PEER REVIEW

by Corresponding Member of BAS Oleg Krastev Mushkarov,
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related to the procedure of application for appointment to the academic post **Associate Professor** in Sofia University "St. Kliment Ohridski", Faculty of Mathematics and Informatics, Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional field: 4.5 Mathematics; Scientific specialty: Geometry., announced in State Gazette (SG), No. 21, March 13, 2020 and the web sites of Sofia University (SU) and Faculty of Mathematics and Informatics(FMI) .

I am presenting my peer review related to this procedure as an assessor of an Academic Board, formed with Order No. РД 38-266/10.07.2020 of the Rector of Sofia University "St. Kliment Ohridski"Prof. D. Sc. Anastas Gerdzhikov. It is prepared according to the requirements of:

- The Law Act for Development of the Academic Staff in the Republic of Bulgaria (LADASRB);
- The Statutes for application of LADASRB;
- The Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in SU.

One applicant has presented the required documentation for participation in the procedure: Assistant Professor Aleksander Vladimirov Petkov, Ph.D. .

I. General description of the presented materials

1. The materials submitted by asst. prof A. Petkov for the competition meet the requirements of The Law Act for Development of the Academic Staff in the Republic of Bulgaria (LADA-SRB), The Statutes for application of LADASRB and The Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in SU. He has submitted 6 research papers, their abstracts and citations, lists of seminars, conferences and projects he has participated in, reference for fulfillment of the minimum requirements of p. 26 of LADASRB, and the other required documents in Ch. 107 of the Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in SU. All submitted documents are prepared correctly.

2. Personal data.

Asst. prof. A. Petkov was born on 17.12.1985 in the town of Montana. He graduated from Financial-Business School in 2004 and the same year he was admitted as a student in the Faculty of Mathematics and Informatics, Sofia University. He graduated SU in 2010 and got masters' degree, speciality "Dinamical Systems and

Geometry". From 2011 to 2014, he has been a Ph.D. student at the Department of Geometry of FMI and in 2014 he successfully defended his Ph.D. thesis "Reimannian and sub-Riemannian manifolds with structures". From February to July 2014 he has been an assistant and from July until now, an assistant professor in the same department. He had a post-doc position at the Institute of Mathematics, Faculty of Mathematics, University of Vienna in April 1-June 30 and October 1-December 31, 2017. He has been a visiting researcher at Department of Mathematics and the Institute of the Mathematical Sciences of the Americas at the University of Miami in January 14-March 14, 2018 and in September 2-December 15, 2019, respectively.

Asst. prof. A. Petkov has published a total of 8 research papers. They have been cited 24 times of which 10 are self-citations , and 13 are in journals with impact-factor. He has delivered talks at 7 scientific seminars in FMI, Institute of Mathematics and Informatics, Sofia and more than 30 international conferences in Bulgaria and abroad. A. Petkov has participated in 3 scientific projects with National Science Fund-Bulgaria, 1 with the Ministry of Education and Science, and 8 with Sofia University.

3. General Characteristics of the Application Materials.

Asst. prof. A. Petkov has submitted for consideration under the application procedure 6 research articles which have not been presented in association to his dissertation. All of them are devoted to important geometric and analytic problems for quaternionic contact(QC) manifolds. I would like to note that 5 of these articles have been published in journals with impact-factor (Compt. Rend. Acad. Bulg. Sci. - 2, Nonlinear Analysis-1, Journal of Geometric Analysis-1, Journal de Mathematiques Pures et Appliquees-1), and 1 in Annual of Sofia University. Two of them are written jointly with S. Ivanov and D. Vassilev, and 1 is joint with S. Ivanov. It is my opinion that in the joint publications the contribution of the applicant is equivalent to each of the co-authors. Asst. prof. A. Petkov has attached a list of 8 citations of the papers submitted for this procedure all of which are in journals with impact-factor. The submitted 6 research papers fulfill the minimum requirements of p. 26, LADASRB and The Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in SU. There is no data for plagiarism.

4. Education Activity.

Asst. prof. A. Petkov has an active academic career. He has taught lectures and held seminar classes in *Mathematics* for students from speciality *Geology*, *Analytic geometry* for students from specialities *Statistics* and *Mathematics and Informatics* (correspondence education). He has also held seminar classes in *Differential geometry* for students from speciality *Mathematics* , *Geometry* for students from specialities *Software engineering* and *Computer sciences*, *Linear algebra and Analytic geometry* for students from specialities *Chemistry* and *Physics*. I have no direct impressions of his pedagogical work, but in the opinion of his colleagues it is at a high level.

