R E V I E W

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of the dissertation **"On some diophantine equations and inequalities"**

of *Zhivko Hristov Petrov*

Order RD 38-255/30.05.2019 of the President of Sofia University "St. Kliment Ohridski" appoints me as a member of the Juri for the defence of the Ph.D. dissertation "On some diophantine equations and inequalities" of Zhivko Hristov Petrov in the professional field 4.5 Mathematics. The thesis pertains to the analytic number theory, studying the distribution of the prime integers by analytic tools. This is a classical area of research, which has evolved from the middle of XIX-th century and has ever been a crucial part of mathematics, both, with its results and the developed techniques for asymptotic description of functions. The problems, solved by the thesis under review can be considered as generalizations of Goldbach, Waring and Twin Primes Conjectures. By an extraordinary ingenuity, the proofs split in several steps and the known techniques are developed appropriately, in order to realize highly non-trivial original research ideas. The recollection of the historical background is correct, diverting and related precisely to the research problems under study.

The thesis consists of five chapters. The first one is an introduction. The second chapter collects some preliminary results and their modifications, which are used in the sequel. Any of the next three chapters reflects the content of an article. The bibliography consists of 88 references. About 80% of these are articles. Twenty three percents of the articles are published in the period 1900 - 1950, $38,5\%$ - in the second half of XX century and $38,5\%$ from 2000 until 2018. This shows that Zhivko Petrov has studied thoroughly, both, the classical and the contemporary achievements of the most prominent specialists. He has mastered quite a lot of techniques from the analytic number theory and developed several of them, in order to obtain prominent original results.

The third chapter is on the solvability of an equation with one prime and one almost prime variable. Throughout, let $[X]$ be the integral part of $X \in \mathbb{R}$, i.e., the largest integer $z \in \mathbb{Z}$ which does not exceed X. In 2009 Kumchev shows that for a real number $1 < c < \frac{16}{15}$ and any sufficiently large natural number N the equation $[p^c] + [m^c] = N$ has a solution for a prime integer p and a natural number m. Since c is close to 1, this result can be viewed as a generalization of the Binary Goldbach Conjecture, asserting that any even integer $N > 2$ decomposes into a sum of two prime numbers. Recall that a natural number m is almost prime of order $r \in \mathbb{N}$, if m has at most r prime factors, counted with their multiplicities. The Binary Goldbach Conjecture is open and the best known result is an article of Chen from 1973, showing that any sufficiently large even integer decomposes into a sum of a prime number p and an almost prime natural number m of order 2. Generalizing the results of Kumchev and Chen, the third chapter shows that for any real number $1 < c < \frac{29}{28}$ and any sufficiently large natural number N the equation $[p^c] + [m^c] = N$ has a solution for a prime integer p and an almost prime natural number m of order $\left[\frac{52}{29-28c}\right] + 