R E V I E W

of dissertation thesis for conferring the **Doctor of Science** degree in
higher education area 4. Natural sciences, mathematics and informatics;
professional direction 4.5 Mathematics; scientific specialty Geometry and Topology

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Theme of the thesis: *The geometry of quaternionic-contact manifolds and the Yamabe problem*

Referee: associate professor PhD Georgi Ganchev, associated member of IMI at BAS, pensioner
By order No РД 38–113 from 19.02.2020 г. by the Rector of Sofia University "Kliment Ohridski"(SU), I was appointed a member of the scientific jury for providing a thesis defence procedure of the present dissertation thesis.

1. General presentation of received materials

The set of materials presented by Ivan Minchev is in accordance with the Rules for the Development of Academic Staff of the Sofia University and includes the following documents:

1) CV in European format;
2) copy of the diploma for higher education;
3) copy of the diploma of educational and scientific degree "PhD";
4) author's summary in English and Bulgarian;
5) dissertation thesis in English;
6) declaration of originality and authenticity of the attached documents;
7) information on meeting the minimum requirements under Art. 2b of the Law on the Development of Academic Staff in Bulgaria;
8) copies of the scientific publications;
9) copy of the habilitation diploma from Germany;
10) CD with attached documents.

2. Short biography of the applicant

Ivan Minchev was born on 12.06.1975 in Sofia. During the period 1995–2001 he completed his higher education in *mathematics* at the Sofia University with a diploma thesis. From 2002 to 2005 he was a PhD student in the Department of Geometry with a dissertation thesis *Differential geometry of metric connections with torsion.* After defending his dissertation in 2006 he received the educational and scientific degree PhD in Geometry and Topology. From October 2006 to November 2008 he specialized at the Humboldt University in Germany, and from November 2008 to March 2012 he specialized at the Philipps Univesity in Marburg, Germany. This specialization ended with habilitation before a reputable international jury in 2012. During the period May 2012 - May 2013, he was the Assistant Professor in the Department of Geometry of the Faculty of Mathematics and Informatics at Sofia University, and since May 2013 he has been an associate professor in the same department. We will also note the specialization at Masaryk University in Brno, Czech Republic from November 2013 to November 2016.

3. Actuality of the topic

The present dissertation thesis is devoted to the differential geometry of quaternionic-contact (QC-) manifolds, with special emphasis on the Yamabe QC-problem. The notion of QC-geometry appeared about 20 years ago and has been a subject of intense research during these two decades. As the known mathematicians who laid the foundations of this modern theory, we will mention LeBrun and Biquard.

4. Knowledge of the problem and research methodology

The construction and organization of the thesis work show a highly professional approach in studying the problem.

The chosen methodology is suitable for solving the tasks. The results obtained unambiguously indicate a high degree of knowledge of the problem and mastery of the specific methods of studying these manifolds.

5. Overview and evaluation of the results in the thesis

In the present thesis it is treated an important and actual topic in QC-geometry. The work has a volume of 212 standard pages. The exposition is divided into 5 chapters, including a bibliography covering 91 titles of papers.

A QC-structure H on a (4n+3)-dimensional manifold M^{4n+3} is a $4n$-dimensional distribution on M, which is locally given as a kernel of an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$, for which the restrictions on H of the three 2-forms $d\eta_i|_H$ are fundamental forms of a quaternionic structure on H. Historically, the 4-dimensional self-dual Einstein metrics can be considered as quaternionic Kähler metrics of dimension 4. Therefore, the QC-geometry can be considered as a natural generalization of the three-dimensional conformal Riemannian geometry to dimensions of the type $4n+3$.

An impressive result in this work is the complete solution to the Yamabe problem for the quaternionic Heisenberg group $\mathbb{G}(H)$.

The QC-conformal transformations of the 1-form η, preserving the QC-structure H, have the form $\tilde{\mu} = \mu\Psi\,\eta$, where μ is a positive function, and Ψ is an $SO(3)$-valued function on M. There exists on M canonically determined conformal class of metrics $[g]$ of the distribution H and a quaternionic structure Q. With any fixed metric g of the class $[g]$ it is associated a uniquely determined linear connection on M (Biquard connection), which preserves the QC-structure and the metric g. An important invariant on a QC-manifold (M, H) is the scalar curvature $Scal$ of the Biquard connection. Generally speaking, the Yamabe problem consists in finding the metrics g from the class $[g]$, for which $Scal = const$.

