdots$	

Here each column can be seen as $F_{(u,v)}$ for some non-commuting pair (u, v). Now the system of isomorphisms from Lemma 5.3.9 allow us to identify each element in one column with a single element from each other column, and modding out by this identification will yield a single copy of F.

Let H be a group isomorphic to $H(F)$. Recalling the natural isomorphisms $f_{(u,v),(u',v')}$ defined in Lemma 5.3.9 for non-commuting pairs (u, v) and (u', v'), we define $D \subseteq H$, a binary relation $\sim$ on D, and ternary relations $\oplus$, $\odot$ (which are binary operations) on D, as follows.

1. D is the set of triples (u, v, x) such that $uv \neq vu$ and $xu = ux$ and $xv = vx$. (Notice that, no matter which non-commuting pair (u, v) is chosen, the set of corresponding elements x is precisely the center $Z(H)$, by Theorem 5.3.7.)

2. $(u, v, x) \sim (u', v', x')$ holds if and only if the isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ to $F_{(u',v')}$ maps x to x'.

3. $\oplus((u, v, x), (u', v', y'), (u'', v'', z''))$ holds if there exist $y, z \in H$ such that
$(u, v, y) \sim (u', v', y')$ and $(u, v, z) \sim (u'', v'', z'')$, and $F_{(u,v)} \models x + y = z$.

4. $\odot((u, v, x), (u', v', y'), (u'', v'', z''))$ holds if there exist $y, z \in H$ such that
$(u, v, y) \sim (u', v', y')$ and $(u, v, z) \sim (u'', v'', z'')$, and $F_{(u,v)} \models x \cdot y = z$.

In Theorem 5.3.13, to eliminate parameters from Maltsev's definition of F in $H(F)$, we gave an interpretation of F in $H(F)$, rather than another definition. (Recall that a definition is an interpretation in which the equivalence relation on the domain is simply equality.) We now demonstrate the impossibility of strengthening the theorem to give a parameter-free definition of F in $H(F)$.

Proposition 5.3.14. There is no parameter-free definition of any field F in its Heisenberg group $H(F)$ by finitary formulas.

5.3.4 Question of bi-interpretability

If $\mathcal{B}$ is interpreted in $\mathcal{A}$, we write $\mathcal{B}^{\mathcal{A}}$ for the copy of $\mathcal{B}$ given by the interpretation of $\mathcal{B}$ in $\mathcal{A}$. The structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are uniformly relatively computable isomorphisms f from $\mathcal{A}$ onto $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and g from $\mathcal{B}$ onto $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. In general, the isomorphism f would map each element of $\mathcal{A}$ to an equivalence class of equivalence classes of tuples in $\mathcal{A}$. We would represent f by a relation R_f that holds for $a, \overline{a}_1, \ldots, \overline{a}_r$ if f maps a to the equivalence class of the tuple of equivalence classes of the $\overline{a}_i$'s. Similarly, the isomorphism g would be represented by a relation R_g that holds for $b, \overline{b}_1, \ldots, \overline{b}_r$ if g maps b to the equivalence class of the tuple of equivalence classes of the $\overline{b}_i$'s. Saying that f and g are uniformly relatively computable is equivalent to saying that the relations R_f, R_g, have generalized computable Σ_1^c definitions without parameters.

For a field F and its Heisenberg group $H(F)$, when we define $H(F)$ in F, the elements of $H(F)$ are represented by triples from F, and we have finitary formulas, quantifier-free or existential, that define the group operation (as a relation). When we interpret F in $H(F)$, the elements of F are represented by triples from $H(F)$, and we have finitary existential formulas that define the field operations and their negations (as ternary relations). Thus, in $F^{H(F)^F}$ (the copy of F interpreted in the copy of $H(F)$ that is defined in F), the elements are equivalence classes of triples of triples. In $H(F)^{F^{H(F)}}$ (the copy of $H(F)$ defined in the copy of F that is interpreted in $H(F)$), the elements are triples of equivalence classes of triples. So, an isomorphism f from F to $F^{H(F)^F}$ is represented by a 10-ary relation R_f on F, and an isomorphism

g from $H(F)$ to $H(F)^{F^{H(F)}}$—it is represented by a 10-ary relation R_g on $H(F)$.

For a Turing computable embedding Θ of K in K' we have *uniform effective bi-interpretability* if there are (generalized) computable Σ_1^c formulas with no parameters that, for all $\mathcal{A} \in K$ and $\mathcal{B} = \Theta(\mathcal{A})$, define isomorphisms from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and from $\mathcal{B}$ to $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$. After a talk by the fifth author, Montalbán asked the following very natural question.

Question 5.3.15. Do we have uniform effective bi-interpretability of F and $H(F)$?

The answer to this question is negative. In particular, $\mathbb{Q}$ and $H(\mathbb{Q})$ are not effectively bi-interpretable. One way to see this is to note that $\mathbb{Q}$ is rigid, while $H(\mathbb{Q})$ is not—in particular, for any non-commuting pair, $u, v \in H(\mathbb{Q})$, there is a group automorphism that takes (u, v) to (v, u). The negative answer to Question 5.3.15 then follows from [Mon, Lemma VI.26(4)], which states that if $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable, then their automorphism groups are isomorphic.

Morozov's result shows which half of effective bi-interpretability causes the difficulties.

Proposition 5.3.16. There is a finitary existential formula that, for all F, defines in F a specific isomorphism k from F to $F^{H(F)^F}$.

The other half of what we would need for uniform effective bi-interpretability is sometimes impossible, as remarked above in the case $F = \mathbb{Q}$. We do not know of any examples where F and $H(F)$ are effectively bi-interpretable: the obstacle for $\mathbb{Q}$ might hold in all cases.

Problem 5.3.17. For which fields F, if any, are the automorphism groups of F and $H(F)$ isomorphic?

Even if there are fields F such that $\mathrm{Aut}(F) \cong \mathrm{Aut}(H(F))$, we suspect that F and $H(F)$ are not effectively bi-interpretable, simply because it is difficult to see how one might give a computable Σ_1^c formula in the language of groups that defines a specific isomorphism from $H(F)$ to $H(F)^{F^{H(F)}}$.

5.3.5 Generalizing the method

Our first general definition and proposition follow closely the example of a field and its Heisenberg group.

Definition 5.3.18. Let $\mathcal{A}$ be a structure for a computable relational language. Assume that its basic relations are R_i, where R_i is k_i-ary. We say that $\mathcal{A}$ is *effectively defined in* $\mathcal{B}$ *with parameters* $\bar{b}$ if there exist $D(\bar{b}) \subseteq \mathcal{B}^{<\omega}$, and $\pm R_i(\bar{b}) \subseteq D(\bar{b})^{k_i}$, defined by a uniformly computable sequence of generalized computable Σ_1^c formulas with parameters $\bar{b}$.

Proposition 5.3.19. Suppose $\mathcal{A}$ is effectively defined in $\mathcal{B}$ with parameters $\overline{b}$. For $\overline{c}$ in the orbit of $\overline{b}$, let $\mathcal{A}_{\overline{c}}$ be the copy of $\mathcal{A}$ defined by the same formulas, but with parameters $\overline{c}$ replacing $\overline{b}$. Then the following conditions together suffice to give an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters:

(1) The orbit of $\overline{b}$ is defined by a computable Σ^c_1 formula $\varphi(\overline{u})$;

(2) There is a generalized computable Σ^c_1 formula $\psi(\overline{u}, \overline{v}, \overline{x}, \overline{y})$ such that for all $\overline{c}, \overline{d}$ in the orbit of $\overline{b}$, the formula $\psi(\overline{c}, \overline{d}, \overline{x}, \overline{y})$ defines an isomorphism $f_{\overline{c},\overline{d}}$ from $\mathcal{A}_{\overline{c}}$ onto $\mathcal{A}_{\overline{d}}$; and

(3) The family of isomorphisms $f_{\overline{c},\overline{d}}$ preserves identity and composition.

Corollary 5.3.20. In the situation of Proposition 5.3.19, if $D(\overline{b})$ is contained in $\mathcal{B}^n$ for some single $n \in \omega$, then the ψ in item (2) and the formulas in Definition 5.3.18 will simply be computable Σ^c_1 formulas (as opposed to generalized computable Σ^c_1 formulas) and the interpretation of $\mathcal{A}$ in $\mathcal{B}$ without parameters will also be by computable (as opposed to generalized) Σ^c_1 formulas. $\square$

Definition 5.3.21. We say that $\mathcal{A}$, with basic relations R_i, k_i-ary, is *effectively interpreted with parameters* $\overline{b}$ in $\mathcal{B}$ if there exist $D \subseteq \mathcal{B}^{<\omega}$, $\equiv \subseteq D^2$, and $R^*_i \subseteq D^{k_i}$ such that

1. $(D, (R^*_i)_i)/_{\equiv} \cong \mathcal{A}$,
2. D, $\pm \equiv$, and $\pm R^*_i$ are defined by a computable sequence of generalized computable Σ^c_1 formulas, with a fixed finite tuple of parameters $\overline{b}$.

Again, in the case where $D \subseteq \mathcal{B}^n$ for some fixed n, the formulas defining the effective interpretation are computable Σ^c_1 formulas of the usual kind, with parameters $\overline{b}$.

Proposition 5.3.22. Suppose that $\mathcal{A}$ (with basic relations R_i, k_i-ary) has an effective interpretation in $\mathcal{B}$ with parameters $\overline{b}$. For $\overline{c}$ in the orbit of $\overline{b}$, let $\mathcal{A}_{\overline{c}}$ be the copy of $\mathcal{A}$ obtained by replacing the parameters $\overline{b}$ by $\overline{c}$ in the defining formulas, with domain $D_{\overline{c}}/\equiv_{\overline{c}}$ containing $\equiv_{\overline{c}}$-classes $[\overline{a}]_{\equiv_{\overline{c}}}$. Then the following conditions suffice for an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ (without parameters):

(1) The orbit of $\overline{b}$ is defined by a computable Σ^c_1 formula $\varphi(\overline{x})$;

(2) There is a relation $F \subseteq \mathcal{B}^{<\omega}$, with a generalized computable Σ^c_1-definition, such that for every $\overline{c}$ and $\overline{d}$ in the orbit of $\overline{b}$, the set of pairs $(\overline{x}, \overline{y}) \in D_{\overline{c}} \times D_{\overline{d}}$ with $(\overline{c}, \overline{d}, \overline{x}, \overline{y}) \in F$ is invariant under $\equiv_{\overline{c}}$ on $\overline{x}$ and under $\equiv_{\overline{d}}$ on $\overline{y}$, and defines an isomorphism $f_{\overline{c},\overline{d}}$ from $\mathcal{A}_{\overline{c}}$ onto $\mathcal{A}_{\overline{d}}$; and

(3) The family of isomorphisms $f_{\overline{c},\overline{d}}$ preserves identity and composition.

We show that our results apply not only to effective interpretations, but to all interpretations using generalized $L_{\omega_1\omega}$ formulas.

Theorem 5.3.23. Let $\mathcal{A}$ be a relational structure with basic relations R_i that are k_i-ary. Suppose there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by generalized $L_{\omega_1\omega}$ formulas, with parameters $\overline{b}$ from $\mathcal{B}$. For $\overline{c}$ in the orbit of $\overline{b}$, let $\mathcal{A}_{\overline{c}}$ be the copy of $\mathcal{A}$ obtained by the interpretation with parameters $\overline{c}$ replacing $\overline{b}$. Assume that there is a generalized $L_{\omega_1\omega}$-definable relation F defining, for each $\overline{c}$ and $\overline{d}$ in the orbit of $\overline{b}$, an isomorphism $f_{\overline{c},\overline{d}} : \mathcal{A}_{\overline{c}} \to \mathcal{A}_{\overline{d}}$ as in Proposition 5.3.22, and that this family is closed under composition, with the identity map as $f_{\overline{c},\overline{c}}$ for all $\overline{c}$.

Then there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ by $L_{\omega_1\omega}$ formulas without parameters. Moreover, the new interpretation satisfies all of the following.

- For each countable ordinal α, if the interpretation in $(\mathcal{B},\overline{b})$ defines D, $\equiv$, and each R_i using Σ_α formulas from $L_{\omega_1\omega}$, and F and the orbit of $\overline{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define these sets.

- For each countable ordinal α, if the interpretation in $(\mathcal{B},\overline{b})$ defines each of D, $\pm\equiv$, and $\pm R_i$ using Σ_α formulas, and F and the orbit of $\overline{b}$ in $\mathcal{B}$ are both defined by Σ_α formulas, then the parameter-free interpretation also uses Σ_α formulas to define its domain, its equivalence relation $\sim$, the complement $\nsim$, and its relations $\pm R_i$. (Defining $\nsim$ and $\neg R_i$ this way is required by the usual notion of effective Σ_α interpretation.)

- Let $X \subseteq \mathbb{N}$. If the interpretation in $(\mathcal{B},\overline{b})$ used X-computable formulas, and F and the orbit of $\overline{b}$ in $\mathcal{B}$ are both defined by X-computable formulas, then the parameter-free interpretation also uses X-computable formulas.

  (With $X = \varnothing$, X-computable formulas are simply computable formulas.)

5.4 Interpreting $ACF(0)$ - C in a special linear group $SL_2(C)$

Let C be an algebraically closed field of characteristic 0 - $ACF(0)$. We write $SL_2(C)$ for the group of 2×2 matrices over C with determinant 1. Clearly, $SL_2(C)$ is defined in C without parameters. Each particular C has a computable copy, and that is effectively interpreted in $SL_2(C)$. But, there are infinitely many non-isomorphic C, differing in transcendence degree. We give finitary existential formulas that (for all C) define C in $SL_2(C)$, with a pair of parameters. Before defining the field as a whole, we look separately at addition and multiplication. This is a work in progress together with Alvir, Knight and Miller [AKMS].

Defining $(C,+)$

Let A be the set of matrices in $SL_2(C)$ of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. Note that on A, matrix multiplication gives addition; that is,

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}.$$

We can define A using the parameter $p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Claim 1: The matrices that commute with p have the forms $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$.

It is easy to check that I is the unique element of $SL_2(C)$ that is its own square. Thus, we can define I by a quantifier-free formula. Now, I has many square roots apart from $\pm I$. However, these do not commute with p—the unique square root of I that is not equal to I and commutes with p is $-I$.

Claim 2: $x \in A$ iff x commutes with p and x has a square root that commutes with p.
Now, $(C,+) \cong (A,*)$, so we have a copy of $(C,+)$ defined in $SL_2(C)$ using the parameter p.

5.4.1 Defining $(C \setminus \{0\}, \cdot)$

Let M be the set of matrices of form $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$. On M, matrix multiplication gives multiplication; that is, $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} * \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & (ab)^{-1} \end{bmatrix}$. We can define M using a parameter $q = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Claim 3: $x \in M$ iff x commutes with q.

We have $(C \setminus \{0\}, \cdot) \cong (M,*)$, so $(C \setminus \{0\}, \cdot)$ is defined in $SL_2(C)$ using quantifier-free formulas with the parameter q.

5.4.2 Defining $(C,+,\cdot)$

To define the field $(C,+,\cdot)$, we represent an element $a \in C$ by a pair of matrices (x,y), where $x \in A$ and $y \in M$. The most natural choice for x is $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. If $a \neq 0$, then we let $y = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$, while if $a = 0$, then we let

$y = I$. For $a = 1$, we choose (p, I), and for $a = 0$, we choose (I, I)—the same second component. Let T be the set of these pairs (x, y) chosen to represent elements of C.

Claim 4: $(x, y) \in T$ iff either $x = y = I$ (so (x, y) represents 0) or else $x \neq I$, $x \in A$, $y \in M$, and there is some z such that $z * z = y$, $z \in M$, and $z * p * z^{-1} = x$. (In the second case, (x, y) represents some $a \neq 0$, and there are just two possibilities for z, corresponding to the two possible square roots of a.)

For $(x, y) \in T$, we define addition and multiplication relations as follows:

1. $(x, y) \oplus (x', y') = (u, v)$ if $x * x' = u$ and $(u, v) \in T$,
2. $(x, y) \otimes (x', y') = (u, v)$ if either at least one of (x, y), (x', y') is (I, I) and $(u, v) = (I, I)$, or else neither of (x, y), (x', y') is (I, I), and then $y * y' = v$ and $(u, v) \in T$.

We have established the following.

Proposition 5.4.1. The field C is defined in $SL_2(C)$ using finitary existential formulas with parameters p and q. (The definition of T is existential, while the definitions of the operations are quantifier-free.)

Question 5.4.2. Are there formulas that, for all algebraically closed fields C of characteristic 0, define an effective interpretation of C in $SL_2(C)$? Are there existential formulas that serve?

Remarks. There are old model theoretic results, due to Poizat [Poi01], that give uniform definability of a copy of C in $SL_2(C)$ using elementary first order formulas without parameters. But we do not know the complexity of the defining formulas. We have a formula $\varphi(u, v)$, saying of the formulas D, $\pm \sim$, $\oplus$, and $\otimes$ that give our interpretation of C in $SL_2(C)$ that they give an field, not of characteristic 2, in which every element has a square root. For any (u, v) satisfying this formula, we get an infinite field $F_{(u,v)}$. The theory of $SL_2(C)$ is ω-stable. By an old result of Macintyre, $F_{(u,v)}$ must be algebraically closed. Poizat's results show that $F_{(u,v)}$ is isomorphic to C and that there are unique definable isomorphisms between the fields $F_{(u,v)}$ corresponding to pairs (u, v) that satisfy $\varphi(u, v)$. These isomorphisms are functorial. So, we have, not necessarily an *effective* interpretation without parameters, but one that is defined by elementary first order formulas. We do not know the complexity of the formulas.