1$. In order to outline the idea of the proof, let $\gamma := c^{-1}$, $P := 10^{-9}N^{\gamma}$ and $\alpha > 0$ be a real constant, which is to be specified later. One forms a sum Γ, whose positiveness implies the existence of a prime $p \in (P, 2P]$ and an almost prime $m \in \mathbb{N}$ of order $\left[\frac{\gamma}{\alpha}\right]$ with $[p^c] + [m^c] = N$. By the lower Rosser functions $\lambda^-(d)$ for the characteristic function ω of the prime integers and Chebishew's Theorem on the asymptotic of the sum $\theta(X)$ of the logarithms of the prime integers $p \leq X$, Γ is bounded below by $\Gamma_0 + \Sigma_0 - \Sigma_1$ for appropriate Γ_0, Σ_0, Σ_1. Towards a lower bound $\text{Const}\frac{N^{2\gamma-1}}{\log N}$ for Γ_0, derived by the means of Mertens formula and the properties of $\lambda^-(d)$ from Rosser's Sieve, one chooses $\alpha := \frac{29\gamma-28}{52} - \varepsilon_o$ for a sufficiently small real $\varepsilon_o > 0$. That provides an upper bound $\left[\frac{52}{29-28c}\right] + 1$ on the number of the prime divisors of m, counted with their multiplicities. The essential part of the argument establishes an upper bound $\text{Const}\frac{N^{2\gamma-1}}{(\log N)^2}$ on Σ_0 and Σ_1, which implies that $\Gamma \geq \text{Const}\frac{N^{2\gamma-1}}{\log N} > 0$ for a sufficiently large N and concludes the proof of the theorem. By the means of a Vaaler's Theorem on the asymptotic of the difference $\rho(t)$ of $\frac{1}{2}$ and the fractional part $\{t\}$ of $t \in \mathbb{R}$, the problem reduces to estimating exponential sums, depending on $[p^c]$. A Lemma of Vinogradov for the existence of a periodic function with bounded Fourier coefficients and a Theorem of van der Corput on exponential sums allow to replace $[p^c]$ by p^c and to bound the arising difference from above. Making use of Chebishev's Theorem on the asymptotic of $\theta(X)$ and Vaughan's identity for a sum, weighted by Mangoldt function, the resulting sums are split and bounded from above by van der Corput's Theorem, Rolle's Theorem and etc.

The fourth chapter shows that for an arbitrary real number $1 < c < \frac{17}{16}$ and a sufficiently large natural number N, the equation $[p_1^c] + [p_2^c] + [p_3^c] = N$ has a solution in prime integers p_1, p_2, p_3, for which $p_1 + 2, p_2 + 2, p_3 + 2$ are almost prime of order $\left[\frac{95}{17-16c}\right]$. This result can be viewed as a generalization of the Ternary Goldbach Conjecture that any odd integer $N > 5$ is a sum of three primes. This conjecture is proved by Helfgott in 2014. On the other hand, the main theorem of chapter four is an analogue of Twin Primes Conjecture, according to which there are infinitely many primes p, for which $p + 2$ is a prime. Peneva proves in 2000 that for sufficiently large natural numbers $N \equiv 3 (\operatorname{mod} 6)$ the equation $p_1 + p_2 + p_3 = N$ has a solution in prime integers p_1, p_2, p_3, for which $p_1 + 2$ and $p_2 + 2$ are almost prime of order 5, while $p_3 + 2$ is an almost prime of order 8. In 2017 Matomaki and Shao improve Peneva's result by showing that for a sufficiently large natural number $N \equiv 3 (\operatorname{mod} 6)$ the equation $p_1 + p_2 + p_3 = N$ has a solution in prime integers p_1, p_2, p_3, for which $p_1 + 2, p_2 + 2, p_3 + 2$ are almost prime of