R E V I E W

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on a competition for occupying the academic position of **Associate Professor**

area of higher education: *4. Natural Sciences, Mathematics and Informatics*, professional field *4.5 Mathematics (Geometry)*

for the needs of Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics

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I am a member of the scientific panel for this procedure according to order No. РД 38-266/10.07.2020 of the Rector of Sofia University "St. Kliment Ohridski" Prof. D.Sc. Atanas Gerdzhikov. Documents for participation in the announced competition have been submitted only by Assistant Professor Dr. Alexander Vladimirov Petkov. As a member of the scientific panel, I have received from Dr. Petkov all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at Sofia University "St. Kliment Ohridski".

Personal data

Dr. Alexander Petkov was born on December 17, 1985. In the period 2004-2010, he was a student in the Faculty of Mathematics and Informatics of Sofia University "St. Kliment Ohridski". He graduated with excellence in 2010 and got master's degree. From 2011 to 2014, he has been a Ph.D. student at the Department of Geometry of the Faculty of Mathematics and Informatics, Sofia. He successfully defended his Ph.D. thesis "Reimannian and sub-Riemannian manifolds with structures" in 2014. From 2014 till now, he has been an assistant professor at the Department of geometry of the Faculty of Mathematics and Informatics, SU. He had a post-doc position at the Institute of Mathematics, Faculty of Mathematics, University of Vienna in 2017, two times, each of 3 months. He was a visiting researcher at the Institute of the Mathematical Sciences of the Americas at the University of Miami in 2018 and 2019 (two times, each visit of 3 months).

General characterization of the scientific works of the applicant.

The candidate A. Petkov has submitted 6 scientific papers for participation in the competition. These papers meet the minimal requirements according to ADASRB. Petkov is the only author of three of them. He has 2 co-authors in 2 of the papers and 1 co-author in one paper. I have known Alexander Petkov and his mathematical work for 6 years. I have been a member of the scientific panel for evaluation of his Ph.D. thesis and have written a peer review of it. I have no doubt that his contribution in the joint papers is of equal worth with the other co-authors.

The six papers submitted by A. Petkov for the competition have been quoted 8 times by other authors. The list of all publications by the candidate includes 8 papers quoted 14 times (without self-citations).

The papers presented by A. Petkov have not been used for awarding the scientific degree "doctor" or occupying an academic position.

I would like also to mention that I have not discovered any plagiarism.

The papers presented by A. Petkov are in the field of differential geometry, more precisely sub-Riemannian geometry. They treat interesting and difficult problems in this area. Most of them are publish in prestigious mathematical journals.

Analysis of the scientific achievements of the applicant

All of the papers submitted by A. Petkov deal with quaternionic contact manifolds, shortly "QC-manifolds". Recall that a quaternionic contact structure on a $(4n+3)$-dimensional smooth manifold M, a notion introduced by O. Biquard, consists of a rank $4n$ subbundle H of TM, a positive definite metric g on H and a rank 3 subbundle Q of $End(H)$ such that, in a neighbourhood U of each point of M, there are 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and a triple $\vartheta = (I_1, I_2, I_3)$ of sections of Q with the following properties: (1) $H|U$ is the kernel of η; (2) The endomorphisms (I_1, I_2, I_3) of H are three almost complex structures locally generating the bundle Q and satisfying the identities of the imaginary unit quaternions: $I_1^2 = I_2^2 = I_3^2 = -Id_H$, $I_1 I_2 = -I_2 I_1 = I_3$; (3) $d\eta_s(X, Y) = 2g(I_s X, Y)$ for $X, Y \in H|U$. Any two triples of sections of Q satisfying condition (2) constitute frames of Q which induce the same orientation, thus the bundle Q has a canonical orientation. This bundle is endowed with the restriction of the standard metric on End(H) induced by the metric g of H. It is a result of Biquard that if $dim\, M > 7$, there are a unique complementary bundle V of H and a unique connection ∇ on M satisfying certain conditions which I omit here for brevity and refer to the papers by the candidate or the original paper by Biquard. One of the properties of ∇ is that it preserves the bundle Q. If $dim\, M = 7$, it is not always possible to find a supplement V to H and a connection ∇ such that Q is preserved by the connection, as D. Duchemin has shown, and one has to assume additionally that ∇ has this property. The bundles H and V are often called "horizontal" and "vertical", respectively. Let the form $\eta = (\eta_1, \eta_2, \eta_3)$ and the almost complex structures $\vartheta = (I_1, I_2, I_3)$ satisfy conditions (1), (2), (3), and let (ξ_1, ξ_2, ξ_3) be the frame of V dual to $(\eta_1|V, \eta_2|V, \eta_3|V)$. The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields and have the property that $\imath_{\xi_s} d\eta_t|H = -\imath_{\xi_t} d\eta_s|H$, $s, t = 1, 2, 3$ (this property is missing in dimension 7 without the additional assumption for ∇). The assignment $\xi_s \to I_s$, $s = 1, 2, 3$, determines a bundle isomorphism $V \to Q$ which does not depend on the particular choice of η and ϑ . The metric and orientation of Q can be transferred to a metric and orientation on V via this isomorphism. Thus, we can define a Riemannian metric on the whole bundle TM, again denoted by g. The decomposition $TM = H \oplus V$ is orthogonal and the connection ∇ is metric with respect to g.

