R E V I E W

of the dissertation "Algorithms for Characterisation of Orthogonal Arrays"

by Tanya Todorova Marinova

for awarding the educational and scientific degree "Doctor"

in Scientific Field 4. Natural Sciences, Mathematics and Informatics,

Professional Direction 4.5 Mathematics

Ph.D. Program "Algebra, Topology and Applications"

1 General characterisation of the dissertation and the provided materials

The dissertation under review contains 117 pages. It consists of an introduction, four chapters and a bibliography. The first chapter collects some preliminaries on orthogonal arrays. The second chapter reflects the results of three articles on the distance distribution of binary orthogonal arrays. The third chapter is devoted to the contributions of two articles on the distance distribution of ternary orthogonal arrays. The fourth chapter concerns the energy of orthogonal arrays and presents the results of one paper. The bibliography consists of 59 items. These include 47 articles, 9 monographs, two data libraries and one directory. Six of the aforementioned 47 articles reflect the results of the thesis. From the remaining 41 articles 17 are published between 2000 and 2021, 16 appeared in the period 1980-1999, 7 are from 1950-1979 and one is from 1947. Two of the cited monographs are published between 2000 and 2021, four in the period 1980-1999, two are from 1960-1979 and one from 1939. The aforementioned is a testimony that Tanya Marinova has gained a thorough knowledge of the orthogonal arrays by a careful study of the classical results and the contemporary achievements in the area.

2 Biographical data and personal impressions

Tanya Todorova Marinova graduated as a Bachelor of Informatics in 2011. In 2013 she was awarded a Masters Degree in Informatics (Discrete and Algebraic Structures). Tanya Marinova is a part-time PhD student in Doctoral Program "Algebra, topology and applications" from 2014. During her Masters and PhD studies, she has worked as a programmer and a senior programmer. In such a way, she developed simultaneously her knowledge in the area of mathematics and her practical skills in programming. These turned to be a firm premise for achievement of significant scientific results in her research.

I know Tanya Todorova Marinova from 2014. In June 2014 I participated in the committee for her PhD Entrance Exam. In April 2015 I was among the examinators of her General PhD Exam in Algebra. During the Winter Quarter of the academic year 2016-2017 I was a part of the committee for her Special PhD Exam in Spherical Codes, Orthogonal Arrays and Designs. At all of the aforementioned exams, Tanya Marinova performed in an excellent manner, both, in exposing theory and solving problems. Her diligent preparation allowed to demonstrate a rigorous knowledge in the basics of abstract algebra and its applications to the theory of orthogonal arrays.

Besides of the aforementioned three PhD exams, I know Tanya Marinova from her talks on five international and domestic scientific conferences and workshops. More precisely, in November 2014 I have witnessed her report "Combinatorial bounds for energies of codes in Hamming spaces" on a joint work with Peter Boyvalenkov, Maya Stoyanova and Mila Sukalinska at the National Coding Workshop with international participants "Professor Stefan Dodunekov" in Veliko Tarnovo. In December 2014 Tanya Marinova presented a report "Computing distance distributions of ternary orthogonal arrays" on a joint work with Maya Stoyanova at the conference "125 Years Mathematics and Natural Sciences in Sofia University "St. Kliment Ohridski". At the Spring Scientific Session of the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski" in March 2015, Tanya Marinova reported on "Bounds on energy of codes and designs in $H(n,q)$". In March 2016, she gave a talk "Non-existence of some binary orthogonal arrays" on a joint work with Peter Boyvalenkov and Maya Stoyanova at the Spring Scientific Session of the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski". At the Eleventh International Workshop on Algebraic and Combinatorial Coding Theory, which took place at Albena in June 2016, Tanya Marinova co-authored two of the presented articles. These were "Non-existence of $(96,9,4)$ and $(112,10,5)$ binary orthogonal arrays" on a joint work of Peter Boyvalenkov and Maya Stoyanova, as well as "Non-existence of $(112,9,4)$ and $(224,10,5)$ binary orthogonal arrays" on results with Maya Stoyanova.

The aforementioned exams and reports left me with very nice impressions for Tanya Marinova. I am convinced that her preparation is thorough and her work is accurate, earnest and diligent.