Chapter 6

Cohesive powers

The ultimate inspiration for this work is Skolem's 1934 construction of a countable non-standard model of arithmetic [Sko34]. Skolem's construction can be described roughly as follows. For sets $X, Y \subseteq \mathbb{N}$, write $X \subseteq^* Y$ if $X \setminus Y$ is finite. First, fix an infinite set $C \subseteq \mathbb{N}$ that is *cohesive* for the collection of arithmetical sets: for every arithmetical $A \subseteq \mathbb{N}$, either $C \subseteq^* A$ or $C \subseteq^* \overline{A}$. Next, define an equivalence relation $=_C$ on the arithmetical functions $f\colon \mathbb{N} \to \mathbb{N}$ by $f =_C g$ if and only if $C \subseteq^* \{n : f(n) = g(n)\}$. Then define a structure on the $=_C$-equivalence classes $[f]$ by $[f]+[g] = [f+g]$, $[f]\times[g] = [f\times g]$ (where $f+g$ and $f \times g$ are computed pointwise), and $[f] < [g] \Leftrightarrow C \subseteq^* \{n : f(n) < g(n)\}$. Using the arithmetical cohesiveness of C, one then shows that this structure is elementarily equivalent to $(\mathbb{N}; +, \times, <)$. The structure is countable because there are only countably many arithmetical functions, and it has non-standard elements, such as the element represented by the identity function.

Think of Skolem's construction as a more effective analog of an ultrapower construction. Instead of building a structure from all functions $f\colon \mathbb{N} \to \mathbb{N}$, Skolem builds a structure from only the arithmetical functions f. The arithmetically cohesive set C plays the role of the ultrafilter. Feferman, Scott, and Tennenbaum [FST59] investigate the question of whether Skolem's construction can be made more effective by assuming that C is only *r-cohesive* (i.e., cohesive for the collection of computable sets) and by restricting to computable functions f. They answer the question negatively by showing that it is not even possible to obtain a model of Peano arithmetic in this way. Lerman [Ler70] investigates the situation further and shows that if one restricts to *cohesive* sets C (i.e., cohesive for the collection of c.e. sets) that are co-c.e. and to computable functions f, then the first-order theory of the structure obtained is exactly determined by the many-one degree of C. Additional results in this direction appear in [Hir75, HW75].

Dimitrov [Dim09] generalizes the effective ultrapower construction to arbitrary computable structures. These *cohesive powers* of computable structures are studied in [Dim08, DH15, DHMM14] in relation to the lattice of c.e. subspaces, modulo finite dimension, of a fixed computable infinite dimensional vector space over $\mathbb{Q}$. In this work, we investigate a question dual to the question studied by Lerman. Lerman fixes a computable presentation of a computable structure (indeed, all computable presentations of the standard model of arithmetic are computably isomorphic) and studies the effect that the choice of the cohesive set has on the resulting cohesive power. Instead of fixing a computable presentation of a structure and varying the cohesive set, we fix a computably presentable structure and a cohesive set, and then we vary the structure's computable presentation. We focus on linear orders, with special emphasis on computable presentations of ω. We choose to work with linear orders because they are a good source of non-computably categorical structures and because the setting is simple enough to be able to completely describe certain cohesive powers up to isomorphism. This work [DHM+19] and a greatly expanded version [DHM+20] is a joint with Dimitrov, Harizanov, Morozov, Shafer, and Vatev.

Our main results are the following, where ω, ζ, and η denote the respective order-types of the natural numbers, the integers, and the rationals.

- If C is cohesive and $\mathcal{L}$ is a computable copy of ω that is computably isomorphic to the standard presentation of ω (i.e., $\mathcal{L}$ has a computable successor function), then the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega+\zeta\eta$. (Corollary 6.4.6.)

- If C is co-c.e. and cohesive and $\mathcal{L}$ is a computable copy of ω, then the finite condensation of the cohesive power $\Pi_C\mathcal{L}$ has order-type $1+\eta$. (Theorem 6.4.4. See Definition 6.3.3 for the definition of *finite condensation*.)

- If C is co-c.e. and cohesive, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega+\eta$. (Corollary 6.5.2.)

- More generally, if C is co-c.e. and cohesive and $X \subseteq \mathbb{N} \smallsetminus \{0\}$ is either a Σ^0_2 set or a Π^0_2 set, thought of as a set of finite order-types, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega+\sigma(X \cup \{\omega+\zeta\eta+\omega^*\})$. Here ω^* denotes the reverse of ω, and σ denotes the shuffle operation of Definition 6.6.1. Furthermore, if X is finite and non-empty, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega+\sigma(X)$. (Theorem 6.6.6.)

The above results provide many examples of pairs of isomorphic computable linear orders with non-elementarily equivalent cohesive powers. We also give examples of computable linear orders that are always isomorphic to their cohesive powers and examples of pairs of non-elementarily equivalent computable linear orders with isomorphic cohesive powers.

6.1 Basic properties

Definition 6.1.1. An infinite set $C \subseteq \mathbb{N}$ is *cohesive* if for every c.e. set W, either $C \subseteq^* W$ or $C \subseteq^* \overline{W}$.

Notice that if C is cohesive and X is either c.e. or co-c.e., then $C \cap X$ being infinite implies that $C \subseteq^* X$. We use quantifiers $\forall^\infty n$ and $\exists^\infty n$ as abbreviations for 'for almost every n' and 'there are infinitely many n'. So for example, $(\forall^\infty n \in C)(n \in X)$ means $C \subseteq^* X$.

Definition 6.1.2 ([Dim09])**.** Let $\mathfrak{L}$ be a computable language. Let $\mathcal{A}$ be a computable $\mathfrak{L}$-structure with non-empty domain $A \subseteq \mathbb{N}$. Let $C \subseteq \mathbb{N}$ be cohesive. The *cohesive power of* $\mathcal{A}$ *over* C, denoted $\Pi_C\mathcal{A}$, is the $\mathfrak{L}$-structure $\mathcal{B}$ defined as follows.

- Let $D = \{\varphi \mid \varphi{:}\mathbb{N} \to A$ is partial computable function and $C \subseteq^* dom(\varphi)\}$.
- For $\varphi, \psi \in D$, let $\varphi =_C \psi$ denote $C \subseteq^* \{x : \varphi(x)\downarrow = \psi(x)\downarrow\}$. The relation $=_C$ is an equivalence relation on D. Let $[\varphi]$ denote the equivalence class of $\varphi \in D$ with respect to $=_C$.
- The domain of $\mathcal{B}$ is the set $B = \{[\varphi] : \varphi \in D\}$.
- Let R be an n-ary predicate symbol of $\mathfrak{L}$. For $[\varphi_0], \dots, [\varphi_{n-1}] \in B$, define

  $$R^{\mathcal{B}}([\varphi_0], \dots, [\varphi_{n-1}]) \Leftrightarrow C \subseteq^* \{x : (\forall i < n)\varphi_i(x)\downarrow \wedge \\ R^{\mathcal{A}}(\varphi_0(x), \dots, \varphi_{n-1}(x))\}.$$

- Let f be an n-ary function symbol of $\mathfrak{L}$. For $[\varphi_0], \dots, [\varphi_{n-1}] \in B$, let ψ be the partial computable function defined by

  $$\psi(x) \simeq f^{\mathcal{A}}(\varphi_0(x), \dots, \varphi_{n-1}(x)),$$

  and notice that $C \subseteq^* dom(\psi)$ because $C \subseteq^* dom(\varphi_i)$ for each $i < n$. Define $f^{\mathcal{B}}$ by $f^{\mathcal{B}}([\varphi_0], \dots, [\varphi_{n-1}]) = [\psi]$.
- Let c be a constant symbol of $\mathfrak{L}$. Let ψ be the total computable function with constant value $c^{\mathcal{A}}$, and define $c^{\mathcal{B}} = [\psi]$.

We often consider cohesive powers of computable structures by co-c.e. cohesive sets. The co-c.e. cohesive sets are exactly the complements of the *maximal* sets, which are the co-atoms of the lattice of c.e. sets modulo finite difference. Such sets exist by a well-known theorem of Friedberg (see [Soa87] Theorem X.3.3). Cohesive powers are intended to be effective analogs of ultrapowers, so in light of this analogy, it makes sense to impose effectivity on the cohesive set, which plays the role of the ultrafilter, as well as on the base structure itself. Technically, it helps to be able to learn what numbers are not in the cohesive set C when building a computable structure $\mathcal{A}$ so as to influence $\Pi_C\mathcal{A}$ in a particular way. Cohesive powers by co-c.e. cohesive sets also have the helpful property that every member of the cohesive power has a total computable representative.

A restricted form of Łos's theorem holds for cohesive powers. If $\mathcal{A}$ is a computable structure, C is a cohesive set, and Φ is a Π_3 sentence, then $\Pi_C\mathcal{A} \vDash \Phi$ implies $\mathcal{A} \vDash \Phi$. In general, this version of Łos's theorem for cohesive powers is the best possible. In Sections 6.4, 6.5, and 6.6, we see several examples of computable linear orders $\mathcal{L}$ where the Σ^0_3 sentence "there is an element with no immediate successor" is true of some cohesive power of $\mathcal{L}$ but not true of $\mathcal{L}$.

Theorem 6.1.3 ([Dim09])**.** Let $\mathcal{A}$ be a computable structure, and let C be a cohesive set.

(1) Let $t(v_0,\dots,v_{n-1})$ be a term. Let $[\varphi_0],\dots,[\varphi_{n-1}] \in |\Pi_C\mathcal{A}|$. Let ψ be the partial computable function $\psi(x) \simeq t^{\mathcal{A}}(\varphi_0(x),\dots,\varphi_{n-1}(x))$. Then $t^{\Pi_C\mathcal{A}}([\varphi_0],\dots,[\varphi_{n-1}]) = [\psi]$.

(2) Let $\Phi(v_0,\dots,v_{n-1})$ be a Boolean combination of Σ^0_1 and Π^0_1 formulas, with all free variables displayed. For any $[\varphi_0],\dots,[\varphi_{n-1}] \in |\Pi_C\mathcal{A}|$,

$$\begin{aligned}\Pi_C\mathcal{A} \vDash \Phi([\varphi_0],\dots,[\varphi_{n-1}]) \quad \Leftrightarrow \quad & C \subseteq^* \{x : (\forall i < n)\varphi_i(x)\downarrow \wedge \\ & \mathcal{A} \vDash \Phi(\varphi_0(x),\dots,\varphi_{n-1}(x))\}.\end{aligned}$$

(3) If Φ is a Π^0_2 sentence or a Σ^0_2 sentence, then $\Pi_C\mathcal{A} \vDash \Phi$ if and only if $\mathcal{A} \vDash \Phi$.

(4) If Φ is a Π_3 sentence and $\Pi_C\mathcal{A} \vDash \Phi$, then $\mathcal{A} \vDash \Phi$.

As with structures and their ultrapowers, a computable structure $\mathcal{A}$ always naturally embeds into its cohesive powers. For $a \in A$, let ψ_a be the total computable function with constant value a. Then for any cohesive set C, the map $a \mapsto [\psi_a]$ embeds $\mathcal{A}$ into $\Pi_C\mathcal{A}$. This map is called the *canonical embedding* of $\mathcal{A}$ into $\Pi_C\mathcal{A}$. If $\mathcal{A}$ is finite and C is cohesive, then every partial computable function $\varphi\colon \mathbb{N} \to |\mathcal{A}|$ with $C \subseteq^* dom(\varphi)$ is eventually constant on C. In this case, every element of $\Pi_C\mathcal{A}$ is in the range of the canonical embedding, and therefore $\mathcal{A} \cong \Pi_C\mathcal{A}$. If $\mathcal{A}$ is an infinite computable structure, then every cohesive power $\Pi_C\mathcal{A}$ is countably infinite: infinite because $\mathcal{A}$ embeds into $\Pi_C\mathcal{A}$, and countable because the elements of $\Pi_C\mathcal{A}$ are represented by partial computable functions. See [Dim09] for further details.

Computable structures that are computably isomorphic have isomorphic cohesive powers. This fact essentially appears in [Dim09], but we include a proof here for reference.

Theorem 6.1.4. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be computable $\mathfrak{L}$-structures that are computably isomorphic, and let C be cohesive. Then $\Pi_C\mathcal{A}_0 \cong \Pi_C\mathcal{A}_1$.

Recall that a computable structure $\mathcal{A}$ is called *computably categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is isomorphic to $\mathcal{A}$ via a computable isomorphism. It follows from Theorem 6.1.4 that if $\mathcal{A}$ is a computably categorical computable structure and C is cohesive, then $\Pi_C\mathcal{A} \cong \Pi_C\mathcal{B}$ whenever $\mathcal{B}$ is a computable structure isomorphic to $\mathcal{A}$.

Corollary 6.1.5. Let $\mathcal{A}$ be a computably categorical computable structure, let $\mathcal{B}$ be a computable structure isomorphic to $\mathcal{A}$, and let C be cohesive. Then $\Pi_C\mathcal{A} \cong \Pi_C\mathcal{B}$.

In Theorem 6.1.4, it is essential that the two structures are isomorphic via a computable isomorphism. In the next section we present a construction with two isomorphic structures which cohesive powers are not isomorphic. In Sections 6.4, 6.5, and 6.6, we see many examples of pairs of computable linear orders that are isomorphic (but not computably isomorphic) to ω with non-elementarily equivalent cohesive powers.

6.2 Non-Isomorphic Cohesive Powers of Isomorphic Structures

Theorem 6.2.1. For every co-maximal set $C \subseteq \mathbb{N}$ there exist two isomorphic computable structures $\mathcal{A}$ and $\mathcal{B}$ such the cohesive powers $\prod_C \mathcal{A}$ and $\prod_C \mathcal{B}$ are not isomorphic.

Let S be the set of the even numbers. For every infinite set $A \subseteq S$, such that $S \setminus A$ is infinite, we construct a computable structure $\mathcal{M}_A = (\mathbb{N}, P)$, where $P(x, w, y)$ says that there is an arrow with label w from x to y (e.g., $x \xrightarrow{w} y$) with several properties, including: the formula

$$\Phi(x, y) = \exists w P(x, w, y) \wedge \neg \exists w_1 P(y, w_1, x)$$

will be satisfied by exactly those $x, y \in A$ such that $x < y$. Moreover for any infinite D, $E \subseteq S$ and such that $S \setminus D$ and $S \setminus E$ are infinite and c.e., we have $\mathcal{M}_D \cong \mathcal{M}_E$. The formula $\Theta(x)$ defines the set A in $\mathcal{M}_A$, where

$$\Theta(x) = (\exists t)\,[\Phi(x, t) \vee \Phi(t, x)]\,.$$

For any structure $\mathcal{M} = (M, P)$ in the language with one ternary predicate symbol let $L_{\mathcal{M}} = \{x \in M | \mathcal{M} \vDash \Theta(x)\}$, and $<_{L_{\mathcal{M}}} = \{(x, y) \in M \times M | \mathcal{M} \vDash \Phi(x, y)\}$.

Fix $A \subseteq S$ such that $S \setminus A$ is infinite and c.e. It follows that the formula $\Phi(x,y)$ defines in $\mathcal{M}_A$ the restriction of the natural order $<$ to A. Clearly, $(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}})$ has order type ω.

Let $\mathcal{M}_A^\sharp = \prod_C \mathcal{M}_A$. For partial computable functions f and g such that $[f], [g] \in dom(\mathcal{M}_A^\sharp)$ we have:
(i) $\mathcal{M}_A^\sharp \vDash \Phi([f],[g]) \Leftrightarrow C \subseteq^* \{i | (f(i) \in A) \wedge (g(i) \in A) \wedge (f(i) < g(i))\}$
(ii) $L_{\mathcal{M}_A^\sharp} = \{[f] \in \mathcal{M}_A^\sharp | \; f(C) \subseteq^* A\}$ and $(L_{\mathcal{M}_A^\sharp}, <_{L_{\mathcal{M}_A^\sharp}})$ is a linear order.

For any $a \in A$ let $f_a(i) = a$ for all $i \in \omega$. We will call the element $[f_a]$ in $\mathcal{M}_A^\sharp$ a constant in $\mathcal{M}_A^\sharp$.

The set of constants $\{[f_a] | a \in A\}$ in the structure $\mathcal{M}_A^\sharp$ forms an initial segment of $(L_{\mathcal{M}_A^\sharp}, <_{L_{\mathcal{M}_A^\sharp}})$ of order type ω.

We now define the following Σ_3^0 sentence

$$\Psi = (\exists x)\,[\Theta(x) \wedge (\forall y)\,[\Theta(y) \Rightarrow \Phi(y,x)]]\,.$$

The intended interpretation of Ψ is that when $\Phi(x,t)$ defines a linear order $(L_{\mathcal{M}}, <_{L_{\mathcal{M}}})$, then the order has a greatest element. Note that $\mathcal{M}_A \vDash \neg\Psi$. This is because $\left(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}}\right)$ has order type ω and hence has no greatest element.

We use the Proposition 2.1 from [Ler70].

Proposition 6.2.2. (Lerman [Ler70]) Let R be a co-r-maximal set, and let f be a computable function such that $f(R) \cap R$ is infinite. Then the restriction $f \restriction R$ differs from the identity function only finitely.

We now fix a co-maximal (hence co-r-maximal) set $C \subseteq S$ and an infinite co-infinite computable set $D \subseteq S$. We have $\mathcal{M}_C \cong \mathcal{M}_D$. Let $\mathcal{M}_C^\sharp = \prod_C \mathcal{M}_C$ and $\mathcal{M}_D^\sharp = \prod_C \mathcal{M}_D$.

It is not hard to show that, since C is co-maximal, for every partial computable function φ for which $C \subseteq^* dom(\varphi)$, there is a computable function f_φ such that $[\varphi] = [f_\varphi]$ (see [DHMM14]).