The main goal of the papers with numbers 1,2,3 in the list of publications of the candidate is to give for QC-manifolds a version of the classical Lichnerowicz-Obata theorem. Lichnerowicz type results are discussed also in paper $\sharp$6. Recall that by a result of A. Lichnerowicz if the Ricci tensor of a compact Riemannian manifold of dimension n satisfies the inequality $Ricci(X,X) \geq kg(X,X)$ where g is the Riemannian metric and k is a positive constant, then the first non-zero eigenvalue of the Laplacian satisfies the inequality $\lambda_1 \geq \frac{n}{n-1}k$. Note that on the unit sphere S^n with its standard metric, $\lambda_1 = n$. M. Obata has proved that, conversely, if $\lambda_1 = n$ on a compact Riemannian manifold of dimension n, the manifold is isometric to S^n. In the case of a QC-manifold, it is natural to considered the sub-Laplacian instead of Laplacian, i.e., the trace of the Hessian over the horizontal spaces with minus sign.

In paper ♯1, the QC-manifolds under consideration are compact and of dimension 7. The condition on the Ricci tensor in the Lichnerowic theorem is replaced by the inequality

$$Ricci(X,X) + 6T^0(X,X) \geq kg(X,X)$$

for every horizontal vector X. In this inequality, *Ricci* means the QC-Ricci tensor, i.e., the trace of the curvature of the Biquard connection over the horizontal spaces and T^0 is a symmetric traceless tensor defined by means of the torsion of the Biquard connection. This condition resembles the assumption $Ricci(X,X) + \frac{n}{2}T(X,X) \geq kg(X,X)$ with respect to the Webster-Tanaka connection on a compact pseudo-Hermitian CR-manifold of dimension $2n+1$ used by Greenleaf (1985) to prove the estimate $\lambda_1 \geq \frac{n}{n+1}k$ for the first eigenvalue of the CR sub-Laplacian in the case $n \geq 3$; for $n = 2$ this was proved by S.-Y. Li – H.-S. Luk (2004). Another assumption in paper ♯1 is that the P-function of every eigenfunction of the sub-Laplacian is non-negative. As the authors of the paper note, the motivation for introducing the P-function comes from the Paneitz operator used in CR-geometry (as well as from the P-function used in elliptic PDE theory). The main result in the paper states that, under the assumptions above, the first eigenvalue λ_1 of the sub-Laplacian satisfies the inequality $\lambda_1 \geq \frac{1}{3}k$. Moreover, if the equality holds for a compact 3-Sasakian manifold of dimension 7, the manifold is the 3-Sasakian unit sphere. The proof of the main result is based on a Bochner type formula (Lemma 3.2).