3 Content analysis of the dissertation

Orthogonal arrays generalize the notions of Latin Square and a mutually orthogonal pair of Latin Squares, which are inspired by works of Leonard Euler. As combinatorial objects with an abundance of symmetries, the orthogonal arrays are studied through the invariance of their parameters under application of specific bijective correspondences. This is in the spirit of Klein's Erlangen Program for classification of geometric objects through their groups of symmetries. The presence of a vast amount of symmetries imposes strong restrictions on the existence of orthogonal arrays and raises substantial obstacles for their classification.

The first chapter of the dissertation collects some preliminaries on orthogonal arrays in a Hamming space $H(n,q)$ of ordered n-tuples over a finite field with q elements. In an article from 1992 Levenshtein introduces the notion of a polynomial compact metric space (X,d) with a normalized measure μ. The Hamming space $H(n,q)$ is a compact metric space with respect to the Hamming distance $d(a,b)$, which equals the number of the different components of $a = (a_1,\dots,a_n), b = (b_1,\dots,b_n) \in H(n,q)$. The normalized measure μ on $H(n,q)$ is given by $\mu(S) := \frac{|S|}{q^n}$ for any subset $S \subseteq H(n,q)$ of cardinality $|S|$. In order to formulate precisely, let us fix a standard substitution $\sigma(d)$, which by definition is a continuous strictly decreasing function on the distance d with values in the closed segment $[-1,1]$. For the Hamming space $H(n,q)$ one takes $\sigma(d) := 1 - \frac{2d}{n}$. Levenshtein shows that there exist a unique system $\{Q_i(x) \,|\, 0 \leq i \leq n\}$ of orthogonal polynomials $Q_i(x)$ of degree $\deg Q_i(x) = i$ with $Q_i(1) = 1$ and uniquely determined constants $r_i \in \mathbb{R}^{>0}$, such that $\frac{r_i}{q^2} \sum\limits_{a,b\in H(n,q)} Q_i(\sigma(d(a,b)))Q_j(\sigma(d(a,b))) = \delta_{ij}$ for Kronecker's delta δ_{ij}. Here $Q_i(x)$ are the normalized Krawtchouk polynomials. Moreover, there are integers $m_i \in \mathbb{N}$ and non-zero continuous functions $w_{ij} : H(n,q) \to \mathbb{C}$, $1 \leq j \leq m_i$, such that $Q_i(\sigma(d(a,b))) = \sum\limits_{j=1}^{m_i} w_{ij}(a)\overline{w_{ij}(b)}$ for all $a,b \in H(n,q)$. The aforementioned properties turn the Hamming space $H(n,q)$ into a polynomial metric space.

Section 1.1 of the dissertation explains some properties of the Hamming space $H(n,q)$, viewed as a polynomial metric space. It describes the normalized Krawtchouk polynomials $Q_i(x) \in \mathbb{R}[x]$,