order 2. On the other hand, in 1995 Laporta and Tolev establish that for an arbitrary real number $1 < c < \frac{17}{16}$ and a sufficiently large natural number N the equation $[p_1^c] + [p_2^c] + [p_3^c] = N$ has a solution in prime integrs p_i. In order to verify that the solution can be chosen in such a way that $p_1 + 2, p_2 + 2, p_3 + 2$ are almost prime of order $\left[\frac{95}{17-16c}\right]$, Zhivko Petrov considers a sum Γ, whose positiveness implies the existence of a solution of $[p_1^c] + [p_2^c] + [p_3^c] = N$ in prime integers $p_1, p_2, p_3 \in (\mu X, X]$ for $X := N^{\frac{1}{c}}$ and some

$0 < \mu < 1$, such that p_1+2, p_2+2, p_3+2 are almost prime of order $\left[\frac{94{,}4}{17-16c}\right]$. After expressing Γ as an exponential sum, one applies the basic inequality of the vector sieve to Rosser functions for $\omega(d) = \frac{d}{\varphi(d)}$ and obtains an inequality $\Gamma \geq 3\Gamma_1 - 2\Gamma_4$ for two other sums Γ_1, Γ_4 of integrals of exponential functions. (Here $\varphi(d)$ is Euler's function, which equals the number of the relatively prime to d residues modulo d.) The aforementioned integrals split in two summands - Γ_1', Γ_4' in a neighborhood of 0 and Γ_1'', Γ_4'' outside this neighborhood in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In the neighborhood of zero, the integrals Γ_1' and Γ_4' are bounded below by Const$\frac{X^{3-c}}{(\log X)^3}$. Outside the neighborhood of 0, the integrals Γ_1'' and Γ_4'' are bounded above by Const$X^{3-c-\varepsilon}$ for a sufficiently small real number $\varepsilon > 0$. Both, Γ_i' and Γ_i'' are treated in a similar vein, as in an article of Tolev from 2017. The resulting estimate $\Gamma \geq \text{Const}\frac{X^{3-c}}{(\log(X)^3} > 0$ for sufficiently large natural number N justifies the existence of primes p_1, p_2, p_3 with almost prime p_1+2, p_2+2, p_3+2 of order $\left[\frac{95}{17-16c}\right]$, satisfying the equation $[p_1^c] + [p_2^c] + [p_3^c] = N$.

The last, fifth chapter studies diophantine inequalities in Piatetskii-Shapiro primes. These can be viewed as generalizations of Waring-Goldbach problem, estimating the minimal natural number $n(k) \in \mathbb{N}$, for which the equation $p_1^k + \ldots + p_{n(k)}^k = N$ has a solution in prime integers p_i for any sufficiently large natural number $N \equiv n(k) (\operatorname{mod} K(k))$. Here $K(k)$ is a function of the primary divisors p^s of k, for which $p-1$ divides k. The best known upper bound on $n(k)$ is $n(k) \leq (4k-2)\log k - (2\log 2 - 1)k - 3$, which is proved by Kumchev and Wooley in 2017. Concerning the minimal $n(c) \in \mathbb{N}$, for which the Diophantine inequality $\left|p_1^c + \ldots + p_{n(c)}^c - N\right| < \varepsilon$ with fixed real $c > 1$, $c \notin \mathbb{N}$, $\varepsilon > 0$ is solvable in primes p_i for sufficiently large natural numbers $N \in \mathbb{N}$, Piatetskii-Shapiro derives in 1952 an asymptotic upper bound $4c\log c$ on $n(c)$ for sufficiently large real c. Tolev studies the solvability of $|p_1^c + p_2^c + p_3^c - N| < \varepsilon(N)$ in prime integers p_1, p_2, p_3 for a fixed real $1 < c < \frac{15}{14}$, an explicit function $\varepsilon(N)$ of N and a sufficiently large natural number $N \in \mathbb{N}$. In 1992 he derives the existence of a solution for $\varepsilon(N) = N^{1-\frac{15}{14c}}(\log