The Bochner type formula proved in paper ♯3 (Theorem 3.1) is used in paper ♯2 where another version of the Lichnerowicz theorem is proved for the sub-Laplacian of a compact QC-manifold of dimension 7. In this version, the condition for the P-function is dropped and the condition on the QC-Ricci tensor is replaced by the inequality

$$Ricci(X,X) - 2T^0(X,X) - \frac{36}{k}A(X) \geq kg(X,X)$$

where $A(X)$ is defined for horizontal vectors by means of the QC-scalar curvature and the torsion and its covariant derivatives with respect to the Biquard connection. A condition of this type involving the Webster-Tanaka connection has been used by S.-Y. Li – H.-S. Luk (2004) in order to obtain the Greenleaf estimate for λ_1 in the 3-dimensional CR-case.

The case of a compact QC-manifold of any dimension $4n+3 > 7$ is discussed in paper ♯3. The assumption in it is that

$$Ricci(X,X) + \frac{2(4n+5)}{2n+1}T^0(X,X) + \frac{6(2n^2+5n-1)}{(n-1)(2n+1)}U(X,X) \geq kg(X,X)$$

where U is a trace-free symmetric tensor defined by means of the torsion of the Biquard connection. The main result of the paper is that $\lambda_1 \geq \frac{n}{n+2}k$. The tensors T^0 and U have been introduced by Ivanov-Minchev- Vasilev. They are defined via symmetric and, correspondingly, anti-symmetric parts of the torsion endomorphisms $V \ni \xi \to T(\xi, .)$. In the case $n = 1$, the tensor U vanishes. Thus, forgetting the fraction in front of U above, we may say that for $n = 1$ the assumption in paper ♯3 reduces to one of the assumptions in paper ♯2. Clearly, in this case, the estimates for λ_1 are the same in both papers. Another result in paper ♯3 is that if the manifold is QC-Einstein with QC-scalar curvature $16n(n+2)$, i.e., $Ricci(X,X) = 4(n+2)g(X,X)$ and if the the lower bound for λ_1 is achieved, i.e., $\lambda_1 = 4n$, then the QC-manifold is the 3-Sasakian sphere of dimension $4n+3$. In particular, a compact 3-Sasakian manifold (endowed with its standard QC-structure) of dimension $4n+3$ with $\lambda_1 = 4n$ is equivalent to S^{4n+3}.

Paper ♯4 deals with the Yamabe problem on a compact QC-manifold M. It consists of finding a new metric $\overline{g} = fg$ in the conformal class of the given metric g on the horizontal bundle H such that the new QC-structure determined by a suitable multiple $\overline{\eta}$ of the given contact form η and the metric $\overline{g}$ is of constant QC-scalar curvature. As in the Riemannian and CR cases, the function f satisfies the Euler-Lagrange equation of a suitably define functional, the Yamabe functional $\Upsilon_M(\eta)$. If f is such a function and $\overline{\eta} = f^{1/n+1}\eta$, then the QC-structure $(\overline{\eta}, \overline{g})$ is of constant QC-scalar curvature. The main result of paper ♯4 is that if a compact QC-manifold of dimension $4n+3$ is not locally equivalent to the QC-structure of the 3-Sasakian sphere S^{4n+3}, then the Yamabe problem has a solution. The idea of the proof is to show that the Yamabe constant $\lambda(M)$ of the manifold, i.e., the minimum of the Yamabe functional, is less than the Yamabe constant $\lambda(S^{4n+3})$ of the sphere S^{4n+3}, the latter being calculated by Ivanov-Minchev-Vasilev. Then the result follows from a theorem by W. Wang. In order to show that $\lambda(M) < \lambda(S^{4n+3})$, the authors find an asymptotic formula for the values $\Upsilon_M(\eta^\varepsilon)$ of the Yamabe functional on contact forms $\eta^\varepsilon = (f^\varepsilon)^{1/n+1}\eta$, $\varepsilon > 0$, where, for a fixed $q \in M$, f^ε are smooth functions suitably defined by means of the so called QC-normal coordinates at q constructed by C. Kunkel. The asymptotic formula looks like $\Upsilon_M(\eta^\varepsilon) = \lambda(S^{4n+3})(1 - c(n))||W^{QC}(q)||^2\varepsilon^4 + O(\varepsilon^5)$ where $c(n)$ is a positive constant and W^{QC} is the QC-conformal curvature tensor introduced by Ivanov-Vasilev who have proved that a QC-manifold is locally equivalent to S^{4n+3} if and only if $W^{QC} = 0$. Since by assumption $W^{QC}(q) \neq 0$ for some $q \in M$, the asymptotic formula implies $\Upsilon_M(\eta^\varepsilon) < \lambda(S^{4n+3})$ for small ε which gives the desired inequality $\lambda(M) < \lambda(S^{4n+3})$. As the authors mention, a similar scheme has been used by D. Jerison - J.M. Lee (1989) to prove the existence of a solution of the CR-Yamabe problem on a compact strictly pseudoconvex CR-manifold ($\equiv$ positive definite Levi form) of dimension $2n+1$ not locally CR-equivalent to S^{2n+1}. It is worth noting that their proof works in dimension $2n+1 > 3$ while the proof in the QC-case works in any dimension. I would like also to note that the Yamabe problem for S^{4n+3} (and, more generally, a compact 3-Sasakian manifold) has been solved by Ivanov-Minchev-Vasilev.