$0 \leq i \leq n$ and lists some properties of their roots. Special attention is paid to the kernels $T_k(x,y) := \sum_{i=0}^{k} r_i Q_i(x) Q_i(y) \in \mathbb{R}[x,y]^{(\mathrm{sym})}$, $0 \leq k \leq n$ of Krawtchouk polynomials $Q_i(x)$, the adjoint Krawtchouk polynomials $Q_i^{p,q}(x) \in \mathbb{R}[x]$ for $p, q \in \{0,1\}$, as well as to the kernels $T_k^{p,q}(x,y) := \sum_{i=0}^{k} r_i Q_i^{p,q}(x) Q_i^{p,q}(y) \in \mathbb{R}[x,y]^{(\mathrm{sym})}$, $0 \leq k \leq n$ of $Q_i^{p,q}(x)$. Section 1.1 includes various equalities and inequalities among the roots of the aforementioned polynomials from $[-1,1]$. Section 1.2 recalls the definition of an orthogonal array C with parameters $(\lambda q^t, n, q, t)$. It is such a matrix $C \in M_{\lambda q^t \times n}(\mathbb{F}_q)$ with λq^t rows and n columns over $\mathbb{F}_q$ that the rows of any submatric with λq^t rows and t columns contains exactly λ times any word from $H(t,q)$. The natural number t is called the strength of C, while the natural number λ is the index of C. Any orthogonal array C with parameters $(\lambda q^t, n, q, t)$ can be considered as a subset of $H(n,q)$ of cardinality λq^t, when the rows of C are counted with their multiplicities. The distance distribution $W(C,c)$ of C with respect to $c \in H(n,q)$ is the ordered $(n+1)$-tuple $W(C,c) = (w_0(c), w_1(c), \dots, w_n(c)) \in (\mathbb{Z}^{\geq 0})^{n+1}$, where $w_i(c)$ stands for the number of the words (rows) of C at Hamming distance $i \in \{0,1,\dots,n\}$ from c. Section 1.2 recalls some known properties of orthogonal arrays, which include the behaviour of their parameters under removing a row, followed by decomposition. It lists certain bijective transformations of orthogonal arrays, which preserve their parameters. Section 1.3 formulates the basic problem of estimating the index of an orthogonal array of fixed strength t in $H(n,q)$. It recalls Delsarte's linear programming upper and lower bounds, Rao's lower and upper bounds, Singleton's lower and upper bounds, Plotkin's upper bound and Levenshtein's lower and upper bounds. Section 1.4 recalls a result of Boyvalenkov and Kulina from 2013, which is a starting point of the algorithms, developed in the next chapters. The result in question is a linear system of equations on the distance distribution of an orthogonal array, whose constant terms depend on the first coefficients of the expansions of $x^0, x^1, \dots, x^t$ with respect to the normalized Krawtchouk polynomials $Q_i(x)$. The rest of Section 1.4 explains why the set $W(\lambda q^t, n, q, t)$ of all the possible distance distributions $W(C,c)$ of orthogonal arrays C of fixed strength $t \in \mathbb{N}$ and index $\lambda \in \mathbb{N}$ in $H(n,q)$ with respect to arbitrary points $c \in H(n,q)$ is depleted by the set of all $W(C,c_o)$ with respect to a fixed point $c_o \in H(n,q)$. Similarly, the sets $P(\lambda q^t, n, q, t)$, respectively, $Q(\lambda q^t, n, q, t)$ of all the possible distance distributions $W(C,c)$ of orthogonal arrays in $H(n,q)$ of fixed strength $t \in \mathbb{N}$ and index $\lambda \in \mathbb{N}$ with respect to arbitrary interior points $c \in C$, respectively, with respect to arbitrary exterior points $c \in H(n,q) \setminus C$ are depleted by the sets of all such $W(C,c_o)$ with a fixed $c_o \in C$, respectively, with a fixed $c_o \in H(n,q) \setminus C$.

The second chapter of the dissertation reflects the original results on the distance distribution of binary orthogonal arrays. Section 2.1 recalls some important properties of the orthogonal arrays in $H(n,2)$ and the available lower bounds on the index of an orthogonal array of strength $2 \leq t \leq 10$ in $H(n,2)$ with $4 \leq n \leq 20$. Section 2.2 discusses constructions of derived orthogonal arrays with smaller parameters. These include a removal of a column of an $(\lambda 2^t, n, 2, t)$-orthogonal array C with $n > t$, which yields an $(\lambda 2^t, n-1, 2, t)$-orthogonal array C'. The collection C_0, respectively, C_1 of the rows of C', whose preimage in C contained 0, respectively, 1 in the deleted column are orthogonal arrays with parameters $(\lambda 2^{t-1}, n-1, 2, t-1)$. Section 2.3 is devoted to some relations among an interior distance distribution of a binary orthogonal array C and the interior distance distribution of the derived arrays C', C_0, C_1. To this end, for any $0 \leq i \leq n$ is considered the i-th block of C, consisting of the words of distance i from the reference point. If y_i, respectively, x_i is the number of the zeroes, respectively, ones in the deleted column of the i-th block then the distance distribution of C_0, respectively, of C_1 is shown to be given by the collection of y_i, respectively, of x_i. The interior distance distributions of C and C' are expressed by the interior distance distributions of C_0, C_1. Let us fix an interior distance distribution $P(C,c)$ of an $(\lambda 2^t, n, 2, t)$-orthogonal array C and consider all the possible distance distributions $P(C_1, c') = (x_1, \dots, x_n)$ of C_1, while varying the distance distribution $P(C', c')$ of C'. Section