5. Analysis of the scientific achievements of the applicant.

The scientific interests of asst. prof. A. Petkov are in the areas of Differential geometry and Geometric analysis and more precisely in Geometry of hyper-Kähler manifolds with torsin and Quaternionic-contac (QC) manifolds, Sub-Riemannian geometry, String theory and related geometric structures, and Complex algebraic geometry.

As we have already noticed all 6 papers presented by A. Petkov are devoted to important geometric and analytic problems of current interest in the theory of QC manifolds. Those with numbers [1, 2, 3] are motivated by the classical results of Lichnerowicz and Obata, giving respectively a sharp lower bound for the first eigenvalue of the Laplace operator of a compact Riemannian manifold under a condition on the Ricci tensor and characterizing the case of equality. More precisely, by the theorem of Lichnerowicz (1958 г.) for any n-dimensional Riemannian manifold with Ricci curvature greater than the Ricci curvature of the n-dimensional sphere $\mathbb{S}^n(1)$ the first eigenvalue of the Laplace operator is greater than the first eigenvalue of the Laplace operator of the sphere. Later, Obata (1962 г.) considered the case of equality and proved that it is achieved only for the the sphere.

The study of the optimal lower bounds for the sub-Laplacians on manifolds has attracted a lot of interest in the past few years. In particular, the most studied example has been that of CR-manifolds. The first analog of the Lichnerowicz theorem was obtained by Greenleaf in 1985 г. under a coondition on the Ricci curvature and the torsion of the Tanaka–Webster connection and dimension ≥ 7. Later this result has been generalized by many authors - Li-Luk (2004 г.), Barletta (2007 г.), Chang-Chiu (2007 г.), Chang-Wu (2010 г.), Aribi-Dragomir-El Soufi (2014 г.), Baudoin-Wang (2014 г.) etc..

The main result in [3] (joint paper with S. Ivanov and D. Vassilev) is the following QC version of Lichnerowicz and Greenleaf theorems for QC manifolds. In this case the role of Tanaka–Webster connection is played by the Biquard connection.

Theorem 1. Let $(M, g, \mathbb{Q})$ be a compact QC-manifolds of dimension $4n+3 > 7$. If the the Ricci tensor and the torsion of the Biquard connection satisfy the inequality

$$Ric(X,X) + \frac{2(4n+5)}{2n+1}T_0(X,X) + \frac{6(2n^2+5n-1)}{(n-1)(2n+1)}U(X,X) \geq k_0 g(X,X)$$

for a positive constant k_0, then for every positive eigenvalue λ of the sub-Laplacian we have

$$\lambda \geq \frac{n}{n+2}k_0.$$

The proof of this theorem uses the classical idea of Lichnerowicz, and for this purpose a Bochner–Weitzenbock type formula for QC manifolds is proved. Let us note that the estimate in the theorem above is sharp since we have equality in the case of the 3-Sasakian sphere. The question for equality (the QC analog of Obata theorem) is solved completely for compact 3-Sasakian manifolds.

Theorem 2. Let $(M, g, \mathbb{Q})$ *be a compact QC Einstein manifold of dimension* $4n + 3 > 7$ *with*

$$Scal = 16n(n+2), Ric(X,Y) = \frac{1}{4n} Scalg(X,Y) = 4(n+2)g(X,Y).$$

The first eigenvalue of the sub-Laplacian is equal to $4n$ *if and only if* $(M, g, \mathbb{Q})$ *is QC equivalent to the* 3*-Sasakian sphere of dimension* $4n + 3 > 7$*. In particular, a compact* 3*-Sasakian manifold of dimension* > 7 *satisfies the above condition if and only if it is QC equivalent to the* 3*-Sasakian sphere*

The second main result in [3] is the Cordes type inequality between the horizontal Hessian and the sub-Laplacian of a function. As an application in nthe case of the quaternionic Heisenberg group it is obtained a sharp limit for the sub-Hessian of a smooth function by means of its sub-Laplacian. These results are important in solving the problem for $\mathcal{C}^{1,\delta}$- regularity of the p sub-Laplacian for p close to 2. The precise interval for these numbers p is also determined.