To finish the proof we will establish the following facts:
$\mathcal{M}_C^\sharp \vDash \Psi$
$\mathcal{M}_D^\sharp \vDash \neg\Psi$

In conclusion, we defined computable isomorphic structures $\mathcal{M}_C$ and $\mathcal{M}_D$ such that $\prod_C \mathcal{M}_C$ and $\prod_C \mathcal{M}_D$ are not even elementary equivalent. The structure $\mathcal{M}_C$ also provides a sharp bound for the fundamental theorem of cohesive powers. Namely, for the Σ_3^0 sentence Ψ, $\mathcal{M}_C \vDash \neg\Psi$ but $\prod_C \mathcal{M}_C \vDash \Psi$.

6.3 Linear orders and their cohesive powers

We investigate the cohesive powers of computable linear orders, with special attention to computable linear orders of type ω. A *linear order* $\mathcal{L} = (L, \prec)$

consists of a non-empty set L equipped with a binary relation $\prec$ satisfying the following axioms.

- $\forall x\,(x \nprec x)$
- $\forall x \forall y\,(x \prec y \ \rightarrow\ y \nprec x)$
- $\forall x \forall y \forall z\,[(x \prec y \wedge y \prec z)\ \rightarrow\ x \prec z]$
- $\forall x \forall y\,(x \prec y\ \vee\ x = y\ \vee\ y \prec x)$

Additionally, a linear order $\mathcal{L}$ is *dense* if $\forall x \forall y \exists z\,(x \prec y\ \rightarrow\ x \prec z \prec y)$ and *has no endpoints* if $\forall x \exists y \exists z\,(y \prec x \prec z)$. Rosenstein's book [Ros82] is an excellent reference for linear orders.

For a linear order $\mathcal{L} = (L, \prec)$, we use the usual interval notation $(a,b)_{\mathcal{L}} = \{x \in L : a \prec x \prec b\}$ and $[a,b]_{\mathcal{L}} = \{x \in L : a \preceq x \preceq b\}$ to denote open and closed intervals of $\mathcal{L}$. Sometimes it is convenient to allow $b \preceq a$ in this notation, in which case, for example, $(a,b)_{\mathcal{L}} = \varnothing$. The notation $|(a,b)_{\mathcal{L}}|$ denotes the cardinality of the interval $(a,b)_{\mathcal{L}}$. The notations $\min_{\prec}\{a,b\}$ and $\max_{\prec}\{a,b\}$ denote the minimum and maximum of a and b with respect to $\prec$.

As is customary, ω denotes the order-type of $(\mathbb{N},<)$, ζ denotes the order-type of $(\mathbb{Z},<)$, and η denotes the order-type of $(\mathbb{Q},<)$. That is, ω, ζ, and η denote the respective order-types of the natural numbers, the integers, and the rationals, each with their usual order. We refer to $(\mathbb{N},<)$, $(\mathbb{Z},<)$, and $(\mathbb{Q},<)$ as the *standard presentations* of ω, ζ, and η, respectively. Recall that every countable dense linear order without endpoints has order-type η (see [Ros82] Theorem 2.8). Furthermore, every computable countable dense linear order without endpoints is computably isomorphic to $\mathbb{Q}$ (see [Ros82] Exercise 16.4).

To help reason about order-types, we use the *sum*, *product*, and *reverse* of linear orders as well as *condensations* of linear orders.

Definition 6.3.1. Let $\mathcal{L}_0 = (L_0, \prec_{\mathcal{L}_0})$ and $\mathcal{L}_1 = (L_0, \prec_{\mathcal{L}_1})$ be linear orders.

- The *sum* $\mathcal{L}_0 + \mathcal{L}_1$ of $\mathcal{L}_0$ and $\mathcal{L}_1$ is the linear order $\mathcal{S} = (S, \prec_{\mathcal{S}})$, where $S = (\{0\} \times L_0) \cup (\{1\} \times L_1)$ and

$$(i,x) \prec_{\mathcal{S}} (j,y) \quad \text{if and only if} \quad (i < j)\ \vee\ (i = j\ \wedge\ x \prec_{\mathcal{L}_i} y).$$

- The *product* $\mathcal{L}_0\mathcal{L}_1$ of $\mathcal{L}_0$ and $\mathcal{L}_1$ is the linear order $\mathcal{P} = (P, \prec_{\mathcal{P}})$, where $P = L_1 \times L_0$ and

$$(x,a) \prec_{\mathcal{P}} (y,b) \quad \text{if and only if} \quad (x \prec_{\mathcal{L}_1} y)\ \vee\ (x = y\ \wedge\ a \prec_{\mathcal{L}_0} b).$$

  Note that, by (fairly entrenched) convention, $\mathcal{L}_0\mathcal{L}_1$ is given by the product order on $L_1 \times L_0$, not on $L_0 \times L_1$.

- The *reverse* $\mathcal{L}_0^*$ of $\mathcal{L}_0$ is the linear order $\mathcal{R} = (R, \prec_{\mathcal{R}})$, where $R = L_0$ and $x \prec_{\mathcal{R}} y$ if and only if $y \prec_{\mathcal{L}_0} x$. We warn the reader that the $*$ in the notation $\mathcal{L}_0^*$ is unrelated to the $*$ in the notation $X \subseteq^* Y$.

If $\mathcal{L}_0$ and $\mathcal{L}_1$ are computable linear orders, then one may use the pairing function $\langle \cdot, \cdot \rangle$ to compute copies of $\mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{L}_0\mathcal{L}_1$. Clearly, if $\mathcal{L}$ is a computable linear order, then so is $\mathcal{L}^*$.

Definition 6.3.2. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a linear order. A *condensation* of $\mathcal{L}$ is any linear order $\mathcal{M} = (M, \prec_{\mathcal{M}})$ obtained by partitioning L into a collection of non-empty intervals M and, for intervals $I, J \in M$, defining $I \prec_{\mathcal{M}} J$ if and only if $(\forall a \in I)(\forall b \in J)(a \prec_{\mathcal{L}} b)$.

The most important condensation is the *finite condensation*.

Definition 6.3.3. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a linear order. For $x \in L$, let $\mathbf{c}_{\mathrm{F}}(x)$ denote the set of $y \in L$ for which there are only finitely many elements between x and y:

$$\mathbf{c}_{\mathrm{F}}(x) = \{y \in L : \text{the interval } [\min_{\prec_{\mathcal{L}}}\{x, y\}, \max_{\prec_{\mathcal{L}}}\{x, y\}]_{\mathcal{L}} \text{ in } \mathcal{L} \text{ is finite}\}.$$

The set $\mathbf{c}_{\mathrm{F}}(x)$ is always a non-empty interval, as $x \in \mathbf{c}_{\mathrm{F}}(x)$. The *finite condensation* $\mathbf{c}_{\mathrm{F}}(\mathcal{L})$ of $\mathcal{L}$ is the condensation obtained from the partition $\{\mathbf{c}_{\mathrm{F}}(x) : x \in L\}$.

For example, $\mathbf{c}_{\mathrm{F}}(\omega) \cong 1$, $\mathbf{c}_{\mathrm{F}}(\zeta) \cong 1$, $\mathbf{c}_{\mathrm{F}}(\eta) \cong \eta$, and $\mathbf{c}_{\mathrm{F}}(\omega + \zeta\eta) \cong 1 + \eta$. Notice that for an element x of a linear order $\mathcal{L}$, the order-type of $\mathbf{c}_{\mathrm{F}}(x)$ is always either finite, ω, ω^*, or ζ.

We often refer to the intervals that comprise a condensation of a linear order $\mathcal{L}$ as *blocks*. For the finite condensation of $\mathcal{L}$, a block is a maximal interval I such that for any two elements of I, there are only finitely many elements of $\mathcal{L}$ between them. For elements a and b of $\mathcal{L}$, we write $a \lll_{\mathcal{L}} b$ if the interval $(a, b)_{\mathcal{L}}$ (equivalently, the interval $[a, b]_{\mathcal{L}}$) in $\mathcal{L}$ is infinite. For $a \prec_{\mathcal{L}} b$, we have that $a \lll_{\mathcal{L}} b$ if and only if a and b are in different blocks. See [Ros82] Chapter 4 for more on condensations.

It is straightforward to directly verify that if $\mathcal{L}$ is a computable linear order and C is cohesive, then $\Pi_C\mathcal{L}$ is again a linear order. Furthermore, one may verify that if $\mathcal{L}$ is a computable dense linear order without endpoints, then $\Pi_C\mathcal{L}$ is again a dense linear order without endpoints. These two facts also follow from Theorem 6.1.3 because linear orders are described by Π_1^0 sentences, and dense linear orders without endpoints are described by Π_2^0 sentences.

The case of $\mathbb{Q} = (\mathbb{Q}, <)$ is curious and deserves a digression. We have seen that if $\mathcal{A}$ is a finite structure, then $\mathcal{A} \cong \Pi_C\mathcal{A}$ for every cohesive set C. For $\mathbb{Q}$, $\Pi_C\mathbb{Q}$ is a countable dense linear order without endpoints, and hence isomorphic to $\mathbb{Q}$, for every cohesive set C. Thus $\mathbb{Q}$ is an example of an infinite computable structure with $\mathbb{Q} \cong \Pi_C \mathbb{Q}$ for every cohesive set C. That $\mathbb{Q}$ is isomorphic to all of its cohesive powers is no accident. By combining Theorem 6.1.3 with the theory of *Fraïssé limits* (see [Hod93] Chapter 6, for example), we see that a uniformly locally finite ultrahomogeneous computable structure for a finite language is always isomorphic to all of its cohesive powers. Recall that a structure is *locally finite* if every finitely-generated substructure is finite and is *uniformly locally finite* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every substructure generated by at most n elements has cardinality at most $f(n)$. Notice that every structure for a finite relational language is uniformly locally finite. Also recall that a structure is *ultrahomogeneous* if every isomorphism between two finitely-generated substructures extends to an automorphism of the structure.

Proposition 6.3.4. Let $\mathcal{A}$ be an infinite uniformly locally finite ultrahomogeneous computable structure for a finite language, and let C be cohesive. Then $\mathcal{A} \cong \Pi_C \mathcal{A}$.

Proposition 6.3.4 implies that if a uniformly locally finite computable structure for a finite language is a Fraïssé limit, then it is isomorphic to all of its cohesive powers. Thus computable presentations of the Rado graph and the countable atomless Boolean algebra are additional examples of computable structures that are isomorphic to all of their cohesive powers. Examples of this phenomenon that cannot be attributed to ultrahomogeneity appear in Sections 6.4 and 6.5.

Returning to linear orders, we recall the following well-known lemma stating that a strictly order-preserving surjection from one linear order onto another is necessarily an isomorphism.

Lemma 6.3.5. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ and $\mathcal{M} = (M, \prec_{\mathcal{M}})$ be linear orders. If $f : L \to M$ is surjective and satisfies $(\forall x, y \in L)[x \prec_{\mathcal{L}} y \;\to\; f(x) \prec_{\mathcal{M}} f(y)]$, then f is an isomorphism.

Cohesive powers commute with sums, products, and reverses.

Theorem 6.3.6. Let $\mathcal{L}_0 = (L_0, \prec_{\mathcal{L}_0})$ and $\mathcal{L}_1 = (L_1, \prec_{\mathcal{L}_1})$ be computable linear orders, and let C be cohesive. Then

(1) $\Pi_C(\mathcal{L}_0 + \mathcal{L}_1) \cong \Pi_C \mathcal{L}_0 + \Pi_C \mathcal{L}_1$,

(2) $\Pi_C(\mathcal{L}_0 \mathcal{L}_1) \cong (\Pi_C \mathcal{L}_0)(\Pi_C \mathcal{L}_1)$, and

(3) $\Pi_C(\mathcal{L}_0^*) \cong (\Pi_C \mathcal{L}_0)^*$.

Sections 6.4, 6.5, and 6.6 concern calculating the order-types of cohesive powers of computable copies of ω. To do this, we must be able to determine when one element of a cohesive power is an immediate successor or immediate predecessor of another, and we must be able to determine when two elements of a cohesive power are in different blocks of its finite condensation.

In a cohesive power $\Pi_C\mathcal{L}$ of a computable linear order $\mathcal{L}$, $[\varphi]$ is the immediate successor of $[\psi]$ if and only if $\varphi(n)$ is the immediate successor of $\psi(n)$ for almost every $n \in C$. Therefore also $[\psi]$ is the immediate predecessor of $[\varphi]$ if and only if $\psi(n)$ is the immediate predecessor of $\varphi(n)$ for almost every $n \in C$.

Lemma 6.3.7. Let $\mathcal{L}$ be a computable linear order, let C be cohesive, and let $[\psi]$ and $[\varphi]$ be elements of $\Pi_C\mathcal{L}$. Then the following are equivalent.

(1) $[\varphi]$ is the $<_{\Pi_C\mathcal{L}}$-immediate successor of $[\psi]$.

(2) $(\forall^\infty n \in C)(\varphi(n)$ is the $<_{\mathcal{L}}$-immediate successor of $\psi(n))$.

(3) $(\exists^\infty n \in C)(\varphi(n)$ is the $<_{\mathcal{L}}$-immediate successor of $\psi(n))$.

Lemma 6.3.8. Let $\mathcal{L}$ be a computable linear order, let C be cohesive, and let $[\psi]$ and $[\varphi]$ be elements of $\Pi_C\mathcal{L}$. Then the following are equivalent.

(1) $[\psi] \ll_{\Pi_C\mathcal{L}} [\varphi]$.

(2) $\lim_{n\in C}|(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$.

(3) $\limsup_{n\in C}|(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$.

The finite condensation of a computable linear order by a co-c.e. cohesive set is always dense.

Theorem 6.3.9. Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be a computable linear order, and let C be co-c.e. and cohesive. Then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ is dense.

Let $[\varphi]$ and $[\psi]$ be elements of $\Pi_C\mathcal{L}$ with $[\psi] \ll_{\Pi_C\mathcal{L}} [\varphi]$. We partially compute a function $\theta\colon \mathbb{N} \to L$ so that $[\theta]$ is an element of $\Pi_C\mathcal{L}$ with $[\psi] \ll_{\Pi_C\mathcal{L}} [\theta] \ll_{\Pi_C\mathcal{L}} [\varphi]$.

By Lemma 6.3.8, $[\psi] \ll_{\Pi_C\mathcal{L}} [\varphi]$ means that $\limsup_{n\in C}|(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$. We define θ by enumerating $\mathrm{graph}(\theta) = \{\langle n, x\rangle : \theta(n) = x\}$. The goal is to arrange $|C \cap dom(\theta)| = \infty$ (so that $C \subseteq^* dom(\theta)$ by cohesiveness), $\limsup_{n\in C}|(\psi(n), \theta(n))_{\mathcal{L}}| = \infty$, and $\limsup_{n\in C}|(\theta(n), \varphi(n))_{\mathcal{L}}| = \infty$. It then follows that $[\psi] \ll_{\Pi_C\mathcal{L}} [\theta] \ll_{\Pi_C\mathcal{L}} [\varphi]$.

6.4 Cohesive powers of computable copies of ω

We investigate the cohesive powers of computable linear orders of type ω. Observe that an infinite linear order has type ω if and only if every element has only finitely many predecessors. We rely on this characterization throughout. Though not part of the language of linear orders, every linear order $\mathcal{L}$ of type ω has an associated successor function $S^{\mathcal{L}}\colon |\mathcal{L}| \to |\mathcal{L}|$ given by $S^{\mathcal{L}}(x)$ = the $<_{\mathcal{L}}$-immediate successor of x. For the standard presentation of ω, the successor function is of course given by the computable function $S(x) = x + 1$. It is straightforward to check that a computable copy $\mathcal{L}$ of ω is computably isomorphic to the standard presentation if and only if $S^{\mathcal{L}}$ is computable.

We show that every cohesive power of the standard presentation of ω has order-type $\omega + \zeta\eta$ (Theorem 6.4.5). This is to be expected because $\omega + \zeta\eta$ is familiar as the order-type of every countable non-standard model of Peano arithmetic (see [Kay91] Theorem 6.4). Therefore, by Theorem 6.1.4, every cohesive power of every computable copy of ω that is computably isomorphic to the standard presentation has order-type $\omega + \zeta\eta$; or, equivalently, every cohesive power of every computable copy of ω with a computable successor function has order-type $\omega + \zeta\eta$. However, being computably isomorphic to the standard presentation (equivalently, having a computable successor function) is not a characterization of the computable copies of ω having cohesive powers of order-type $\omega + \zeta\eta$. We show that there is a computable copy of ω that is not computably isomorphic to the standard presentation, yet still has every cohesive power isomorphic to $\omega + \zeta\eta$ (Theorem 6.4.8). Thus to compute a copy of ω having a cohesive power not of type $\omega + \zeta\eta$, one must do more than simply arrange for the successor function to be non-computable. We show that for every cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that the cohesive power $\Pi_C\mathcal{L}$ does not have order-type $\omega + \zeta\eta$ (Theorem 6.4.9). However, we also show that whenever $\mathcal{L}$ is a computable copy of ω and C is a co-c.e. cohesive set, the finite condensation $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ of the cohesive power $\Pi_C\mathcal{L}$ always has order-type $1 + \eta$ (Theorem 6.4.4).

First, a cohesive power of a computable copy of ω always has an initial segment of order-type ω.

Lemma 6.4.1. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, and let C be cohesive. Then the image of the canonical embedding of $\mathcal{L}$ into $\Pi_C\mathcal{L}$ is an initial segment of $\Pi_C\mathcal{L}$ of order-type ω.

Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $\varphi\colon \mathbb{N} \to L$ be any total computable bijection. Then $[\varphi]$ is not in the image of the canonical embedding of $\mathcal{L}$ into $\Pi_C\mathcal{L}$, so it must be $\prec_{\Pi_C\mathcal{L}}$-above every element in the image of the canonical embedding. Thus $\Pi_C\mathcal{L}$ is of the form $\omega + \mathcal{M}$ for some non-empty linear order $\mathcal{M}$. By analogy with the terminology for models of Peano arithmetic, we call the elements of the ω-part of $\Pi_C\mathcal{L}$ (i.e., the image of the canonical embedding) *standard* and the elements of the $\mathcal{M}$-part of $\Pi_C\mathcal{L}$ *non-standard*.

Lemma 6.4.2. Let $\mathcal{L} = (L, \prec_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $[\varphi]$ be an element of $\Pi_C\mathcal{L}$. Then $[\varphi]$ is non-standard if and only if $\liminf_{n \in C} \varphi(n) = \infty$ (equivalently, $\lim_{n \in C} \varphi(n) = \infty$).

Lemma 6.4.3. Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be a computable copy of ω, let C be cohesive, and let $[\varphi]$ be a non-standard element of $\Pi_C\mathcal{L}$. Then there are non-standard elements $[\psi^-]$ and $[\psi^+]$ of $\Pi_C\mathcal{L}$ with $[\psi^-] \preccurlyeq_{\Pi_C\mathcal{L}} [\varphi] \preccurlyeq_{\Pi_C\mathcal{L}} [\psi^+]$.

Lemmas 6.4.1 and 6.4.3 imply that if $\mathcal{L}$ is a computable copy of ω and C is cohesive, then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \mathcal{M}$ for some infinite linear order $\mathcal{M}$. We call the block corresponding to 1 the *standard block* and the blocks corresponding to $\mathcal{M}$ *non-standard blocks*. If we further assume that C is co-c.e., then we obtain that $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \eta$.

Theorem 6.4.4. Let $\mathcal{L}$ be a computable copy of ω, and let C be co-c.e. and cohesive. Then $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L})$ has order-type $1 + \eta$.

By Lemma 6.4.1, the standard elements of $\Pi_C\mathcal{L}$ form an initial block. By Theorem 6.3.9 and Lemma 6.4.3, the non-standard blocks of $\Pi_C\mathcal{L}$ form a countable dense linear order without endpoints. Thus $\mathbf{c}_{\mathrm{F}}(\Pi_C\mathcal{L}) \cong 1 + \eta$.

Thinking in terms of blocks, showing that a linear order $\mathcal{M}$ has type $\omega + \zeta\eta$ amounts to showing that $\mathcal{M}$ consists of an initial block of type ω followed by densely (without endpoints) ordered blocks of type ζ.

Theorem 6.4.5. Let $\mathbb{N}$ denote the standard presentation of ω, and let C be cohesive. Then $\Pi_C\mathbb{N}$ has order-type $\omega + \zeta\eta$.

By Lemma 6.4.1, $\Pi_C\mathbb{N}$ has an initial segment of order-type ω. To show that the non-standard blocks each have order-type ζ, we show that every element of $\Pi_C\mathbb{N}$ has an $<_{\Pi_C\mathbb{N}}$-immediate successor and that every element of $\Pi_C\mathbb{N}$ except the first element has an $<_{\Pi_C\mathbb{N}}$-immediate predecessor.

By Lemma 6.4.3, there is neither a least nor a greatest non-standard block of $\Pi_C\mathbb{N}$. We cannot use Theorem 6.3.9 to conclude that the non-standard blocks are densely ordered because we do not assume that C is co-c.e. So suppose $[\varphi]$ and $[\psi]$ are such that $[\psi] \ll_{\Pi_C\mathbb{N}} [\varphi]$. Then $\lim_{n\in C}|(\psi(n), \varphi(n))| = \infty$ by Lemma 6.3.8. Define a partial computable function θ by $\theta(n) \simeq \lfloor(\varphi(n) + \psi(n))/2\rfloor$. Then $\lim_{n\in C}|(\psi(n), \theta(n))| = \infty$ and $\lim_{n\in C}|(\theta(n), \varphi(n))| = \infty$, so $[\psi] \ll_{\Pi_C\mathbb{N}} [\theta] \ll_{\Pi_C\mathbb{N}} [\varphi]$. Thus the non-standard blocks of $\Pi_C\mathbb{N}$ form a dense linear order without endpoints. Thus $\Pi_C\mathbb{N} \cong \omega + \zeta\eta$.

Corollary 6.4.6. Let $\mathcal{L}$ be a computable copy of ω with a computable successor function, and let C be cohesive. Then $\Pi_C\mathcal{L}$ has order-type $\omega + \zeta\eta$.

We can calculate the order-types of the cohesive powers of many other computable presentations of linear orders by combining Theorems 6.1.4, 6.3.6, 6.4.5, and the fact that $\Pi_C\mathbb{Q} \cong \eta$.

Example 6.4.7. Let C be a cohesive set. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the standard presentations of ω, ζ, and η.

(1) $\Pi_C\mathbb{N}^* \cong \zeta\eta + \omega^*$: This is because

$$\Pi_C\mathbb{N}^* \;\cong\; (\Pi_C\mathbb{N})^* \;\cong\; (\omega + \zeta\eta)^* \;\cong\; \zeta\eta + \omega^*.$$

(2) $\Pi_C\mathbb{Z} \cong \zeta\eta$. This is because $\mathbb{Z}$ is computably isomorphic to $\mathbb{N}^* + \mathbb{N}$, so

$$\begin{aligned}\Pi_C\mathbb{Z} \;\cong\; \Pi_C(\mathbb{N}^* + \mathbb{N}) \;\cong\; \Pi_C(\mathbb{N})^* + \Pi_C(\mathbb{N}) \;&\cong\; (\zeta\eta + \omega^*) + (\omega + \zeta\eta)\\ &\cong\; \zeta\eta + \zeta + \zeta\eta \;\cong\; \zeta\eta.\end{aligned}$$

(3) $\Pi_C(\mathbb{Z}\mathbb{Q}) \cong \zeta\eta$. This is because

$$\Pi_C(\mathbb{Z}\mathbb{Q}) \;\cong\; (\Pi_C\mathbb{Z})(\Pi_C\mathbb{Q}) \;\cong\; (\zeta\eta)\eta \;\cong\; \zeta\eta.$$

(4) $\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q}) \cong \omega + \zeta\eta$. This is because

$$\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q}) \;\;\cong\;\; \Pi_C(\mathbb{N}) + \Pi_C(\mathbb{Z}\mathbb{Q}) \;\;\cong\;\; (\omega + \zeta\eta) + \zeta\eta \;\;\cong\;\; \omega + \zeta\eta.$$

Recall that, by Proposition 6.3.4, an ultrahomogeneous computable structure for a finite relational language, like the computable linear order $\mathbb{Q}$, is isomorphic to each of its cohesive powers. Notice, however, that the computable linear orders $\mathbb{Z}\mathbb{Q}$ and $\mathbb{N} + \mathbb{Z}\mathbb{Q}$ are not ultrahomogeneous, yet nevertheless are isomorphic to each of their respective cohesive powers. Thus it is also possible for a non-ultrahomogeneous computable structure to be isomorphic to each of its cohesive powers.

Notice also that $\Pi_C\mathbb{N}$ and $\Pi_C(\mathbb{N} + \mathbb{Z}\mathbb{Q})$ both have order-type $\omega + \zeta\eta$. Similarly, $\Pi_C\mathbb{Z}$ and $\Pi_C(\mathbb{Z}\mathbb{Q})$ both have order-type $\zeta\eta$. Thus it is possible for non-isomorphic linear orders to have isomorphic cohesive powers. In Section 6.5, we give an example of a pair of non-elementarily equivalent linear orders with isomorphic cohesive powers.

Now we give an example of a computable copy of ω that is not computably isomorphic to the standard presentation, yet still has all its cohesive powers isomorphic to $\omega + \zeta\eta$.

Theorem 6.4.8. There is a computable copy $\mathcal{L}$ of ω such that

- $\mathcal{L}$ is not computably isomorphic to the standard presentation of ω, yet
- for every cohesive set C, the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \zeta\eta$.

We use a classic example of a computable copy of ω with a non-computable successor function.

Let C be cohesive. We show that $\Pi_C\mathcal{L} \cong \omega + \zeta\eta$. As in the proof of Theorem 6.4.5, it suffices to establish the following.

(a) Every element of $\Pi_C\mathcal{L}$ has a $<_{\Pi_C\mathcal{L}}$-immediate successor.

(b) Every element of $\Pi_C\mathcal{L}$ that is not the $\prec_{\Pi_C\mathcal{L}}$-least element has a $\prec_{\Pi_C\mathcal{L}}$-immediate predecessor.

(c) If $[\psi],[\varphi] \in |\Pi_C\mathcal{L}|$ satisfy $[\psi] \preccurlyeq_{\Pi_C\mathcal{L}} [\varphi]$, then there is a $[\theta] \in |\Pi_C\mathcal{L}|$ with $[\psi] \preccurlyeq_{\Pi_C\mathcal{L}} [\theta] \preccurlyeq_{\Pi_C\mathcal{L}} [\varphi]$.

Lastly, we show that for every cohesive set C, there is a computable copy $\mathcal{L}$ of ω such that $\Pi_C\mathcal{L}$ is not isomorphic, indeed, not elementarily equivalent, to $\omega+\zeta\eta$. The strategy is to arrange for the element [id] of $\Pi_C\mathcal{L}$ represented by the identity function $\mathrm{id}: \mathbb{N} \to \mathbb{N}$ to have no $\prec_{\Pi_C\mathcal{L}}$-immediate successor. This exhibits an elementary difference between $\Pi_C\mathcal{L}$ and $\omega + \zeta\eta$ because every element of $\omega + \zeta\eta$ has an immediate successor. This also shows that Theorem 6.1.3 part (4) is tight: "there is an element with no immediate successor" is a Σ^0_3 sentence that is true of $\Pi_C\mathcal{L}$ but not of $\mathcal{L}$.

Theorem 6.4.9. Let C be any cohesive set. Then there is a computable copy $\mathcal{L}$ of ω for which $\Pi_C\mathcal{L}$ is not elementarily equivalent (and hence not isomorphic) to $\omega+\zeta\eta$.

Corollary 6.4.10. Theorem 6.1.3 item (4) is tight in general: There is a cohesive set C, a computable linear order $\mathcal{L}$, and a Σ^0_3 sentence Φ such that $\Pi_C\mathcal{L} \vDash \Phi$, but $\mathcal{L} \nvDash \Phi$.

Corollary 6.4.10 may also be deduced from Lerman's proof of Feferman, Scott, and Tennenbaum's theorem that no cohesive power of the standard model of arithmetic is a model of Peano arithmetic (see [Ler70] Theorem 2.1). Lerman gives a somewhat technical example of a Σ^0_3 sentence Φ invoking Kleene's T predicate that fails in the standard model of arithmetic but is true in every cohesive power. Our proof of Corollary 6.4.10 is more satisfying because it witnesses the optimality of Theorem 6.1.3 item (4) with a natural Σ^0_3 sentence in the simple language of linear orders.

In the next section, we enhance the construction of Theorem 6.4.9 in order to compute a copy $\mathcal{L}$ of ω with $\Pi_C\mathcal{L} \cong \omega+\eta$ for a given co-c.e. cohesive set C.

6.5 A cohesive power of order-type $\omega + \eta$

Given a co-c.e. cohesive set, we compute a copy $\mathcal{L}$ of ω for which $\Pi_C\mathcal{L}$ has order-type $\omega + \eta$. In order to help shuffle various linear orders into cohesive powers in Section 6.6, we in fact compute a linear order $\mathcal{L} = (\mathbb{N}, \prec_\mathcal{L})$ along with a coloring function $F: \mathbb{N} \to \mathbb{N}$ that colors the elements of $\mathcal{L}$ with countably many colors. The coloring F induces a coloring $\widehat{F}$ of $\Pi_C\mathcal{L}$ in the following way. Colors of elements of $\Pi_C\mathcal{L}$ are represented by partial computable functions $\delta: \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* dom(\delta)$. As in Definition 6.1.2, write $\delta_0 =_C \delta_1$ if $(\forall^\infty n \in C)(\delta_0(n)\downarrow = \delta_1(n)\downarrow)$, and write $[\![\delta]\!]$ instead of $[\delta]$

for the $=_C$-equivalence class of δ when thinking in terms of colors. Then $\widehat{F}$ is given by $\widehat{F}([\varphi]) = [\![F \circ \varphi]\!]$. So, for example, elements $[\varphi]$ and $[\psi]$ of $\Pi_C\mathcal{L}$ have the same $\widehat{F}$-color if and only if $\varphi(n)$ and $\psi(n)$ have the same F-color for almost every $n \in C$.

Call a color $[\![\delta]\!]$ a *solid color* if there is an $x \in \mathbb{N}$ such that $(\forall^\infty n \in C)(\delta(n) = x)$. Otherwise, call $[\![\delta]\!]$ a *striped color*. Observe that if $[\![\delta]\!]$ is striped, then $\lim_{n \in C}\delta(n) = \infty$. We compute $\mathcal{L}$ and F so that $\Pi_C\mathcal{L} \cong \omega + \eta$ and every solid color occurs densely in the η-part. Between any two distinct elements of the η-part there is also an element with a striped color, but we do not ask for every striped color to occur densely. In Section 6.6, we show that replacing each point of $\mathcal{L}$ by some finite linear order depending on its color has the effect of shuffling these finite orders into the non-standard part of $\Pi_C\mathcal{L}$.

Theorem 6.5.1. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{L} = (\mathbb{N}, \prec_{\mathcal{L}})$ of ω and a computable coloring $F{:}\,\mathbb{N} \to \mathbb{N}$ of $\mathcal{L}$ with the following property. Let $[\varphi]$ and $[\psi]$ be any two non-standard elements of $\Pi_C\mathcal{L}$ with $[\psi] \prec_{\Pi_C\mathcal{L}} [\varphi]$. Then for every solid color $[\![\delta]\!]$, there is a $[\theta]$ in $\Pi_C\mathcal{L}$ with $[\psi] \prec_{\Pi_C\mathcal{L}} [\theta] \prec_{\Pi_C\mathcal{L}} [\varphi]$ and $\widehat{F}([\theta]) = [\![\delta]\!]$. Also, there is a $[\theta]$ in $\Pi_C\mathcal{L}$ with $[\psi] \prec_{\Pi_C\mathcal{L}} [\theta] \prec_{\Pi_C\mathcal{L}} [\varphi]$, where $\widehat{F}([\theta])$ is a striped color.

We are working with a co-c.e. cohesive set, so recall that in this situation every element $[\varphi]$ of $\Pi_C\mathcal{L}$ has a total representative by the discussion following Definition 6.1.2. Recall also that an element $[\varphi]$ of $\Pi_C\mathcal{L}$ is non-standard if and only if $\lim_{n \in C}\varphi(n) = \infty$ by Lemma 6.4.2.

The goal of the construction of $\mathcal{L}$ is to arrange, for every pair of total computable functions φ and ψ with $\lim_{n \in C}\varphi(n) = \lim_{n \in C}\psi(n) = \infty$, that

$$\begin{aligned}&(\forall^\infty n \in C)(\psi(n)\downarrow \prec_{\mathcal{L}} \varphi(n)\downarrow \Rightarrow \\ &(\forall d \le \max_{<}\{\varphi(n), \psi(n)\})(\exists k)[(\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)) \wedge (F(k) = d)]). \qquad (*)\end{aligned}$$

Suppose we achieve $(*)$ for φ and ψ, where $\lim_{n \in C}\varphi(n) = \lim_{n \in C}\psi(n) = \infty$ and $(\forall^\infty n \in C)(\varphi(n)\downarrow \prec_{\mathcal{L}} \psi(n)\downarrow)$. Fix any color d, and let δ be the constant function with value d. Partially compute a function $\theta(n)$ by searching for a k with $\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)$ and $F(k) = d$. If there is such a k, let $\theta(n)$ be the first such k. Property $(*)$ and the assumption $\lim_{n \in C}\varphi(n) = \lim_{n \in C}\psi(n) = \infty$ ensure that there is such a k for almost every $n \in C$. Therefore $C \subseteq^* dom(\theta)$, $[\psi] \prec_{\Pi_C\mathcal{L}} [\theta] \prec_{\Pi_C\mathcal{L}} [\varphi]$, and $\widehat{F}([\theta]) = [\![\delta]\!]$. Likewise, we could instead define $\theta(n)$ to search for a k with $\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)$ and $F(k) = \varphi(n)$ and let $\theta(n)$ be the first (if any) such k found. In this case we would have $[\psi] \prec_{\Pi_C\mathcal{L}} [\theta] \prec_{\Pi_C\mathcal{L}} [\varphi]$ and $\widehat{F}([\theta]) = [\![\varphi]\!]$, which is a striped color because $\lim_{n \in C}\varphi(n) = \infty$. Thus achieving $(*)$ suffices to prove the theorem, provided we also arrange $\mathcal{L} \cong \omega$. The tension in the construction is between achieving $(*)$ and ensuring that for every z, there are only finitely many x with $x \prec_{\mathcal{L}} z$.

Corollary 6.5.2. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C\mathcal{L}$ has order-type $\omega + \eta$.

Example 6.5.3. Let C be a co-c.e. cohesive set, and let $\mathcal{L}$ be a computable copy of ω with $\Pi_C\mathcal{L} \cong \omega + \eta$ as in Corollary 6.5.2.