2.3 expresses $P(C,c)$ by all the possible $P(C_1,c')$ and their multiplicities. The aforementioned relations among the distance distributions of C, C', C_0, C_1 provide a first algorithm for reduction of the set $P(\lambda 2^t, n, 2, t)$. Section 2.4 discusses a second basic algorithm, which reduces the set $W(\lambda 2^t, n, 2, t)$ of all (interior and exterior) distance distributions of orthogonal arrays of strength t and index λ in $H(n,2)$. It provides two formulae for the distance distributions of C_0, C_1 in terms of the ones of C, C'. Any $(\lambda 2^t, n, 2, t)$-orthogonal array C has an antipodal array $\overline{C}$ with the same parameters, which is obtained from C by exchanging the zeroes with ones. The distance distributions of C and $\overline{C}$ are symmetric to each other with respect to their middle point. The exchange of the zeroes and ones in a fixed l-th column of C produces an $(\lambda 2^t, n, 2, t)$-orthogonal array $C_l^{1,0}$. The distance distribution of $C_l^{1,0}$ is relateed to the distance distributions of C_0, C_1, whereas to the distance distributions of C, C'. Let us fix a distance distribution $W(C,c)$ of an $(\lambda 2^t, n, 2, t)$-orthogonal array C and consider all the possible distance distributions $W(C',c')$ of the $(\lambda 2^t, n-1, 2, t)$-orthogonal array C', which is obtained from C by removing a column. Then all the possible $W(C_1,c')$ and their multiplicities suffice for expressing $W(C,c)$. The relations among the distance distributions of C, C', C_0, C_1, derived in Section 2.4, are used for construction of a second, basic algorithm for reduction of $W(\lambda 2^t, n, 2, t)$, which is explained at the end of this section. Section 2.5 develops a third algorithm for reduction of $W(\lambda 2^t, n, 2, t)$ with $n-1 > t$. It is based on removing two columns from the initial $(\lambda 2^t, n, 2, t)$-orthogonal array C. The relations from Section 2.4 among the distance distributions of the original and its derived orthogonal arrays are applied by removing first the l-th and then the m-th column of C, as well as by removing first the m-th and then the l-th column of C. The aforementioned constructions are composed with exchanging zeroes and ones only in the l-th column of C, as well as with exchanging zeroes and ones simultaneously in the l-th and the m-th column of C. Section 2.5 concludes with a brief explanation of a generalized third algorithm for reduction of $W(\lambda 2^t, n, 2, t)$. Section 2.6 provides a detailed account of the third algorithm and some ideas for optimization of its speed. Section 2.7 illustrates the constructed three algorithms on the set $W(5.2^3, 20, 2, 3)$ of the distance distributions of the orthogonal arrays of strength $t=3$ and index $\lambda=5$ in $H(20,2)$. The final Section 2.8 of Chapter 2 announces the original non-existence results, obtained by the means of the developed three algorithms. These include orthogonal arrays of strength $4 \leq t \leq 10$ and index $\lambda \in \{6,7,10,11,13\}$ in $H(n,2)$ with $10 \leq n \leq 15$, associated with 42 sets of parameters. The aforementioned non-existence results for binary orthogonal arrays yield exact lower bounds $\Lambda(n,2,4)$ on the index of orthogonal arrays of strength 4 in $H(n,2)$ with $9 \leq n \leq 12$, as well as exact lower bounds $\Lambda(n,2,5)$ on the index of orthogonal arrays of strength 5 in $H(n,2)$ with $10 \leq n \leq 13$.