In paper ♯5, the heat equation on a QC-manifold, i.e. the equation $\frac{\partial u}{\partial t} = -\Delta u$ where Δ is the sub-Laplacian is considered. A integral formula for the time derivative of the corresponding energy functional is derived. This entropy formula is used to prove that if the Lichnerowicz type condition in paper ♯3 holds for $k = 0$, then the energy functional is time monotone non-increasing on the solutions of the heat equation provided $n > 1$. In the case $n = 1$, this holds under an additional assumption for the P-function.

The entropy formula obtained in paper ♯5 is used in paper ♯6 to reprove the Lichnerowicz-type results in papers ♯1 and ♯3. Also, a lower bound for the first eigenvalue λ_1 of the sub-Laplacian on a compact QC-manifold is obtained under the assumption in paper ♯3 stated above, but with arbitrary $k \in \mathbb{R}$ and an assumption on the so-called C-operator for a QC-manifold introduced in paper ♯1. The method of the proof is inspired, as the author mentions, by a paper of S.-C. Chang – C.-T. Wu (2010) for pseudo-Hermitian CR-manifolds, but of course it has some specific characteristics and difficulties.

At the end of this survey, I would like to emphasize that the computations in the papers discussed above are non-trivial and require high rank mathematical skills.

Educational activity

In the Faculty of Mathematics and Informatics, Alexander Petkov has conducted problem solving sessions on Differential geometry and on Geometry. He has also had such classes on Linear algebra and analytic geometry in the Faculty of Physics and Faculty of Chemistry and Pharmacy of Sofia University. He had lectures and exercise classes on Analytic geometry for students of Mathematical statistics as well as of Mathematics and Informatics (correspondence education). He has taught Mathematics for students of Geology, lectures and exercise classes.

Participation in scientific projects

A. Petkov has taken part in 12 scientific projects. Three of them has been supported by Bulgarian National Science Fund, eight by University of Sofia, one by Ministry of Education and Science.

Participation in scientific conferences

A. Petkov has presented his scientific works at international conferences held in many European countries, USA and Mexico. He has given talks at 17 conferences and posted posters at 5 conference. He has also given talks at 4 seminars in Bulgaria and 3 seminars in USA.

CONCLUSION

The documents and materials presented by Dr. Alexander Petkov meet the requirements of the Act on the Development of the Academic Staff in the Republic of Bulgaria , the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying academic positions at Sofia University “St. Kliment Ohridski".

The results obtained by A. Petkow are important contributions to a difficult field of the contemporary differential geometry. They overpass the usual standards for holding the academic position of associate professor.

Based on the comments above, I give a positive assessment of the scientific work of Dr. Alexander Petkov and recommend to the scientific jury to advise the competent body at the Faculty of Mathematics and Informatics, Sofia University "St. Kl. Ohridski" to appoint Assistant Professor Dr. Alexander Vladimirov Petkov as an Associate Professor in area of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics, scientific speciality "Geometry".

03.09.2020 Reviewer:

(Prof. D.Sc. Johann Davidov)