(1) There is a countable collection of computable copies of ω whose cohesive powers over C are pairwise non-elementarily equivalent. Let $k \geq 1$, and let k denote the k-element linear order $0 < 1 < \cdots < k-1$ as well as its order-type. Then $k\mathcal{L}$ has order-type ω because $\mathcal{L}$ has order-type ω, and $\Pi_C k \cong k$ by the discussion following Theorem 6.1.3. Using Theorem 6.3.6, we calculate

$$\Pi_C(k\mathcal{L}) \;\cong\; (\Pi_C k)(\Pi_C\mathcal{L}) \;\cong\; k(\omega + \eta) \;\cong\; \omega + k\eta.$$

The linear orders $\omega + k\eta$ for $k \geq 1$ are pairwise non-elementarily equivalent. The sentence "there are $x_0 \prec \cdots \prec x_{k-1}$ such that every other y satisfies either $y \prec x_0$ or $x_{k-1} \prec y$; if $y \prec x_0$, then there is a z with $y \prec z \prec x_0$; and if $x_{k-1} \prec y$, then there is a z with $y \prec z \prec x_{k-1}$" expressing that there is a maximal block of size k is true of $\omega + k\eta$, but not of $\omega + m\eta$ if $m \neq k$. Thus $1\mathcal{L}, 2\mathcal{L}, \ldots$ is a sequence of computable copies of ω whose cohesive powers $\Pi_C(k\mathcal{L})$ are pairwise non-elementarily equivalent.

(2) It is possible for non-elementarily equivalent computable linear orders to have isomorphic cohesive powers. Consider the computable linear orders $\mathcal{L}$ and $\mathcal{L} + \mathbb{Q}$. They are not elementarily equivalent because the sentence "every element has an immediate successor" is true of $\mathcal{L}$ but not of $\mathcal{L} + \mathbb{Q}$. However, using Theorem 6.3.6 and the fact that $\Pi_C\mathbb{Q} \cong \eta$, we calculate

$$\Pi_C(\mathcal{L} + \mathbb{Q}) \;\cong\; \Pi_C\mathcal{L} + \Pi_C\mathbb{Q} \;\cong\; (\omega + \eta) + \eta \;\cong\; \omega + \eta \;\cong\; \Pi_C\mathcal{L}.$$

Thus the cohesive powers $\Pi_C\mathcal{L}$ and $\Pi_C(\mathcal{L} + \mathbb{Q})$ of $\mathcal{L}$ and $\mathcal{L} + \mathbb{Q}$ are isomorphic.

6.6 Shuffling finite linear orders

The *shuffle* $\sigma(X)$ of an at-most-countable non-empty collection X of order-types is obtained by densely coloring $\mathbb{Q}$ with $|X|$-many colors, assigning each order-type in X a distinct color, and replacing each $q \in \mathbb{Q}$ by a copy of the linear order whose type corresponds to the color of q.

Definition 6.6.1. Let X be a non-empty collection of linear orders with $|X| \leq \aleph_0$, let $(\mathcal{L}_i)_{i<|X|}$ be a list of the elements of X, and write $\mathcal{L}_i = (L_i, \prec_{\mathcal{L}_i})$ for each $i < |X|$. Let $F\colon \mathbb{Q} \to |X|$ be a coloring of $\mathbb{Q}$ in which each color occurs densely. Define a linear order $\mathcal{S} = (S, <_{\mathcal{S}})$ by replacing each $q \in \mathbb{Q}$ by a copy of $\mathcal{L}_{F(q)}$. Formally, let $S = \{(q, \ell) : q \in \mathbb{Q} \wedge \ell \in L_{F(q)}\}$ and

$$(p, \ell) <_{\mathcal{S}} (q, r) \quad \text{if and only if} \quad (p < q) \vee (p = q \wedge \ell <_{\mathcal{L}_{F(p)}} r).$$

Because every color occurs densely, the order-type of $\mathcal{S}$ does not depend on the particular choice of F or on the order in which X is enumerated. For this reason, $\mathcal{S}$ is called the *shuffle* of X and is denoted $\sigma(X)$. We typically think of X as a collection of order-types instead of as a collection of concrete linear orders.

Let C be co-c.e. and cohesive, let $\mathcal{L}$ be the linear order from Corollary 6.5.2 for C, and consider the linear order $2\mathcal{L}$ from Example 6.5.3 item (1). We can think of $2\mathcal{L}$ as being obtained from $\mathcal{L}$ by replacing each element of $\mathcal{L}$ by a copy of 2. This operation of replacing each element by a copy of 2 is reflected in the cohesive power, and we have that $\Pi_C(2\mathcal{L}) \cong \omega + 2\eta$.

Let us now consider this same $\mathcal{L} = (L, <_{\mathcal{L}})$ along with its coloring $F\colon L \to \mathbb{N}$ from Theorem 6.5.1. Collapse F into a coloring $G\colon L \to \{0, 1\}$, where $G(x) = 0$ if $F(x) = 0$ and $G(x) = 1$ if $F(x) \geq 1$. Then the coloring $\widehat{G}$ of $\Pi_C\mathcal{L}$ induced by G uses exactly two colors: $[\![0]\!]$ represented by the constant function with value 0, and $[\![1]\!]$ represented by the constant function with value 1. Both of these colors occur densely in the non-standard part of $\Pi_C\mathcal{L}$. Compute a linear order $\mathcal{M}$ by starting with $\mathcal{L}$, replacing each $x \in L$ with $G(x) = 0$ by a copy of 2, and replacing each $x \in L$ with $G(x) = 1$ by a copy of 3. The cohesive power $\Pi_C\mathcal{M}$ reflects this construction, and we get the linear order obtained from $\Pi_C\mathcal{L}$ by replacing each point of $\widehat{G}$-color $[\![0]\!]$ by a copy of 2 and replacing each point of $\widehat{G}$-color $[\![1]\!]$ by a copy of 3. Thus we have a computable copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(\{2, 3\})$. Using this strategy, we can shuffle any finite collection of finite linear orders into a cohesive power of a computable copy of ω.

Theorem 6.6.2. Let $k_0, \ldots, k_N$ be non-zero natural numbers. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(\{k_0, \ldots, k_N\})$.

For the remainder of this section, let α denote the order-type $\omega + \zeta\eta + \omega^*$. Ultimately, we want to use the method of Theorem 6.6.2 to show that if $X \subseteq \mathbb{N} \setminus \{0\}$ is either Σ^0_2 or Π^0_2, then, thinking of X as a set of finite order-types, there is a cohesive power of ω with order-type $\omega + \sigma(X \cup \{\alpha\})$. We first consider the particular case $X = \mathbb{N} \setminus \{0\}$ to illustrate how α naturally appears when shuffling infinitely many finite order-types.

Theorem 6.6.3. Let X be the set of all finite non-zero order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. Let $\mathcal{M} = (M, <_{\mathcal{M}})$ be the computable linear order obtained by replacing each $x \in L$ by a copy of $x+1$ if $F(x) = 0$ and by a copy of $F(x)$ if $F(x) > 0$. Then $\mathcal{M}$ is a computable linear order of type ω.

Consider the projection condensation $\mathbf{c}_\pi(\Pi_C \mathcal{M})$ of $\Pi_C \mathcal{M}$ as colored by $\widehat{F}$. By Theorem 6.5.1, the non-standard elements of $\mathbf{c}_\pi(\Pi_C \mathcal{M})$ form a linear order of type η in which the solid $\widehat{F}$-colors occur densely. Furthermore, between any two distinct non-standard elements of $\mathbf{c}_\pi(\Pi_C \mathcal{M})$ there is a non-standard element with a striped color. If $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has solid color $[\![k]\!]$ for some $k > 0$, then its order-type is k. We show that if a non-standard $\mathbf{c}_\pi([\chi]_{\mathcal{M}})$ has either solid color $[\![0]\!]$ or a striped color, then its order-type is α. It follows that the non-standard elements of $\Pi_C \mathcal{M}$ have order-type $\sigma(X \cup \{\alpha\})$, so $\Pi_C \mathcal{M}$ has the desired order-type $\omega + \sigma(X \cup \{\alpha\})$.

Finally, to shuffle Σ^0_2 or Π^0_2 sets of finite order-types into cohesive powers of ω, it is convenient to work with linear orders whose domains are c.e. Cohesive powers of partial computable structures are defined exactly as in Definition 6.1.2, the only difference being that the domain A of the partial computable structure $\mathcal{A}$ is now c.e. instead of computable. It happens that if we wish to show that there is a computable copy of ω having a cohesive power of a certain order-type, it suffices to show that there is a partial computable copy of ω having a cohesive power of the desired order-type.

Theorem 6.6.4. Let $X \subseteq \mathbb{N} \smallsetminus \{0\}$ be a Π^0_2 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C \mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

We arrange $\mathcal{M}$ to shuffle k into $\Pi_C \mathcal{M}$ when a Π^0_2 property holds of k and to shuffle a fixed k_0 into $\Pi_C \mathcal{M}$ when a Π^0_2 property fails of k. Assume that $X \neq \varnothing$, as otherwise we can compute a copy $\mathcal{M}$ of ω with $\Pi_C \mathcal{M} \cong \omega + \sigma(\{\alpha\})$ by combining the proofs of Theorems 6.6.2 and 6.6.3. Let R be a computable predicate for which $X = \{k : \forall a \exists b R(k, a, b)\}$. Let $k_0 > 0$ be the $<$-least element of X. Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. We construct a partial computable copy $\mathcal{M}$ of ω with $\Pi_C \mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. We define $\mathcal{M}$ from $\mathcal{L}$ as follows. If $x \in L$ has $F(x) < k_0$, then replace x by a copy of $x + 1$ as is done with color 0 in the proof of Theorem 6.6.3. If $x \in L$ has $F(x) \geq k_0$, then first replace x by a copy of k_0. Then for each $a \leq x$, search for a b such that $R(F(x), a, b)$. If $(\forall a \leq x)(\exists b) R(F(x), a, b)$, then add further elements to replace x by a copy of $F(x)$ instead of by a copy of k_0. The ultimate effect of this procedure is that if $F(x) \in X$, then we shuffle $F(x)$ into $\Pi_C \mathcal{M}$; whereas if $F(x) \notin X$, then we shuffle k_0 into $\Pi_C \mathcal{M}$.

Theorem 6.6.5. Let $X \subseteq \mathbb{N} \smallsetminus \{0\}$ be a Σ^0_2 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C \mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$.

The proof is similar to that of Theorem 6.6.4. In this proof, we want to shuffle k into $\Pi_C\mathcal{M}$ when a Π^0_2 property fails of k and to shuffle α into $\Pi_C\mathcal{M}$ when a Π^0_2 property holds of k. Let $X \subseteq \mathbb{N} \smallsetminus \{0\}$ be Σ^0_2. Let R be a computable predicate for which $\overline{X} = \{k : \forall a \exists b R(k, a, b)\}$. Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be the linear order from Theorem 6.5.1 for C, along with its coloring $F\colon L \to \mathbb{N}$. Again, it suffices to produce a partial computable copy $\mathcal{M}$ of ω with $\Pi_C\mathcal{M} \cong \omega + \sigma(X \cup \{\alpha\})$. We define $\mathcal{M}$ from $\mathcal{L}$ as follows. If $x \in L$ has $F(x) = 0$, then replace x by a copy of $x + 1$ as is done in the proof of Theorem 6.6.3. If $x \in L$ has $F(x) > 0$, then first replace x by a copy of $F(x)$. Then for each $a \leq x$, search for a b such that $R(F(x), a, b)$. If $x \geq F(x)$ and $(\forall a \leq x)(\exists b) R(F(x), a, b)$, then add further elements to replace x by a copy of $x + 1$ instead of a copy of $F(x)$.

We combine the results of this section into a single statement.

Theorem 6.6.6. Let $X \subseteq \mathbb{N} \smallsetminus \{0\}$ be either a Σ^0_2 set or a Π^0_2 set, thought of as a set of finite order-types. Let C be a co-c.e. cohesive set. Then there is a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X \cup \{\alpha\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{M}$ of ω where the cohesive power $\Pi_C\mathcal{M}$ has order-type $\omega + \sigma(X)$.

Chapter 7

On Cototality and the Skip Operator

In this chapter we present the notions of cototality and skip operator in the enumeration degrees. The degree structures as $\mathcal{D}_T$ and $\mathcal{D}_e$ with $\leq$, $\oplus$ and jump operator are also abstract structures. Here we consider a subclass of the enumeration degrees $\mathcal{D}_e$ called cototal degrees. We started this project in 2015 together with Hristo Ganchev, Steffen Lempp, Joseph Miller, and Mariya Soskova in Sofia, after the CiE 2015 in Bucharest, when Joseph Miller and Steffen Lempp from University of Madison, Wisconsin, visited Sofia. In 2016 Uri Andrews and Rutger Kuyper from the same university also joined the project and we present here the results from the paper [AGK+19], in the journal *Transaction of the American Mathematical Society*.

A set $A \subseteq \mathbb{N}$ is *cototal* if it is enumeration reducible to its complement, $\overline{A}$. The *skip* of A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality and related properties, using the skip operator as a tool in our investigation. We give many examples of classes of enumeration degrees that either guarantee or prohibit cototality. We also study the skip for its own sake, noting that it has many of the nice properties of the Turing jump, even though the skip of A is not always above A (i.e., not all degrees are cototal). In fact, there is a set that is its own double skip.

For an arbitrary set $A \subseteq \mathbb{N}$, the enumeration degree of A and the enumeration degree of $\overline{A}$, the complement of A, need not be comparable. By requiring that they are comparable, we can isolate two interesting subclasses of the enumeration degrees. The first was introduced at the same time as the enumeration degrees themselves. We know that a set $A \subseteq \mathbb{N}$ is *total* if $\overline{A} \leq_e A$, and we call an enumeration degree *total* if it contains a total set. Note that A is total if and only if $A \equiv_e A \oplus \overline{A}$, where $\oplus$ denotes the effective disjoint union of sets. Since every set of the form $A \oplus \overline{A}$ is total, the total degrees are exactly the degrees of sets $A \oplus \overline{A}$ for some $A \subseteq \mathbb{N}$. We know that the map $\iota : A \mapsto A \oplus \overline{A}$ induces an order-preserving isomorphism between the Turing degrees and the total enumeration degrees. The name "total" is due to the fact that an enumeration degree is total if and only if it contains the graph of a total function. In particular, if A is a total set, then $d_{\mathrm{e}}(A)$ contains the graph of the characteristic function of A.

It is important to note that total degrees[1] always contain nontotal sets as well. For example, all c.e. sets have total degree because they are all enumeration equivalent to the empty set, but only computable c.e. sets are total.

7.1 Cototality

Let remain that a set $A \subseteq \mathbb{N}$ is *cototal* if $A \leq_{\mathrm{e}} \overline{A}$, and call an enumeration degree *cototal* if it contains a cototal set. While we are the first to isolate this property under this name, both the property and the name have appeared in the literature. The name was essentially first used, as far as we are aware, in an abstract of A.V. Pankratov from 2000 [Pan00]. Pankratov used "кототальное" (Russian for "cototal") to refer to what *we* call the graph-cototal degrees, which turns out to be a proper subclass of the cototal degrees: For any total function $f : \mathbb{N} \to \mathbb{N}$, let $G_f = \{\langle n, m\rangle : f(n) = m\}$ be the graph of f. It is easy to see that $\overline{G_f} \leq_{\mathrm{e}} G_f$, so $\overline{G_f}$ is a cototal set. If an enumeration degree contains a set of the form $\overline{G_f}$, then we call it *graph-cototal*.

The graph-cototal sets and degrees were further studied by Solon, Pankratov's advisor. In [Sol06], he used "co-total" to refer to what we call "graph-cototal". However, in the Russian version [Sol05] of the same paper, Solon used "кототальное" for a different property: Call a degree $\mathbf{a}$ *weakly cototal* if it contains a set A such that $\overline{A}$ has total enumeration degree. It is clear that every cototal degree is weakly cototal, since if $A \leq_{\mathrm{e}} \overline{A}$, then $\overline{A}$ is a total set. So we have

$$\text{graph-cototal} \implies \text{cototal} \implies \text{weakly cototal.}$$

We show that these three properties are distinct. The harder separation is given in Section 7.6, where we use an infinite-injury argument relative to $\mathbf{0}'$ to construct a cototal degree that is not graph-cototal. In Section 7.5, we give examples of weakly cototal degrees that are not cototal, as well as enumeration degrees that are not weakly cototal. Of these properties, we believe that there is a strong case that *cototal* is the most fundamental.

Our study of cototality was motivated by two examples of cototal sets that were pointed out to us by Jeandel [Jea15]. He showed that the set of non-identity words in a finitely generated simple group is cototal (see also Thomas and Williams [TW16]). Jeandel also gave an example from symbolic

[1] We sometimes use the term *degree* to refer to an enumeration degree.

dynamics: The set of words that appear in a minimal subshift is cototal. This is particularly interesting because the Turing degrees of elements of a minimal subshift are exactly the degrees that enumerate the set of words that appear in the subshift, so understanding the enumeration degree of this set is closely related to understanding the Turing degree spectrum of the subshift.

In Section 7.3, we explain Jeandel's examples in more detail, and we give several other examples of cototal sets and degrees. We show that every Σ^0_2-set is cototal, in fact, graph-cototal. We show that the complement of a maximal independent subset of a computable graph is cototal, and that every cototal degree contains the complement of a maximal independent subset of $\omega^{<\omega}$. Ethan McCarthy proved that the same is true of complements of maximal antichains in $\omega^{<\omega}$. We show that joins of nontrivial K-pairs are cototal, and that the natural embedding of the continuous degrees into the enumeration degrees maps into the cototal degrees. Finally, we note that Harris [Har10] proved that sets with a good approximation have cototal degree.