The third chapter of the dissertation reflects the original results on the distance distribution of the ternary orthogonal arrays. Section 3.1 recalls some properties of the ternary orthogonal arrays and provides a table of the available lower bounds on the index of an orthogonal array of strength $2 \leq t \leq 10$ in $H(n,3)$ with $4 \leq n \leq 25$. Section 3.2 derives relations among the interior distance distribution of a ternary orthogonal array and its derived subarrays. For an arbitrary $(\lambda 3^t, n, 3, t)$-orthogonal array with $n > t$ and any $1 \leq l \leq n$, $0 \leq i \leq n$, let us denote by y_i, z_i, u_i the number of the components of the l-th column of the i-th block of C, which equal, respectively, to $0, 1, 2$. Then $\overline{y_i} = z_i + u_i$ is the number of the non-zero components in the l-th column of the i-th block of C. The $(\lambda 3^t, n-1, 3, t)$-orthogonal array C', obtained from C by deleting the l-th column decomposes into a union of the $(\lambda 3^{t-1}, n-1, 3, t-1)$-orthogonal arrays C_0, C_1, C_2, where the rows of C_j collects the puncturings of those rows of C, which have contained j in their deleted column. The distance distributions of C_0, C_1, C_2 consist of the numbers y_i, z_i, u_i and the distance distribution of the union $\overline{C_0}$ of C_1, C_2 is given by $\overline{y_i}$. Section 3.2 expresses the interior distance distributions of C, C' by the ones of $C_0, \overline{C_0}$. It provides also two formulae for the distance distributions of $C_0, \overline{C_0}$ in terms of the ones of C, C'. The final, third section of Chapter 3 is devoted to a modified, sixth algorithm for reduction of the set of the interior distance distributions of the ternary orthogonal arrays with fixed parameters. By the means of algorithms five and six is derived the non-existence of an $(4.3^3, 17, 3, 3)$-orthogonal array. That improves the lower bound on the index of an orthogonal array of strength 3 in $H(17, 3)$.

The last, fourth chapter derives combinatorial lower and upper bounds on the energy of an orthogonal array. For an arbitrary potential $h : [-1, 1] \to (0, +\infty)$, the energy of an $(\lambda q^t, n, q, t)$-orthogonal array C is defined as $\mathcal{E}(C, h) := \frac{1}{|C|} \sum\limits_{x,y \in C,\, x \neq y} h\left(1 - \frac{2d(x,y)}{n}\right)$. For any interior point $x \in C$, let us denote by $p(x) = (p_0(x), p_1(x), \dots, p_n(x)) \in (\mathbb{Z}^{\geq})^{n+1}$ the distance distribution of C with respect to x. Section 4.1 expresses $\mathcal{E}(C, h)$ by $p_1(x), \dots, p_n(x)$ and relates the problem of estimating $\mathcal{E}(C, h)$ for all the $(\lambda q^t, n, q, t)$-orthogonal arrays C to the explicit knowledge of the set $P(\lambda q^t, n, q, t)$. Section 4.2 gives a combinatorial lower bound $\mathcal{L}(\lambda q^t, n, q, t, h)$ and a combinatorial upper bound $\mathcal{U}(\lambda q^t, n, q, t, h)$ on $\mathcal{E}(C, h)$. An article of Boyvalenkov, Dragnev, Hardin, Saff and Stoyanova from 2017 derives universal lower bounds $\mathcal{L}'(\lambda q^t, n, q, t, h)$ on the energy of an $(\lambda q^t, n, q, t)$-orthogonal array with respect to a potential h. Making use of a seventh algorithm for obtaining $\mathcal{L}(\lambda q^t, n, q, t, h)$ and $\mathcal{U}(\lambda q^t, n, q, t, h)$, Section 4.3 compares the combinatorial lower bound $\mathcal{L}(\lambda q^t, n, q, t, h)$ to the universal lower bound $\mathcal{L}'(\lambda q^t, n, q, t, h)$ for five binary orthogonal arrays and one ternary orthogonal array with respect to three potentials h_i, $1 \leq i \leq 3$. In four of the cases of binary orthogonal arrays, the combinatorial lower bound $\mathcal{L}(\lambda q^t, n, q, t, h)$ is sharper than the universal lower bound $\mathcal{L}'(\lambda q^t, n, q, t, h)$. In the remaining two cases, all the bounds under consideration coincide, i.e. $\mathcal{L}'(3.2^3, 12, 2, 3, h_i) = \mathcal{L}(3.2^3, 12, 2, 3, h_i) = \mathcal{U}(3.2^3, 12, 2, 3, h_i)$, respectively, $\mathcal{L}'(3.3^5, 12, 3, 5, h_i) = \mathcal{L}(3.3^5, 12, 3, 5, h_i) = \mathcal{U}(3.3^5, 12, 3, 5, h_i)$ for all $1 \leq i \leq 3$.