Chapter 2 is devoted to the Yamabe QC-problem in QC-geometry. Precursors to this problem in the classical theory are the corresponding problems in the Riemannian case and the CR-case. An important role in solving the two classic cases plays the investigation of the flat model, which is given by the corresponding Heisenberg group. This role in QC-geometry is played by the quaternionic Heisenberg group. In Chapter 2 it is given a partial solution to the Yamabe problem for the quaternionic Heisenberg group. The Yamabe problem has played an important role in the research of a number of leading mathematicians in recent decades. It is worth noting that successful research in the contemporary version of this topic becomes with the decisive participation of the author and a number of bulgarian differential geometers.

Main theorems in this chapter are Theorem A, Theorem B and Theorem C.

In Theorem A are treated conformal transformations $\Theta = \dfrac{1}{2h}\tilde{\Theta}$ of the standard structure on $G(H)$. Under the condition, that the new structure is again an Einstein structure, there are found explicitly all global functions h on the quaternionic Heisenberg group with this property. The research method here is to study in detail the properties of the Biquard connection, based on excellent knowledge of classical cases.

The Cayley transformation is the analogue of the stereographic projection in Riemannian geometry. This transformation gives a natural identification of S^{4n+3} without a point with the quaternionic Heisenberg group $\mathbb{G}(H)$ of dimension $4n+3$.

Using the Cayley transformation and Theorem A, the author proves Theorem B, which gives a partial (except for one incomplete case) solution to the Yamabe QC-problem on the sphere S^{4n+3}. Theorem B concerns conformal transformations $\eta = f\,\tilde{\eta}$ under the assumption, that the new 1-form η on S^{4n+3} has a constant QC-scalar curvature $Scal = const$. For $n > 1$ Theorem B gives a complete description of the functions f. In the case $n = 1$ these functions are described under the additional assumption, that the vertical distribution, corresponding to η, is integrable.

Theorem C treats a QC-manifold M with a positive QC-scalar curvature $scal > 0$. In the case $n = 1$, the additional assumption $Scal = const$ is made. Then the assumptions:

i) M is a QC-Einstein manifold;

ii) M is locally a 3-Sasakian manifold;

iii) the Biquard connection is symmetric,

are equivalent.

In the course of the proof of the above theorem we will mention the important result Theorem 5.9:

The scalar QC-curvature of any QC-Einstein manifold $(n \geq 0)$ is a constant.

The theorems are characterized by maximum conciseness and simplified formulations, which is achieved with excellent knowledge of the literature and in-depth study of the problems posed. In this context we will also mention Theorem 8.10, which proves that any of the Reeb fields is a QC-vector field if and only if when the QC-structure of the manifold is homothetic to a 3-Sasakian structure.

Chapter 3. The incomplete cases of Chapter 2 are completed here. The main theorem in this chapter is Theorem D, which proves that the QC-scalar curvature on any 7-dimensional QC-Einstein manifold is constant. In the case $n = 1$, it is proved as a corollary of Theorem D that the corresponding to η vertical distribution is integrable. This result completes Theorem B.

Theorem D and Theorem 5.9 give a complete proof of the fact that any QC-Einstein manifold of an arbitrary dimension has a constant QC-scalar curvature.

This result places the author and his (bulgarian) co-authors among the classics of this subject.

In Theorem C the additional requirement for the QC-Einstein manifolds of dimension 7 is dropped out. The research method here is very interesting, as it involves the construction of a special linear connection $\tilde{\nabla}$ on the canonical vertical distribution V of the distribution H. With the aid of this linear connection the author proves the remarkable Theorem 11.3:

Any QC-manifold is QC-Einsteinian if and only if the linear connection $\tilde{\nabla}$ is flat.

In this way, Theorem C becomes complete and covers all cases. The second condition in Theorem C is completed as follows: If the QC-manifold M is with a negative QC-scalar curvature $Scal < 0$, then (M, h) turns out locally a negative 3-Sasakian manifold. If $Scal = 0$, then the Reeb fields of the QC-structure are again Killing fields with respect to the Riemannian metric h, and in this case they commute.

Chapter 4. Here the main theorems are Theorem E and Theorem F.

Let $\tilde{\eta} = \dfrac{1}{2h}\eta$ be a conformal deformation of the standard 1-form $\tilde{\eta}$ on the unit sphere S^{4n+3}, $n > 1$. If the 1-form η is with constant QC-scalar curvature, then in Theorem B it was proved that, up to a multiplicative constant η is a 1-form of the type $\phi^*(\tilde{\eta})$, where ϕ is a conformal QC-automorphism on the sphere. In the case $n = 1$, this statement was derived under the additional assumption that the vertical complement of η is an integrable distribution on the sphere. The additional assumption in Theorem E is eliminated and the Yamabe problem for S^7 is completely solved.