The earliest reference to a cototality notion seems to be in Case's dissertation [Cas69, p. 14] from 1969; he wrote "The author does not know if there are sets A such that A lies in a total partial degree and $\overline{A}$ lies in a non-total partial degree, but he conjectures that there are no such sets." In our language, Case is conjecturing that if $\overline{A}$ has weakly cototal degree, then it has total degree. The same question also appears in the journal version [Cas71, p. 426]. Gutteridge [Gut71, Chapter II] disproved this conjecture by constructing a quasiminimal graph-cototal degree. Recall that an enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total degree below $\mathbf{a}$ is $\mathbf{0}_{\mathrm{e}} = d_{\mathrm{e}}(\varnothing)$; in particular, quasiminimal degrees are nontotal. At least two other independent constructions of nontotal cototal degrees appear in the literature: Sanchis [San78], apparently unaware of Case's conjecture, gave an explicit construction of a cototal set that is not total. Aware of Case's conjecture but not Gutteridge's example, Sorbi [Sor88] constructed a quasiminimal cototal degree. Neither of these constructions explicitly produce a graph-cototal degree.

As mentioned above, Pankratov [Pan00] claimed that there is a graph-cototal Σ^0_2-enumeration degree that forms a minimal pair with every incomplete Π^0_1-enumeration degree.[2] The graph-cototal degrees were studied more extensively by Solon [Sol05, Sol06].[3] He proved that every total enumeration degree above $\overline{K}$ contains the graph G_f of a total function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\overline{G_f}$ is quasiminimal. He also showed that for every total enumeration degree $\mathbf{b}$, there is a graph-cototal enumeration degree $\mathbf{a}$ that is quasiminimal over $\mathbf{b}$. Finally, Solon proved that for every total enumeration

[2] This result does not appear to be published and we do not know the proof that Pankratov had in mind, but note that graph-cototality is free because every Σ^0_2-enumeration degree is graph-cototal.

[3] We note here a slight confusion in Solon's papers between cototal sets and cototal degrees, which does not, however, affect his main results.

degree $\mathbf{b}$ above $\overline{K}$, there is a graph-cototal quasiminimal enumeration degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{b}$ (see below for more about the enumeration jump). This strengthens a result of McEvoy [McE85], who proved that the quasiminimal enumeration degrees have all possible enumeration jumps. Note that all three of Solon's results can also be seen as generalizations of Gutteridge's construction of a quasiminimal graph-cototal degree.

7.2 The skip

Cototality is closely related, as we mentioned, to the skip operator. Let $\{\Gamma_e\}_{e\in\omega}$ be an effective list of all enumeration operators and let $K_A = \bigoplus_{e\in\omega}\Gamma_e(A) = \{\langle e,x\rangle \mid x \in \Gamma_e(A)\}$. Note that $K_A \equiv_{\mathrm{e}} A$. We define the *skip* of A to be $A^\lozenge = \overline{K_A}$. It is easy to see that the skip is degree invariant, so it induces an operator on enumeration degrees. We use $\mathbf{a}^\lozenge$ to denote the skip of $\mathbf{a}$. Note that the complements of elements of $d_{\mathrm{e}}(A)$ are enumeration reducible to $A^\lozenge$; indeed, they are columns of $A^\lozenge$. In other words, $d_{\mathrm{e}}(A^\lozenge)$ is the maximum possible degree of the complement of an element of $d_{\mathrm{e}}(A)$. One consequence of this characterization is the connection between the skip and cototality:

Proposition 7.2.1. A set $A \subseteq \mathbb{N}$ has cototal degree if and only if $A \leq_{\mathrm{e}} A^\lozenge$.

This connection is quite useful; the separations we prove in Section 7.5 rely on our study of the skip operator in Section 7.4.

In some ways, the skip is analogous to the jump operator in the Turing degrees. For example, a standard diagonalization argument shows that $A^\lozenge \nleq_{\mathrm{e}} A$. In Proposition 7.4.1, we restate the well-known fact that $A \leq_{\mathrm{e}} B$ if and only if $A^\lozenge \leq_1 B^\lozenge$, mirroring the jump in the Turing degrees. Finally, in Theorem 7.4.3, we prove a skip inversion theorem analogous to Friedberg jump inversion: If $S \geq_{\mathrm{e}} \overline{K}$, then there is a set A such that $A^\lozenge \equiv_{\mathrm{e}} S$.

The biggest difference between the skip and the Turing jump is that it is not always the case that $A \leq_{\mathrm{e}} A^\lozenge$ (because not all enumeration degrees are cototal). In fact, as we will see in Section 7.4.2, there is a *skip* 2*-cycle*, i.e., a set $A \subseteq \mathbb{N}$ such that $A = A^{\lozenge\lozenge}$. If we modify the skip to ensure that it is increasing in the enumeration degrees, then we recover the definition of the enumeration jump as introduced by Cooper[4] [Coo84].

The *enumeration jump* of a set $A \subseteq \mathbb{N}$ is $A'_{\mathrm{e}} = K_A \oplus \overline{K_A} \equiv_{\mathrm{e}} A \oplus A^\lozenge$. (We also use A' to denote A'_{e}). So A has cototal degree if and only if $A'_{\mathrm{e}} \equiv_{\mathrm{e}} A^\lozenge$. Of course, the enumeration jump is degree invariant and induces an operator on the enumeration degrees; we use $\mathbf{a}'$ for the jump of $\mathbf{a}$. The definition of the enumeration jump ensures that $A <_{\mathrm{e}} A'_{\mathrm{e}}$, as we expect from a jump. On

[4] Cooper [Coo84] thanks his student McEvoy for helping provide the correct definition of the enumeration jump operator. Sorbi recalled (in private communication) that Cooper's original "incorrect" definition was actually our definition of the skip operator.

the other hand, we lose two of the properties that the skip shares with the Turing jump. The enumeration jump is always total, so it cannot possibly map onto all enumeration degrees above $\mathbf{0}'_{\mathrm{e}}$. However, by Friedberg jump inversion, it does map onto the total degrees above $\mathbf{0}'_{\mathrm{e}}$. $A'_{\mathrm{e}} \leq_1 B'_{\mathrm{e}}$ does not necessarily imply that $A \leq_{\mathrm{e}} B$. So neither the skip nor the enumeration jump is the perfect analogue of the Turing jump; we believe that both have a role in the study of the enumeration degrees.

7.3 Examples of cototal sets and degrees

7.3.1 Total degrees

For any set $A \subseteq \mathbb{N}$, the set $A \oplus \overline{A}$ is clearly cototal. Therefore, every total degree is cototal.

7.3.2 The complement of the graph of a total function

As we have noted, if $f\colon \mathbb{N} \to \mathbb{N}$ is total, then $\overline{G}_f$, the complement of the graph of f, is a cototal set. This is because $\langle n, m\rangle \in \overline{G}_f$ if and only if there is $m' \neq m$ such that $\langle n, m'\rangle \in G_f$. The class of graph-cototal enumeration degrees turns out to lie strictly between the total degrees and the cototal degrees. The hard part is showing that there is a cototal degree that is not graph-cototal. We do that in Section 7.6. To see that every total degree is graph-cototal, recall that each total degree contains the graph of the characteristic function χ_A of some total set A; it also contains the complement of the graph of χ_A. We already saw that $\overline{G}_{\chi_A} \leq_{\mathrm{e}} G_{\chi_A}$. But now since $\langle n, m\rangle \in G_{\chi_A}$ if and only if $m \in \{0,1\}$ and $\langle n, 1-m\rangle \in \overline{G}_{\chi_A}$, we have that $\overline{G}_{\chi_A} \equiv_{\mathrm{e}} G_{\chi_A}$. The next result implies that there are nontotal graph-cototal degrees.

Proposition 7.3.1. Every enumeration degree $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$ is graph-cototal.

The enumeration degrees below $\mathbf{0}'_{\mathrm{e}}$ consist entirely of Σ^0_2-sets. So, fix an enumeration degree $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$ and a Σ^0_2-set $A \in \mathbf{a}$. We show that there is a set $\overline{G} \equiv_{\mathrm{e}} A$ that is the complement of the graph G of a total function, using Σ^0_2-approximation $\{A_s\}_{s<\omega}$ to the set A.

It is worth pointing out that the argument above cannot be extended to further levels of the arithmetical hierarchy. In Section 7.5, we show that there are Π^0_2-sets that do not even have cototal enumeration degree. On the other hand, it is easy to see that every Π^0_2-set has weakly cototal degree. This is because every set A is enumeration equivalent to $A \oplus K$, where K is the halting set. So, if A is Π^0_2 then $\overline{A \oplus K} = \overline{A} \oplus \overline{K} \equiv_{\mathrm{e}} \overline{K} \in \mathbf{0}'_{\mathrm{e}}$. As for higher levels of the arithmetical hierarchy, we see in Section 7.5 that there are Δ^0_3-sets that are not even weakly cototal.

Let $\overline{G}$ be the complement of the graph G of a total function. If $x \in \overline{G}$, then there is a unique axiom in Γ that enumerates x into $\Gamma(G)$. We say that $\overline{G}$ reduces to G via *a unique axiom reduction.* We will next see that this property characterizes the graph-cototal enumeration degrees among all cototal enumeration degrees.

Proposition 7.3.2 (Unique Axiom Characterization)**.** An enumeration degree $\mathbf{a}$ is graph-cototal if and only if it contains a cototal set A that reduces to $\overline{A}$ via a unique axiom reduction.

We can make this characterization even tighter by noting that the reduction Γ used to witness that $\overline{G} \leq_{\mathrm{e}} G$ is furthermore a *singleton operator*: every axiom in Γ is of the form $\langle a, \{b\} \rangle$ where $a \neq b$.

We therefore are interested in finding examples of cototal enumeration degrees that do not satisfy the Unique Axiom Characterization, as we would like to separate the cototal degrees from the graph-cototal degrees. The next example, which comes from graph theory, is motivated by this desire.

7.3.3 Complements of maximal independent sets

Recall that an (undirected) graph is a pair $G = (V, E)$, where V is a set of vertices and E is a set of unordered pairs of vertices, called the edge relation.

Definition 7.3.3. An *independent set* for a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that no pair of distinct vertices in S is connected by an edge. An independent set is *maximal* if it has no proper independent superset.

In other words, an independent set S is maximal if and only if every vertex $v \in V$ is either in S or is connected by an edge to an element of S. The maximal independent sets for the graph of the cube are illustrated in the figure below, courtesy of David Eppstein and Wikipedia.

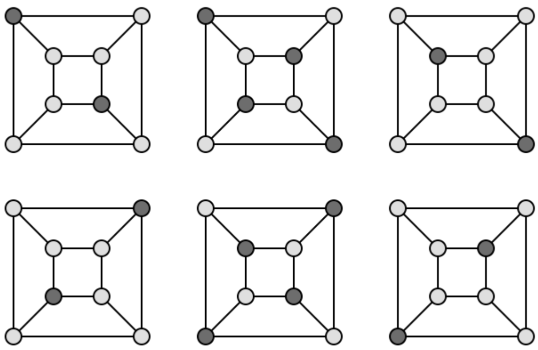

Figure 7.1: Maximal independent sets for the cube

Consider an infinite graph $G = (\mathbb{N}, E)$ with a computable edge relation. For example, we can think of the tree $\omega^{<\omega}$ as a computable graph on the natural numbers by fixing an effective coding of the finite sequences of natural numbers and putting an edge between any non-root node and its immediate predecessor. If S is a maximal independent set for G, then S can enumerate its complement:

$$\overline{S} = \{v \mid (\exists u \in S)[\{u, v\} \in E]\}.$$

It follows that complements of maximal independent sets in computable graphs on $\mathbb{N}$ are cototal. Our main reason for considering this example is that, in general, this reduction does not have the unique axiom property. This is well illustrated by Figure 7.1: the maximal independent set in the middle of the first row, for example, would enumerate each element of its complement with three distinct correct axioms. Hence we might hope that complements of maximal independent sets allow us to move beyond the graph-cototal degrees. They do, and in fact, they are universal for the cototal enumeration degrees.

Theorem 7.3.4. Every cototal degree contains the complement of a maximal independent set for $\omega^{<\omega}$.

Fix a cototal set A and let $A = \Gamma(\overline{A})$. We build a set $G \subseteq \omega^{<\omega}$ which is the complement of a maximal independent set for $\omega^{<\omega}$.

7.3.4 Complements of maximal antichains in $\omega^{<\omega}$

A closely related example comes from simply considering maximal antichains in $\omega^{<\omega}$. In this case, the partial ordering on finite sequences of natural numbers is defined by $\sigma \leq \tau$ if and only if $\sigma \preceq \tau$. An *antichain* is a subset of $\omega^{<\omega}$ such that no two elements in it are comparable, and an antichain is *maximal* if it cannot be extended to a proper superset that is also an antichain. Examples of computable maximal antichains are easy to come up with: For any fixed n, the set of all elements of $\omega^{<\omega}$ of length n is a maximal antichain.

If S is a maximal antichain, then $\overline{S} \leq_{\mathrm{e}} S$ as $\sigma \in \overline{S}$ if an only if there is some $\tau \in S$ that is comparable with σ. As in the example above, this reduction does not have the unique axiom property. Consider for example the maximal antichain of all strings of length n. Then every string of length $m < n$ has infinitely many reasons to be enumerated into the complement of this maximal antichain. Ethan McCarthy has shown that complements of maximal antichains are also universal for the cototal enumeration degrees.

Theorem 7.3.5 (McCarthy [McC18])**.** Every cototal degree contains the complement of a maximal antichain in $\omega^{<\omega}$.

7.3.5 The set of words that appear in a minimal subshift

We will next give a more detailed account of our motivating examples, introduced by Jeandel [Jea15]. The first one requires us to recall some definitions from symbolic dynamics.

Definition 7.3.6. Let $X \subseteq 2^\omega$ be closed in the usual topology on Cantor space.

1. X is a *subshift* if X is closed under the shift operation, which removes the first bit in a binary sequence, i.e., $a\alpha \in X$ implies $\alpha \in X$.

2. If X is a subshift then the *language of* X is the set

$$\mathcal{L}_X = \{\sigma \in 2^{<\omega} : (\exists \alpha \in X)[\sigma \text{ is a subword of } \alpha]\}.$$

   The set $\overline{\mathcal{L}_X}$ is called the *set of forbidden words.*

3. A subshift X is *minimal* if it has no nonempty proper subset that is also a subshift. This is equivalent to saying that every $\sigma \in \mathcal{L}_X$ is a subword of every $\alpha \in X$.

Jeandel discovered an interesting relationship between the enumeration degree of the language of a minimal subshift and the Turing degrees of the elements of the subshift: The Turing degrees of elements in X are exactly the Turing degrees that enumerate $\mathcal{L}_X$. This fact is particularly interesting if one takes into account Selman's characterization of enumeration reducibility. For an arbitrary set A, let $\mathcal{E}_A$ denote the set of all Turing degrees whose elements compute enumerations of A. Selman [Sel71] proved that $A \leq_e B$ if and only if $\mathcal{E}_B \subseteq \mathcal{E}_A$. Thus, the enumeration degree of the set $\mathcal{L}_X$ can be characterized by $\mathcal{E}_{\mathcal{L}_X}$, which turns out to be exactly the set of Turing degrees that compute elements of the minimal subshift X. It is then natural to ask what additional properties an enumeration degree must have in order to be the enumeration degree of the language of a minimal subshift. The following theorem shows that it must be cototal.

Theorem 7.3.7 (Jeandel [Jea15])**.** $\mathcal{L}_X \leq_e \overline{\mathcal{L}_X}$.

Ethan McCarthy has very recently shown that, in fact, cototality precisely characterizes the enumeration degrees of languages of minimal subshifts.

Theorem 7.3.8 (McCarthy [McC18])**.** If A is cototal, then $A \equiv_e \mathcal{L}_X$ for some minimal subshift X.

7.3.6 The non-identity words in a finitely generated simple group

The second example from Jeandel [Jea15] is related to group theory.

Definition 7.3.9. Let G be a group.

1. G is *finitely generated* if there are finitely many elements in G, called generators, such that every element in G can be expressed as a product of these generators. (For convenience, we will assume that the set of generators is closed under inverses.)

2. G is *simple* if its only normal subgroups are G and the trivial group.

3. The set of *identity words* of G is the set $\mathcal{W}_G$ of all words (i.e., finite sequences of generators) that represent the identity element.

4. A *presentation* of G is a pair $\langle F \mid R \rangle$ such that F is a set of generators and $\mathcal{W}_G$ is the normal closure of $R \subset \mathcal{W}_G$.

The word problem for a group G is the problem of deciding the set $\mathcal{W}_G$. Kuznetsov [Kuz58] showed that if G is a finitely generated simple group with a presentation $\langle F \mid R \rangle$ such that R is computable, then it has a decidable word problem. Jeandel considered the collection of all finitely generated simple groups without restricting the complexity of their presentation. He showed that the set of non-identity words in a finitely generated simple group is cototal. This was also independently observed by Thomas and Williams [TW16].

Theorem 7.3.10 (Jeandel [Jea15]; Thomas and Williams [TW16])**.** If G is a finitely generated simple group then $\overline{\mathcal{W}_G} \leq_{\mathrm{e}} \mathcal{W}_G$.

This generalizes Kuznetsov's result, as if a group $G = \langle F \mid R \rangle$ has a computable set of relations R, then $\mathcal{W}_G$ is automatically c.e. The fact that $\overline{\mathcal{W}_G} \leq_{\mathrm{e}} \mathcal{W}_G$ shows that $\overline{\mathcal{W}_G}$ is also c.e. and hence $\mathcal{W}_G$ is computable.

7.3.7 Joins of nontrivial $\mathcal{K}$-pairs

Our next example relates to a class of pairs of enumeration degrees that have been recently shown to play an important role when it comes to the first-order definability of relations on $\mathcal{D}_{\mathrm{e}}$.

Definition 7.3.11. A pairs of sets $\{A, B\}$ form a $\mathcal{K}$*-pair* if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is *nontrivial* if neither of its components is c.e.