The papers [1] and [2] deal with the 7-dimensional case of QC manifolds which is not covered by the results in [3]. Here optimal lower bounds for the eigenvalues of the sub-Laplacian are obtained under stronger conditions than that in [3]. In [1] the additional requirement is the positivity of the P-function for any eigenfunction of the sub-Laplacian. We have to note that the motivation for introducing the so-called P–form, P–function and C–operator are the Paneitz operators in Riemannian and CR geometries. An Obata type theorem for compact 7-dimensional 3-Sasacian manifolds is also proved in [1]. It says that the lower bound is attained for the 7-dimensional 3-Sasacian sphere only.

In [2] the optimal estimate in [1] is obtained under conditions involving the Ricci tensor and the torsion of the Biquard connection and some of the covariant derivatives of the torsion. As a consequence a sharp estimate for the QC scalar curvature of a compact 7–dimensional QC manifold is proved in the extremal case under additional conditions on the torsion of the Biquard connection. The proof of the main Theorem 1.2 in [2] uses an appropriate vertical variant of Bochner's formula.

The main purpose of [5] is to obtain an entropy formula for the QC heat equation which is used to show the important fact that that the QC energy functional is monotone non-increasing under the same conditions as in [1] and [3] respectively in dimension 7 and greater than 7. This results are analogs of that obtained by Chang and Wu for CR manifolds. Other applications of this formula are obtained in [6]. The first one is an improvement of the estimates in [1, 2, 3] related to the so-called "essential positivity"of the C-operator introduced in [1]. The second application is a new uniform proof of the main results in [1] and [3].

The paper [4] is devoted to the solution of the Yamabe problem in QC geometry. Recall that the classical Yamabe problem says that if (M, g) is a compact Riemannian manifold of dimension ≥ 3, then there exists a metric of constant scalar curvature in the conformal class of g. The solution of this problem, due to the works of Yamabe, Trudinger, Aubin, and Schoen, is a milestone in the developement of the theory of nonlinear PDE's. In the complex case the analog of Yamabe problem is for strongly pseudoconvex CR manifolds. Here the Levi form plays the role of a metric,

that of a conformal metric is played by a contact 1-form which vanishes on the Levi distribution(pseudohermitian structure), and the scalar curvature is the scalar curvature of a pseudohermitian structure introduced independently by Webster and Tanaka in 1978. In these terms the CR-Yamabe problem was posed and solved in the general case by Jerison and Lee in 1987. The QC Yamabe problem is the quaternionic version of that for CR manifolds. In this case one seeks a conformal change of the canonical $\mathbb{R}^3$-valued contact 1-form whose Biquard connection has constant scalar curvature. This problem was solved by Wang in 2005 in the sub-critical case when the Yamabe constant of the QC manifolds is less than that of the canonical QC structure of the quaternionic Heisenberg group of the same dimension. The main result in [4] is the derived asymptotic expression of the QC Yamabe functional on a special class of "test"functions. It implies that the Yamabe constant of a compact QC manifold which is not locally spherical is strictly less than the Yamabe constant of the standard 3–Sasacian sphere. This together with the above mentioned result of Wang solves the Yamabe problem in the non-spherical case.

In conclusion, I would like to point out that to obtain the results in the presented papers A. Petkov has overcome a number of technical and conceptual difficulties. He has used practically the whole apparatus of the contemporary differential geometry and geometric analysis and some new ideas with potential applications in solving interesting open problems in sub-Riemannian geometry.

6. Critical remarks

I have no critical remarks on the presented materials of the applicant.

7. Personal impressions of the applicant

I have known A. Petkov since he was a Ph.D. student and I have excellent impressions of his mathematical skills and purely human qualities.

II. Conclusion.

The materials submitted by asst. prof. Ph.D. Alexander Vladimirov Petkov for the procedure demonstrate that he satisfies the requirements of The Law Act for Development of the Academic Staff in the Republic of Bulgaria (LADASRB), The Statutes for application of LADASRB and The Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in Sofia University for occupying the academic post "Associate Professor". There is no data for plagiarism. I assess very positively his research and pedagogical activity and recommend with conviction to the honorable jury to propose to the the competent body at the Faculty of Mathematics and Informatics, Sofia University "St. Kl. Ohridski"to elect Ass.Prof. Ph.D. Alexander Vladimirov Petkov as an "Associate Professor" in the Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional field: 4.5 Mathematics; Scientific specialty: Geometry.

September 4, 2020 Signed:

(Oleg Mushkarov)