Evidence of thorough knowledge of the subject treated by the author is finding a connection between the Yamabe problem on S^7 and the inequality of Foland and Stein on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$.

The inequality of Foland and Stein for $\mathbf{G}(\mathbb{H})$ is the analogue of the Sobolev inequality for $\mathbb{R}^n$: There exists a constant $S > 0$, such that for any smooth function u with a compact support on $\mathbf{G}(\mathbb{H})$ the inequality (3.3) from the thesis is true. The author poses and solves the following natural problem:

Find the best constant $S > 0$ in the inequality of Foland and Stein and find the functions u, which transfer the inequality into equality.

Translations and dilatations in $\mathbf{G}(\mathbb{H})$ are transformations of the group, preserving the distribution H. These transformations have the common name QC-automorphisms of the group.

In Theorem F it is found the best constant S_2 for the L^2-inequality of Foland and Stein on the 7-dimensional Heisenberg group $\mathbf{G}(\mathbb{H})$ and the function v, giving a concrete extremal of this inequality. All non-negative extremals of the inequality are obtained from v through a conformal authomorphism of the group.

In Chapter 5 the author treats the L^2-inequality of Foland and Stein on the quaternionic group $\mathbf{G}(\mathbb{H})$ of an arbitrary dimension. Here, the main theorem is Theorem G, in which it is found the best constant S_2 and a concrete family of non-negative functions F, which turn the inequality into equality; all non-negative functions minimizing the inequality are obtained from the functions F through conformal automorphisms of the group. It was also found the Yamabe QC-constant λ for the standard QC-sphere S^{4n+3}.

6. Contributions and importance of the results obtained

The main contributions of the author have the character of justification, formulation and problem solving, which are an important part of the contemporary development of this topic in the international aspect. These contributions can be combined as follows:

- An explicit description of the conformal deformations of the standard QC-structure on the quaternionic Heisenberg group, transforming it again into a QC-Einsten structure.

- A complete description of the conformal transformations of the standard contact 1-form on the sphere S^{4n+3}, which transform it into a 1-form with constant QC-scalar curvature; this gives a complete solution to the Yamabe QC-problem on S^{4n+3}.

- The proof of the fact that the QC-scalar curvature of any QC-Einstein manifold is constant and the characterization of the QC-Einstein manifolds as 3-Sasakian manifolds.

- The finding of the best constant in the inequality of Foland and Stein and the description of the functions transforming the inequality into an equality.

- The finding of the Yamabe QC-constant for the standard QC-sphere S^{4n+3}.

7. Publications and citations

The present dissertation thesis meets the specific requirements of the Faculty of Mathematics and Informatics at Sofia University for the degree **Doctor of Science** in Mathematics. The author has presented two papers published in the following scientific journals:

[IMV]: Memoirs of the American Mathematical Society, IF 1,727, Q1;

[IMV]: Mathematical Research Letters, IF 0,716, Q2.

The above papers are joint papers with two co-authors. Undoubtedly, the author's contributions in the joint papers are at least equivalent to those of his co-authors. The submitted works are highly rated and rank the author among the leading authors in the subject.

The author has presented 13 impressive citations in articles from quartiles only Q1 and Q2.

8. The author's summary

The author's summary is done according to the rules and reflects accurately and completely the main results, obtained in the thesis.

9. Personal impressions of the applicant

I know the author of the thesis from his procedure for obtaining the scientific title "Associate Professor" during 2013, from which I have very good direct impressions.

Ivan Minchev has established himself as a highly educated young scientist with a very good prospect of developing and gaining international recognition.

Conclusion:

The present thesis contains scientific results that are an original contribution to mathematics and meet all the requirements of the Law on Development of Academic Staff in the Republic of Bulgaria (Low), the rules of application of the Law and the corresponding rules of Sofia University. The presented scientific papers and thesis completely meet the specific requirements of the Faculty of Mathematics and Informatics adopted in connection with the Rules of Sofia University of application of the Law.

The thesis shows that Ivan Minchev possesses deep theoretical knowledge and professional skills in scientific specialty **Geometry and Topology** and has obtained original and significant contributions with wide international response.

For the foregoing I am convinced of my positive assessment of the present work presented above and **strongly recommend** the Honorable Scientific Jury **to confer** the degree "**Doctor of Science**" on **Ivan Minchev Minchev** in higher education area 4. Natural sciences, mathematics and informatics; professional direction 4.5 Mathematics; scientific specialty: Geometry and Topology.

20.04.2020 г.,
Sofia

Referee:..
(Associate Prof. PhD Georgi Ganchev)