$\mathcal{K}$-pairs were introduced by Kalimullin [Kal03]. He showed that they are first-order definable in the structure of the enumeration degrees and used them to give a first-order definition of the enumeration jump. Cai, Ganchev, Lempp, Miller, and M. Soskova [CGL+16] used $\mathcal{K}$-pairs to define the class of total enumeration degrees. It is therefore reasonable to always keep an eye on the class of $\mathcal{K}$-pairs as it might hold the key to the first-order definability of relations that we are considering in this article as well: cototal enumeration degrees and the skip operator. In the next section, $\mathcal{K}$-pairs will give us a wide variety of examples of sets that do not have cototal degree. When one considers the join $A \oplus B$, however, of a nontrivial $\mathcal{K}$-pair $\{A, B\}$, one always gets a cototal set. To see this, we need to review an important property of $\mathcal{K}$-pairs.

Proposition 7.3.12 (Kalimullin [Kal03])**.** If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then

- $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$;
- $\overline{B} \leq_e A \oplus \overline{K}$ and $\overline{A} \leq_e B \oplus \overline{K}$.

It follows from the first part that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $A \oplus B \leq_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

We would like to point out that this example generalizes the fact that every total degree is cototal, as by Cai, Ganchev, Lempp, Miller, and M. Soskova [CGL+16], the total degrees are exactly the ones that contain the join of a particular kind of a $\mathcal{K}$-pair. The joins of nontrivial $\mathcal{K}$-pairs therefore form a first-order definable class of cototal enumeration degrees that contains the total enumeration degrees. Unfortunately, they do not contain all cototal degrees. Ahmad [Ahm89] showed that there are nonsplitting Σ^0_2-enumeration degrees, i.e. degrees that are not the least upper bound of any pair of strictly smaller degrees. So, even though, as we have already seen, all Σ^0_2-enumeration degrees are cototal, the nonsplitting ones cannot be joins of nontrivial $\mathcal{K}$-pairs.

7.3.8 Continuous degrees

Motivated by a question of Pour-El and Lempp from computable analysis, Miller [Mil04] introduced a degree structure that captures the complexity of elements of computable metric spaces, such as $\mathcal{C}[0,1]$ and $[0,1]^\omega$. This structure naturally embeds into the enumeration degrees, and the range of this embedding is strictly between the class of total enumeration degrees and the class of all enumeration degrees.

As an example, consider the metric space $\mathcal{C}[0,1]$ of continuous functions on the unit interval with the standard metric

$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

A computable presentation of a metric space $\mathcal{M}$ consists of a fixed dense sequence $Q^{\mathcal{M}} = \{q_n\}_{n<\omega}$ on which the metric is computable as a function on indices. For a computable presentation of $\mathcal{C}[0,1]$ we can fix, for example, a reasonable enumeration of the polygonal functions having segments with rational endpoints. A *name* n_f for a continuous function f is a code (say, as an element of ω^ω) that gives a way to approximate f. Specifically, a name n_f should code a function taking a rational number $\varepsilon > 0$ and producing an index $n_f(\varepsilon)$ such that $d(f, q_{n_f(\varepsilon)}) < \varepsilon$. For $f, g \in \mathcal{C}[0,1]$, we say that f is *reducible* to g if every name for g computes a name for f. In the same way, we can compare the complexity of elements from arbitrary metric spaces. This reducibility induces a degree structure, the *continuous degrees*. It turns out that every continuous degree contains an element of $\mathcal{C}[0,1]$.

In order to understand the embedding of the continuous degrees into the enumeration degrees, it is easier to focus on another computable metric space: The *Hilbert cube* is $[0,1]^\omega$ along with the metric

$$d(\alpha,\beta) = \sum_{n\in\omega} 2^{-n}|\alpha(n) - \beta(n)|.$$

A dense set witnessing that $[0,1]^\omega$ is computable is, for example, a reasonable enumeration of the rational sequences with finite support. As was the case with $\mathcal{C}[0,1]$, every continuous degree contains an element of $[0,1]^\omega$.

Miller gave a way to assign to a sequence $\alpha \in [0,1]^\omega$ a set A_α such that $\mathcal{E}_{A_\alpha}$ (defined in Section 7.3.5) is the set of all Turing degrees that compute names of α. This induces an embedding of the continuous degrees into the enumeration degrees.

Definition 7.3.13 (Miller [Mil04])**.** For $\alpha \in [0,1]^\omega$, let

$$A_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q >_{\mathbb{Q}} \alpha(i)\}).$$

It is not hard to see that A_α has the desired property: Computing a name for α is exactly as hard as enumerating A_α. We say that the enumeration degree of A_α is *continuous*. By showing that there is a nontotal continuous enumeration degree, Miller proved that there are continuous functions that do not have a name of least Turing degree, which answered Pour-El and Lempp's question.

Note that if α does not have any rational entries, then A_α is a total set. If, on the other hand, α does have rational entries, then every component of A_α is nonuniformly equivalent to a total set. The existence of nontotal continuous enumeration degrees shows that this nonuniformity is significant. We are nevertheless able to show that all continuous degrees are cototal.

Proposition 7.3.14. Every continuous degree is cototal.

7.3.9 Sets with good approximations have cototal degree

Lachlan and Shore [LS92] introduced the following general notion of an approximation to a set.

Definition 7.3.15. Let A be a set of natural numbers. A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ (given by canonical indices) is a *good approximation to* A if

- for every n, there is a stage s such that $A \upharpoonright n \subseteq A_s \subseteq A$; and
- for every n, there is a stage s such that for every $t > s$, if $A_t \subseteq A$ then $A \upharpoonright n \subseteq A_t$.

This definition can be seen as a generalization of Cooper's notion of a Σ^0_2-approximation with infinitely many thin stages, used to show the density of the Σ^0_2-enumeration degrees [Coo84]. Lachlan and Shore [LS92] introduced the hierarchy of the n-c.e.a. sets. A set is 1-c.e.a. if it is c.e., and $(n+1)$-c.e.a. if it is the join of an n-c.e.a. set X and a set Y c.e. in X. It is not difficult to see that the enumeration degrees of the 2-c.e.a. sets are exactly the Σ^0_2-enumeration degrees. Lachlan and Shore proved that every set that is n-c.e.a. has a good approximation and then showed that the enumeration degrees of the n-c.e.a. sets are dense. Harris [Har10] proved that sets that have good approximations always have cototal enumeration degrees.

Proposition 7.3.16 (Harris [Har10, Proposition 4.1])**.** If A has a good approximation, then $K_A \leq_{\mathrm{e}} \overline{K_A}$.

In particular, we obtain that the enumeration degrees of n-c.e.a. sets are cototal. Very recently J. Miller and M. Soskova [MS18] proved that the cototal enumeration degrees are exactly the enumeration degrees of sets with good approximations and they are dense.

7.4 The skip

In the previous section, we saw many examples of cototal sets and enumeration degrees. In this section, we study the skip operator, in part to provide a wide variety of examples of degrees that are not cototal. Recall that the *skip* of a set $A \subseteq \mathbb{N}$ is $A^\Diamond = \overline{K_A}$. As we saw in the introduction, the skip gives us an easy way to determine whether or not a degree is cototal. For the reader's convenience, we restate that result:

Proposition 7.2.1. *A set $A \subseteq \mathbb{N}$ has cototal degree if and only if $A \leq_{\mathrm{e}} A^\Diamond$.*

In addition to being a tool in our study of cototality, the skip is a natural operator in its own right. As we discussed in the beginning, the enumeration jump fails to have some of the nice properties of the Turing jump. For example, it is well-known that $A \leq_T B$ if and only if $K_A \leq_1 K_B$, where K_A denotes the halting set relative to A. The analogous property does not hold, in general, for the enumeration jump. It is true that $A \leq_{\mathrm{e}} B$ implies $K_A \oplus \overline{K}_A \leq_1 K_B \oplus \overline{K}_B$, but the reverse implication can fail. The skip, on the other hand, gives us an embedding of the enumeration degrees into the 1-degrees.

Proposition 7.4.1. $A \leq_{\mathrm{e}} B$ if and only if $A^\Diamond \leq_1 B^\Diamond$.

This shows that we can define the skip operator on degrees.

Definition 7.4.2. The *skip* of the enumeration degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = d_{\mathrm{e}}(A^\Diamond)$ for any member $A \in \mathbf{a}$.

7.4.1 Skip inversion

It follows from Proposition 7.2.1 that an enumeration degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^{\Diamond}$, if and only if $\mathbf{a}^{\Diamond} = \mathbf{a}'$. The definition of the enumeration jump operator restricts its range to the total enumeration degrees and by monotonicity to the total enumeration degrees in the cone above $\mathbf{0}_{\mathrm{e}}'$. By transferring the Friedberg Jump Inversion Theorem through the standard embedding into the enumeration degrees, we see that every total enumeration degree above $\mathbf{0}_{\mathrm{e}}'$ is in the range of the jump operator. The range of the skip operator is also restricted by monotonicity to enumeration degrees above $\mathbf{0}_{\mathrm{e}}^{\Diamond} = \mathbf{0}_{\mathrm{e}}'$. We show that this is the only restriction on the range of the skip operator, thereby providing a further analogy between the skip and the Turing jump. Recall that $\overline{K}$, the complement of the halting set, is a representative of the degree $\mathbf{0}_{\mathrm{e}}'$.

Theorem 7.4.3. For any set $S \geq_{\mathrm{e}} \overline{K}$, there is a set A such that $A^{\Diamond} \equiv_{\mathrm{e}} S$. (In fact, we also have $S \equiv_{\mathrm{e}} \overline{A} \equiv_{\mathrm{e}} \overline{A} \oplus \overline{K}$ and $\overline{S} \leq_{\mathrm{e}} A \oplus \overline{K}$.)

Given a set $S \geq_{\mathrm{e}} \overline{K}$, we build a set A such that $S \equiv_{\mathrm{e}} \overline{A} \leq_{\mathrm{e}} A^{\Diamond} \leq_{\mathrm{e}} \overline{A} \oplus \overline{K}$.

Theorem 7.4.6. Let $n \geq 2$. For any Π^0_n-set $S \geq_{\mathrm{e}} \overline{K}$, there is a Σ^0_n-set A such that $A^{\Diamond} \equiv_{\mathrm{e}} S$. Furthermore, for any Σ^0_n-set $S \geq_{\mathrm{e}} \overline{K}$, there is a Π^0_n-set A such that $A^{\Diamond} \equiv_{\mathrm{e}} S$.

From Definition 2.3.9 we know that an enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total enumeration degree bounded by $\mathbf{a}$ is $\mathbf{0}_{\mathrm{e}}$.

McEvoy [McE85] proved that the enumeration jump restricted to the quasiminimal degrees has the same range as the unrestricted jump operator. We show that the skip has the same property. Actually, we prove with Soskov [SS13] the same property for the degree spectrum: every element of the jump spectrum is a jump of a quasi minimal degree with respect to the spectrum and co-spectrum.

Corollary 7.4.8. For any set $S \geq_{\mathrm{e}} \overline{K}$, there is a set A of quasiminimal degree such that $A^{\Diamond} \equiv_{\mathrm{e}} S$.

7.4.2 Further properties of the skip operator and examples

We will now investigate the possible behavior of the iterated skip operator.

Definition 7.4.9. Fix $A \subseteq \mathbb{N}$. We inductively define $A^{\langle n \rangle}$, the *n-th skip of A*.

- $A^{\langle 0 \rangle} = A$,
- $A^{\langle n+1 \rangle} = (A^{\langle n \rangle})^{\Diamond}$.

The *n-th skip of* $d_{\mathrm{e}}(A)$ is $d_{\mathrm{e}}(A)^{\langle n \rangle} = d_{\mathrm{e}}(A^{\langle n \rangle})$.

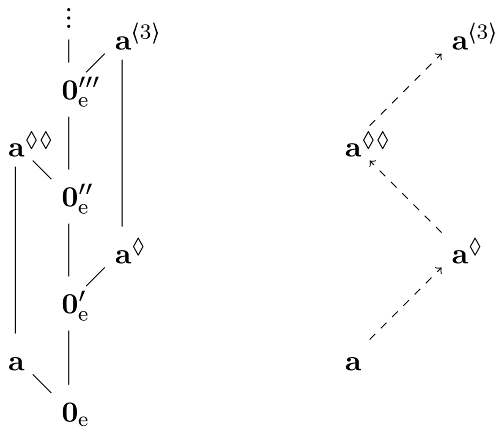

Figure 7.2: Iterated skips of a degree: the zig-zag

If $\mathbf{a}$ is a cototal enumeration degree, then every iteration of the skip of $\mathbf{a}$ agrees with the corresponding iteration of the jump of $\mathbf{a}$, i.e., for all $n < \omega$, we have that $\mathbf{a}^{\langle n\rangle} = \mathbf{a}^{(n)}$. Theorem 7.4.3 proves that there are non-cototal enumeration degrees, e.g., the skip invert of a nontotal enumeration degree. It is natural to ask what we can say in general about the sequence $\{\mathbf{a}^{\langle n\rangle}\}_{n\in\omega}$. One immediate observation is that even though the skip of A need not be above A, its double skip always is: For any set A, we know that $\overline{A} \leq_1 A^{\Diamond}$. Applying this twice, we have $A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$, so a fortiori $A \leq_e A^{\Diamond\Diamond}$. It follows that $\mathbf{a}^{\langle n\rangle} \leq \mathbf{a}^{\langle n+2\rangle}$ for all n. In addition, by monotonicity, we have that for every n, $\mathbf{0}_e^{(n)} \leq \mathbf{a}^{\langle n\rangle}$. If $\mathbf{a}^{\langle n\rangle}$ is not cototal for every natural number n, then we have a form of *zig-zag* behavior of the skip, illustrated in Figure 7.2. We search for examples of degrees whose skips have this general behavior.

Skips of generic sets

We start by investigating the skip for the class of enumeration degrees of 1-generic sets. Definition 2.3.8 defines a relativized form of 1-genericity, suitable for the context of the enumeration degrees. Let me remind that we use the notation "relative to $\langle X\rangle$" to denote "relative to the enumeration degree of X".

From Definition 2.3.10 we know that the degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and every total enumeration degree $\mathbf{x}$ bounded by $\mathbf{a}$ is below $\mathbf{b}$.

We proved in Proposition 2.3.11 the following properties of 1-generic relative to $\langle X\rangle$ set G:

1. $d_e(G \oplus X)$ is a strong quasiminimal cover of $d_e(X)$.

2. $\overline{G}$ is 1-generic relative to $\langle X\rangle$.

We know from Corollary 2.2.8 that the Turing jump of a 1-generic set has a nice characterization: $K_G \equiv_T G \oplus K$, or, in other words, G is generalized low. This property relativizes: If G is 1-generic relative X, then $K_{G\oplus X} \equiv_T G \oplus K_X$. A similar property is true of the skip of a 1-generic set G relative to $\langle X \rangle$.

Proposition 7.4.10. labelprop:genskip If G is 1-generic relative to $\langle X \rangle$, then $(G \oplus X)^{\Diamond} \equiv_e \overline{G} \oplus X^{\Diamond}$.

Now, we can easily give an example of a set G whose iterated skips form a zig-zag. Consider G to be a set that is arithmetically generic, i.e., G is 1-generic relative to $\langle \varnothing^{(n)} \rangle$ for every natural number n. Note that $\overline{G}$ has the same property. Then by induction using the characterization above we can show that for all $n < \omega$:

- If n is odd then $G^{\langle n \rangle} \equiv_e \overline{G} \oplus \varnothing^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e G \oplus \varnothing^{(n)}$.
- If n is even then $G^{\langle n \rangle} \equiv_e G \oplus \varnothing^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e \overline{G} \oplus \varnothing^{(n)}$.

Furthermore, all iterates of the skip for both sets G and $\overline{G}$ are not total, as their degrees are quasiminimal covers of the corresponding iterate of the jump of $\mathbf{0}_e$. It follows that they also do not have cototal degree, as by Proposition 7.2.1 sets H of cototal degree have total skips: $K_H \equiv_e H \leq_e H^{\Diamond} = \overline{K_H}$. This gives an example of a double zig-zag as in Figure 7.3. It is worth noting that only the reductions implied by the diagram occur. For example, $G \nleq_e G^{\langle 3 \rangle}$; otherwise $G^{\langle 3 \rangle} \equiv_e G \oplus G^{\langle 3 \rangle} \equiv_e G \oplus \overline{G} \oplus \varnothing^{(3)}$ would be total.

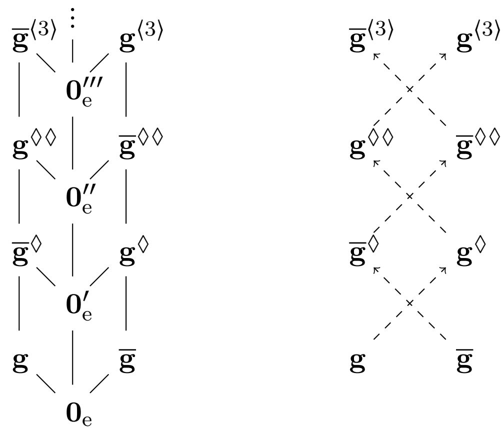

Figure 7.3: The iterated skips of the degrees of an arithmetically generic set and its complement: double zig-zag

Skips of nontrivial $\mathcal{K}$-pairs.

Kalimullin [Kal03] relativized the notion of a $\mathcal{K}$-pair in a way similar to how we relativized the notion of 1-genericity.

Definition 7.4.11. A pair of sets of natural numbers $\{A, B\}$ forms a *$\mathcal{K}$-pair relative to* $\langle X\rangle$ if there is a set $W \leq_{\mathrm{e}} X$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. The pair $\{A, B\}$ is a *nontrivial $\mathcal{K}$-pair relative to* $\langle X\rangle$ if, in addition, $A \nleq_{\mathrm{e}} X$ and $B \nleq_{\mathrm{e}} X$.