N)^9$, while in 2017 he shows the existence of primes p_1, p_2, p_3 with $|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E}$, for which p_1+2, p_2+2, p_3+2 are almost prime of order $\left[\frac{369}{180-168c}\right]$, for any arbitrarily large fixed constant $E > 0$. Let $\mathcal{P} = \{p_n\}_{n=1}^{\infty}$ be the sequence of the prime integers. Many problems in additive number theory are solved for thin sequence $S \subset \mathcal{P}$ of primes. These are the ones, for which the ratio of the number of the primes $p \leq X$ from S to the number of all primes $p \leq X$ tends to 0 for sufficiently large real X. In 1953 Piatetskii-Shapiro establishes that for $\frac{11}{12} < \gamma < 1$ the set $\mathcal{N}_\gamma := \left\{\left[m^{\frac{1}{\gamma}}\right] \,\middle|\, m \in \mathbb{N}\right\}$ has a thin intersection $\mathcal{N}_\gamma \cap \mathcal{P}$ with the set $\mathcal{P}$ of all prime integers. Moreover, he obtains an asymptotic formula $\frac{N^\gamma}{\log N}\left[1 + O\left(\frac{1}{\log N}\right)\right]$ for the number of the primes $p \leq N$ from $\mathcal{N}_\gamma \cap \mathcal{P}$. In 1992 Balog and Friedlander show the existence of prime integers $p_1, p_2, p_3 \in \mathcal{N}_\gamma \cap \mathcal{P}$ with $p_1 + p_2 + p_3 = N$ for any sufficiently large $N \in \mathbb{N}$, provided $\frac{20}{21} < \gamma < 1$. Making use of probabilistic methods, in 1986 Wirsing establishes the existence of a thin subset $S \subset \mathcal{P}$ with an upper bound Const$(X\log X)^{\frac{1}{3}}$ on the number $|S \cap (1, X]|$ of the elements of S, which do not exceed a sufficiently large X, such that $N = p_1 + p_2 + p_3$ is solvable in $p_1, p_2, p_3 \in S$ for a sufficiently large $N \in \mathbb{N}$. The set S from the aforementioned result is implicit. The fifth chapter of the thesis studies the diophantine inequalities $|p_1^c + \ldots + p_s^c - N| < (\log N)^{-1}$ in Piatetskii-Shapiro primes $p_i \in \mathcal{N}_\gamma \cap \mathcal{P}$ for sufficiently large natural numbers $N \in \mathbb{N}$. In order to apply a Hardy-Littlewood method for decreasing the number of the variables, one puts $s = 2t + 2u + 1$, introduces the sequence of real numbers $X_0 := \frac{N^{\frac{1}{c}}}{3u}$, $X_1 := N^{\frac{1}{c}}$, $X_j := \frac{1}{2} X_{j-1}^{1-\frac{1}{c}}$ for $2 \leq j \leq t$ and considers prime integers $p_1, \ldots, p_{2u+1} \in \mathcal{N}_\gamma \cap \left(\frac{X_0}{2}, X_0\right]$, $p_{2u+2j}, p_{2u+2j+1} \in \mathcal{N}_\gamma \cap \left(\frac{X_j}{2}, X_j\right]$. Towards an application of Davenport-Heilbronn's form of the circle method is considered a smooth kernel $K(x)$, supported by $[-1, 1]$ and a sum $R(N)$ over the prime solutions p_i of $\log N \, |p_1^c + \ldots + p_s^c - N| < 1$ with the aforementioned restrictions. The sum $R(N)$, depending on $K(\log N (p_1^c + \ldots + p_s^c - N))$ is designed in such a way that $R(N) > 0$ suffices for the solvability of $|p_1^c + \ldots + p_s^c - N| < (\log N)^{-1}$ in Piatetskii-Shapiro primes p_i. By Fourier inversion formula, the sum $R(N)$ is expressed as an integral of a product of sums $S(\theta, X_j)$, $0 \leq j \leq t$ of $\log p \exp(2\pi i \theta p^c)$ for $p \in \mathcal{N}_\gamma \cap \mathcal{P} \cap \left(\frac{X}{2}, X\right]$ over $\theta \in \mathbb{R}$. For a fixed