To sum up, the basic contributions of the dissertation include three algorithms for reduction of the set of distance distributions of binary orthogonal arrays with fixed parameters. These provide exact lower bounds on the index of an orthogonal array of strength $4 \leq t \leq 5$ in $H(n, 2)$ with $9 \leq n \leq 13$. The original results comprise also two algorithms for reduction of the set of distance distributions of ternary orthogonal arrays with fixed parameters. Algorithms four and five allow to establish the non-existence of an $(4.3^3, 17, 3, 3)$-orthogonal array and to improve the lower bound on the index of an orthogonal array in $H(17, 3)$ of strength 3. Besides, the dissertation provides an algorithm for computing the energy of an orthogonal array with respect to a given potential.

4 Approbation of the results

The dissertation under review reflects the results of six articles. Two of them are in specialized journals with IF from the third quartile, one is in a journal with SJR, two are in indexed and refereed journals and one is in a conference proceedings. The publications of Tanya Marinova earn her 156 points, according to Decree 26/13.02.2019 on the amendments of the Rules of implementation of the Law on Development of Academic Staff of Republic Bulgaria. This exceeds more than five times the required 30 points for acquisition of the educational and scientific degree "Doctor". More precisely, each of the two articles with IF from the third quartile provide 45 points, the article in a journal with SJR earns her 30 points and each of the two articles in indexed and refereed journals costs 18 points. All of the publications of Tanya Marinova are joint works. According to the submitted declarations, the contributions of all co-authors are commensurable. A computer test establishes that the results of the dissertation are original and there is no plagiarism. The aforementioned six articles are cited 12 times. Ten of the citations are in articles, listed in Web of Science or Scopus. The scientific contributions of the dissertation are reported at eight international and national conferences and workshops.

The aforementioned scientometric criteria and content analysis convinced me that the dissertation "Algorithms for Characterisation of Orthogonal Arrays" of Tanya Todorova Marinova complies with the minimal national requirements of Decree 26/13.02.2019 on the amendments of the Rules of implementation of the Law on Development of Academic Staff of Republic Bulgaria, as well as with the requirements of the Rules on the terms and conditions for acquisition of academic degrees and occupation of academic positions at Sofia University "St. Kliment Ohridski".

5 Qualities of the abstract

Both abstracts - in Bulgarian and in English reflect appropriately the content and the scientific contributions of the dissertation. The articles of other authors are presented accurately in the dissertation and its abstracts by providing the exact formulations and a detailed information for the corresponding source.

6 Critical remarks and suggestions

There are some misprints in the dissertation and its abstracts, which do not affect their high quality. I recommend Tanya Marinova to pursue her further research with the willpower and vigour, she manifested as a graduate student.

7 Conclusion

According to my impressions for the dissertation and its corresponding scientific works, and based on the above analysis of their scientific significance and applicability, I confirm that the presented dissertation and its corresponding scientific publications, together with the quality and the originality of their results and achievements, comply with the requirements of the Law on Development of Academic Staff of Republic Bulgaria, the Rules on its implementation and the Rules on the terms and conditions for acquisition of academic degrees and occupation of academic positions at Sofia University "St. Kliment Ohridski" for acquisition of the educational and scientific degree "Doctor" in Scientific Field 4. Natural Sciences, Mathematics and Informatics, Professional Directions 4.5 Mathematics. In particular, Tanya Todorova Marinova complies with the minimal national requirements in the professional direction and no plagiarism was found in the presented scientific works.

Based on the aforementioned, I assess positively and strongly recommend the Scientific Juri to award Tanya Todorova Marinova the educational and scientific degree "Doctor" in Scientific Field 4. Natural Sciences, Mathematics and Informatics, Professional Directions 4.5 Mathematics (Algebra, Topology and Applications).

April 12, 2021

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