Note that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to every $\langle X\rangle$ such that $A, B \nleq_{\mathrm{e}} X$. We summarize some properties of relativized $\mathcal{K}$-pairs below.

Proposition 7.4.12 (Kalimullin [Kal03])**.** Let $A, B, X \subseteq \mathbb{N}$ and suppose that $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$.

1. If $C \leq_{\mathrm{e}} B$ then $\{A, C\}$ forms a $\mathcal{K}$-pair relative to $\langle X\rangle$.
2. $A \leq_{\mathrm{e}} \overline{B} \oplus X$.
3. $\overline{A} \leq_{\mathrm{e}} B \oplus X^{\Diamond}$.
4. $d_{\mathrm{e}}(A \oplus X)$ and $d_{\mathrm{e}}(B \oplus X)$ are strong quasiminimal covers of $d_{\mathrm{e}}(X)$.
5. For every $Z \subseteq \mathbb{N}$, the degrees $d_{\mathrm{e}}(A \oplus X \oplus Z)$ and $d_{\mathrm{e}}(B \oplus X \oplus Z)$ have a greatest lower bound, and it is $d_{\mathrm{e}}(X \oplus Z)$.

Note that items (1), (2) and (3) are symmetrically true if we swap A and B.

The skip of a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$ has the following properties:

Proposition 7.4.13. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$, then

$$(A \oplus X)^{\Diamond} \leq_{\mathrm{e}} B \oplus X^{\Diamond} \quad \text{and} \quad (B \oplus X)^{\Diamond} \leq_{\mathrm{e}} A \oplus X^{\Diamond}.$$

The oracle set X is of cototal degree if and only if for every nontrivial $\mathcal{K}$-pair $\{A, B\}$ relative to $\langle X\rangle$,

$$(A \oplus X)^{\Diamond} \equiv_{\mathrm{e}} B \oplus X^{\Diamond} \quad \text{and} \quad (B \oplus X)^{\Diamond} \equiv_{\mathrm{e}} A \oplus X^{\Diamond}.$$

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair and both A and B are not arithmetical, then $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle\varnothing^{(n)}\rangle$ for every natural number n. As every set $\varnothing^{(n)}$ is of (co)total enumeration degree, it follows by Proposition 7.4.13 that the iterated skips of A and B also form a double zigzag: For all $n < \omega$,

- if n is odd then $A^{\langle n\rangle} \equiv_{\mathrm{e}} B \oplus \varnothing^{(n)}$ and $B^{\langle n\rangle} \equiv_{\mathrm{e}} A \oplus \varnothing^{(n)}$, and
- if n is even then $A^{\langle n\rangle} \equiv_{\mathrm{e}} A \oplus \varnothing^{(n)}$ and $B^{\langle n\rangle} \equiv_{\mathrm{e}} B \oplus \varnothing^{(n)}$.

Furthermore, by Proposition 7.4.12, for every natural number n, $\{d_{\mathrm{e}}(A)^{\langle n\rangle}, d_{\mathrm{e}}(B)^{\langle n\rangle}\}$ forms a minimal pair of quasiminimal degrees above $\mathbf{0}_{\mathrm{e}}^{(n)}$.

A pair of enumeration degrees $\{\mathbf{a},\mathbf{b}\}$ forms a *$\mathcal{K}$-pair* (relative to $\mathbf{x}$) if there are representatives $A \in \mathbf{a}$ and $B \in \mathbf{b}$ that form a $\mathcal{K}$-pair (relative to $\mathbf{x}$). We use the characterization of the skips of $\mathcal{K}$-pairs along with the following theorem of Ganchev and Sorbi [GS16] to give an example of degrees whose iterated skips behave quite differently.

Theorem 7.4.14. (Ganchev, Sorbi [GS16]) For every enumeration degree $\mathbf{x} > \mathbf{0}_{\mathrm{e}}$, there is a degree $\mathbf{a} \leq \mathbf{x}$ such that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.

Using this we construct an example of an enumeration degree such that all iterations of its skip are total enumeration degrees, but mismatch its iterations of the jump by one iteration:

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3\rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1\rangle} < \cdots.$$

We end this discussion with some thoughts about the definability of the skip operator. Kalimullin [Kal03] proved that the relation "$\{\mathbf{a},\mathbf{b}\}$ forms a $\mathcal{K}$-pair relative to $\mathbf{x}$" is first-order definable with parameter $\mathbf{x}$. Using this result, he showed that the enumeration jump operator is first-order definable. Combining these results with the characterization of the skip operator for nontrivial $\mathcal{K}$-pairs, we immediately obtain the following result.

Corollary 7.4.15. The relation

$$SK = \{(\mathbf{a},\mathbf{a}^{\Diamond}) \mid \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair } \}$$

is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

It remains an open question whether or not the skip operator is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

A skip 2-cycle

As seen above, the skip can exhibit a form of *zig-zag* behavior. We now show that there is another extreme case that could occur: The double skip $\mathbf{a}^{\Diamond\Diamond}$ of an enumeration degree $\mathbf{a}$ could be equal to $\mathbf{a}$ itself. Perhaps surprisingly, this degree is not constructed in a way that is common in computability theory. Instead, we use the following theorem due to Knaster and Tarski.

Theorem 7.4.16 (Knaster–Tarski Fixed Point Theorem)**.** Let L be a complete lattice and let $f\colon L \to L$ be monotone, i.e., for all $x, y \in L$, we have that $x \leq y$ implies that $f(x) \leq f(y)$. Then f has a fixed point. In fact, the fixed points of f form a complete lattice.

We apply the Knaster–Tarski theorem to a function on 2^{ω}, which we view as the power set lattice of $\mathbb{N}$, ordered by subset inclusion.

Theorem 7.4.17. There is a set A such that $A^{\Diamond\Diamond} = A$.

Let $f\colon 2^\omega \to 2^\omega$ be the double skip operator, i.e., $f(A) = A^{\Diamond\Diamond}$. Note that if $A \subseteq B$, then $K_A \subseteq K_B$, so $A^\Diamond \supseteq B^\Diamond$. Applied twice, we obtain $A^{\Diamond\Diamond} \subseteq B^{\Diamond\Diamond}$, so f is monotone. Hence, by the Knaster–Tarski Fixed Point Theorem, there is an A such that $A^{\Diamond\Diamond} = A$.

Note that we do not just have that A and $A^{\Diamond\Diamond}$ are enumeration equivalent, but they are equal as sets. However, we will mainly be interested in the fact that the enumeration degree $\mathbf{a}$ of A satisfies $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$. If we have such a degree $\mathbf{a}$, then we will say that $\mathbf{a}$ and $\mathbf{a}^\Diamond$ form a *skip* 2*-cycle*.

As we show next, skip 2-cycles are computationally very complicated; namely, they compute all hyperarithmetic sets.

Proposition 7.4.18. Let $\mathbf{a}$ and $\mathbf{a}^\Diamond$ form a skip 2-cycle. Then $\mathbf{a} \geq \mathbf{b}$ for every total hyperarithmetic degree $\mathbf{b}$.

Given the fact that we have shown the existence of a skip 2-cycle, it is only natural to consider whether (proper) skip n-cycles exist for any other natural number $n \geq 1$. This turns out to be false.

Proposition 7.4.19. Let $n \in \omega$ be nonzero such that $\mathbf{a}^{\langle n \rangle} = \mathbf{a}$. Then $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$.

The set A we obtained in Theorem 7.4.17 allows us to give the example of a pair of sets A and $B = A^\Diamond = \overline{K_A}$ that illustrate the flaw in the enumeration jump mentioned in the last paragraph of Section 7.2.

Proposition 7.4.20. labelprop:flaw $A'_{\mathrm{e}} \equiv_1 B'_{\mathrm{e}}$ does not necessarily imply $A \equiv_{\mathrm{e}} B$.

7.5 Separating cototality properties

7.5.1 Degrees that are not weakly cototal

Let us begin by showing that the weakest cototality property we introduced, aptly named *weakly cototal*, is nontrivial, i.e., that there are degrees that are not weakly cototal. We present three different examples in this section. First, we note that sufficiently generic sets are not weakly cototal.

Proposition 7.5.1. If $\mathbf{a}$ is a 2-generic enumeration degree, then $\mathbf{a}$ is not weakly cototal.

Next, we show that we can also get such examples using $\mathcal{K}$-pairs.

Proposition 7.5.2. Let $\mathbf{a}, \mathbf{b} \not\leq_{\mathrm{e}} \mathbf{0}'_{\mathrm{e}}$ form a nontrivial $\mathcal{K}$-pair. Then $\mathbf{a}$ is not weakly cototal.

For our final example of a degree that is not weakly cototal, recall from Theorem 7.4.17 that there is a degree $\mathbf{a}$ such that $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$. Such a degree is not weakly cototal.

Proposition 7.5.3. Let $\mathbf{a}$ be such that $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$. Then $\mathbf{a}$ is not weakly cototal.

7.5.2 Weakly cototal degrees that are not cototal

We prove the next separation using the skip inversion we proved in Theorem 7.4.3 above.

Proposition 7.5.4. There is a degree $\mathbf{a}$ that is weakly cototal, but not cototal.

Let $B \geq_{\mathrm{e}} \overline{K}$ be any total set, and let $S = K_B$. Then note that $S \equiv_{\mathrm{e}} B$, so the *degree* of S is total, but S is not total *as a set*. Now apply Theorem 7.4.3 to obtain an A such that $A^{\Diamond} \equiv_{\mathrm{e}} S$ and $\overline{S} \leq_{\mathrm{e}} A \oplus \overline{K}$.

Then A is weakly cototal since $A \equiv_{\mathrm{e}} K_A$ and $\overline{K_A} = A^{\Diamond} \equiv_{\mathrm{e}} S$, which has total degree. Let $\mathbf{a}$ be the degree of A. We claim that $\mathbf{a}$ is not cototal. By Proposition 7.2.1, it suffices to show that $A \not\leq_{\mathrm{e}} A^{\Diamond}$. Towards a contradiction, assume that $A \leq_{\mathrm{e}} A^{\Diamond}$. Since $A^{\Diamond} \geq_{\mathrm{e}} \overline{K}$ always holds, we now see that

$$S \equiv_{\mathrm{e}} A^{\Diamond} \geq_{\mathrm{e}} A \oplus \overline{K} \geq_{\mathrm{e}} \overline{S}$$

so S would be a total set, which is a contradiction.

The proof above combined with Theorem 7.4.6 yields the promised Π^0_2 degree that is not cototal. Of course, as noted earlier, such a degree can be obtained using a theorem of Badillo and Harris [BH12] proving the existence of a Π^0_2-enumeration degree that contains only properly Π^0_2-sets. As all Π^0_2 enumeration degrees are weakly cototal, this gives us a more concrete separation result.

An alternative way to separate the weakly cototal degrees from the cototal degrees is given by the following proposition.

Proposition 7.5.5. If $\mathbf{b} \not\leq \mathbf{0}'_{\mathrm{e}}$ but forms a nontrivial $\mathcal{K}$-pair with $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$, then $\mathbf{b}$ forms a minimal pair with $\mathbf{b}^{\Diamond}$.

Towards a contradiction, assume there is a nonzero degree $\mathbf{c}$ such that $\mathbf{c} \leq \mathbf{b}$ and $\mathbf{c} \leq \mathbf{b}^{\Diamond}$. The fact that $\mathbf{c} \leq \mathbf{b}$ gives us that $\mathbf{a}$ and $\mathbf{c}$ form a $\mathcal{K}$-pair by Proposition 7.4.12(1). Using this, Proposition 7.4.12(2), and Proposition 7.4.13 twice, we have

$$\mathbf{b} \leq \mathbf{a}^{\Diamond} = \mathbf{c} \oplus \mathbf{0}'_{\mathrm{e}} \leq \mathbf{b}^{\Diamond} = \mathbf{a} \oplus \mathbf{0}'_{\mathrm{e}} = \mathbf{0}'_{\mathrm{e}}.$$

So $\mathbf{b} \leq \mathbf{0}'_{\mathrm{e}}$, which is a contradiction.

Corollary 7.5.6. If $\mathbf{b}$ is as in the previous proposition, then $\mathbf{b}$ is weakly cototal, but not cototal.

The only separation left to prove is the separation of the cototal degrees from the graph-cototal degrees. We prove this result in the next section.

7.6 There is a cototal degree that is not graph-cototal

Theorem 7.6.1. There is a cototal enumeration degree that is not graph-cototal.

We fix the undirected graph $\mathcal{G} = (\omega^{<\omega}, E)$, where the edge relation is given by $E(a, b)$ if and only if $a^- = b$ or $a = b^-$ (i.e., a is an immediate successor of b or the immediate predecessor of b). We build the complement of a maximal independent set for the graph $\mathcal{G}$. Recall that this is a subset $A \subseteq \omega^{<\omega}$ with the property that every element $a \in \omega^{<\omega}$ is either outside A or is connected by an edge to an element outside A, but not both.

Our other condition on the set A will be that it is not enumeration equivalent to a graph-cototal set. We construct A as such using a construction in the framework of a $0'''$-priority construction over $0'$. We start by listing an infinite sequence of requirements that collectively ensure that we meet our goal. We then make use of a tree of strategies. Strategies on the tree inherit the standard ordering of nodes: We use $\alpha \leq \beta$ to denote that α is a prefix of β and $\alpha <_L \beta$ to denote that α is to the left of β in the tree. Every strategy is assigned one of the requirements. At every stage we build a finite path through this tree, activating strategies along it and injuring all strategies to the right of it. Activated strategies perform actions towards satisfaction of their requirements. Injured strategies are *initialized*—they must start over as if they were never activated before. The intention is that there is a *true path*, a leftmost infinite path of nodes visited at infinitely many stages, such that every strategy along this path succeeds in satisfying the requirement that is assigned to it. We refer the reader to Soare [Soa87] for a more detailed introduction to priority arguments and the tree method. We warn the reader that our argument differs from standard infinite-injury arguments in a couple of ways: There are some strategies α which intentionally injure other strategies β with $\alpha < \beta$, and this will cause injury along the true path. Also, we have strategies β which cause strategies $\alpha < \beta$ to revert to a previous state in α's construction, though for every α each state in α's construction will only be susceptible to reversion by finitely many $\beta > \alpha$. Finally, we make use of the notion of *moment* to refer to substages in the construction. We assume that actions that strategies make, such as injury and initialization, have immediate effect during moments in the construction, rather than at the end of a stage.

Our set $A \subseteq \omega^{<\omega}$ we construct, satisfies the following requirements, for all $a \in \omega^{<\omega}$ and all enumeration operators Φ and Ψ.

Requirements:

global : $(\forall x, y \in \omega^{<\omega} \setminus A)[\neg\, xEy]$

$\mathcal{N}_a$: $a \notin A$ or $(\exists x)[xEa \wedge x \notin A]$

$\mathcal{R}_{\Phi,\Psi}$: $A = \Psi(\Phi(A)) \implies \Phi(A) \neq \overline{G_f}$ for any total function $f\colon \mathbb{N} \to \mathbb{N}$

Clearly, our global requirement and the $\mathcal{N}_a$-requirements and $\mathcal{R}_{\Phi,\Psi}$-requirements ensure that A is of cototal (see Section 7.3.3) but not of graph-cototal enumeration degree.

7.7 Open questions

In this section, we collect the open questions arising from this paper, some of which have already been asked.

Definability

As mentioned above, Kalimullin [Kal03] showed that the enumeration jump is first-order definable. Is this also true for the skip?

Question 7.7.1. Is the skip first-order definable in the enumeration degrees?

Furthermore, we have discussed several cototality notions in this paper. Which of these are definable?

Question 7.7.2. Which cototality notions are first-order definable in the enumeration degrees?

Note that a positive answer to the first question would imply, by Proposition 7.2.1, that the cototal degrees are definable.

Arithmetical zigzag

In Section 7.4.2, we have shown that the skip can exhibit a form of *zigzag behavior*: There are degrees $\mathbf{a}$ such that none of the finite skips of $\mathbf{a}$ are total. However, the examples constructed there are not arithmetical. We suspect that this is not a coincidence.

Conjecture 2. If $\mathbf{a}$ is an arithmetical enumeration degree, then $\mathbf{a}^{\langle n\rangle}$ is total for some $n \in \omega$.

Graph-cototal degrees

Theorem 7.6.1 constructed a cototal Δ^0_3-degree that is not graph-cototal. On the other hand, Proposition 7.3.1 proves that every Σ^0_2-degree is graph-cototal. This leaves the following open:

Question 7.7.3. Is every Π^0_2 cototal enumeration degree graph-cototal?

We do not know of a simpler proof of the existence of a cototal enumeration degree that is not graph-cototal. A more informative separation result would be derived from a positive answer to the following question:

Question 7.7.4. Is there a continuous enumeration degree that is not graph-cototal?

Skip cototality

Let us say that a degree $\mathbf{a}$ is *skip cototal* if $\mathbf{a}^\Diamond$ is total. Notice that every skip cototal degree $\mathbf{a}$ is weakly cototal, and that every cototal degree is skip cototal. Furthermore, note that in the proofs of Proposition 7.5.4 and Corollary 7.5.6, we in fact constructed a degree $\mathbf{a}$ that is skip cototal but not cototal. Even the alternative example of a weakly cototal degree given by Badillo and Harris [BH12]—the degree that is entirely composed of properly Π^0_2-sets—is also a skip cototal degree.

Conjecture 3. Every weakly cototal degree $\mathbf{a}$ is skip cototal.

As mentioned above, every Π^0_2-degree is weakly cototal. Therefore, a proof of our conjecture would in particular imply that the skip of every Π^0_2-degree is total, which is also open.

Chapter 8

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