sufficiently small real number $\delta = \delta(c, \gamma) > 0$, the integration set $\mathbb{R}$ splits in a large arc $\mathfrak{M} = (-X^{\gamma - c - \delta}, X^{\gamma - c - \delta})$, two finite small arcs $\mathfrak{m} = (-X^\delta, -X^{\gamma - c - \delta}) \cup (X^{\gamma - c - \delta}, X^\delta)$ and two infinite arcs $\mathfrak{m}_\infty = (-\infty, -X^\delta) \cup (X^\delta, +\infty)$. The existence of a solution $p_1, \ldots, p_s \in \mathcal{N}_\gamma \cap \mathcal{P}$ of $|p_1^c + \ldots + p_s^c - N| < (\log N)^{-1}$ follows from $R(N) \geq \mathrm{Const}\,\Xi$ for $\Xi := N^{-1} (\log N)^{-1} \left(X_1^2 \ldots X_t^2 N^{\frac{2u+1}{c}}\right)^\gamma > 0$. The integrals over $\mathfrak{m}_\infty$ and $\mathfrak{m}$ are bounded above by $\mathrm{Const}\,\Xi N^{-\frac{1}{c}}$, respectively, $\mathrm{Const}\,\Xi N^{-\frac{\delta}{c}}$. The lower bound $\mathrm{Const}\,\Xi$ for the integral over $\mathfrak{M}$ is obtained by replacing the exponential sums under consideration with appropriate integrals. In such a way is derived the main result of the fifth chapter, asserting that if $c \in (5, +\infty) \setminus \mathbb{N}$, $1 - (8c^2 + 12c + 12)^{-1} < \gamma < 1$ and $s \geq 4c\log c + \frac{4}{3}c + 10$, $s \in \mathbb{N}$ then for any sufficiently large natural number $N \in \mathbb{N}$ the inequality $|p_1^c + \ldots + p_s^c - N| < (\log N)^{-1}$ has a solution $p_1, \ldots, p_s \in \mathcal{N}_\gamma \cap \mathcal{P}$. The aforementioned sum $R(N)$ over $p_1, \ldots, p_s \in \mathcal{N}_\gamma \cap \mathcal{P} \cap \left(\frac{X}{2}, X\right]$ is used towards the solvability of the diophantine inequalities $|p_1^c + \ldots + p_s^c - N| < (\log N)^{-1}$ for $s = 3$ or $s = 4$ in Piatetskii-Shapiro primes p_i. The integrals over $\mathfrak{m}_\infty$ and $\mathfrak{m}$ are bounded above by $\mathrm{Const}(\log N)^{-1} \left(\frac{N}{2}\right)^{\frac{s\gamma - c - 1}{c}}$, respectively, $\mathrm{Const}(\log N)^{-1} \left(\frac{N}{2}\right)^{\frac{s\gamma - c - \delta}{c}}$, while the integral over $\mathfrak{M}$ is bounded below by $\mathrm{Const}(\log N)^{-1} \left(\frac{N}{2}\right)^{\frac{s\gamma - c}{c}}$. In such a way is established that for $\gamma < 1 < c$, $15(c-1) + 28(1-\gamma) < 1$ and a sufficiently large natural number $N \in \mathbb{N}$ there exist $p_1, p_2, p_3 \in \mathcal{N}_\gamma \cap \mathcal{P}$ with $|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-1}$. Similarly, if $\gamma < 1 < c$ and $8(c-1) + 21(1-\gamma) < 1$ then $|p_1^c + p_2^c + p_3^c + p_4^c - N| < (\log N)^{-1}$ has a solution $p_1, p_2, p_3, p_4 \in \mathcal{N}_\gamma \cap \mathcal{P}$ for a sufficiently large natural number $N \in \mathbb{N}$. The inequality $|p_1^c + p_2^c - N| < (\log N)^{-1}$ is not necessarily solvable in $p_1, p_2 \in \mathcal{N}_\gamma \cap \mathcal{P}$ for $\gamma < 1 < c$ sufficiently close to 1 and a sufficiently large natural number $N \in \mathbb{N}$. That motivates the study of the Lebesgue measure $|\mathcal{E}(Z)|$ of the set $\mathcal{E}(Z)$ of the natural numbers $N \in \left(\frac{Z}{2}, Z\right] \cap \mathbb{N}$, for which $|p_1^c + p_2^c - N| \geq (\log N)^{-1}$ for all $p_1, p_2 \in \mathcal{N}_\gamma \cap \mathcal{P}$. After embedding $\mathcal{E}(Z)$ in a set $\mathcal{E}_o(Z)$, over which the integral over the large arc $\mathfrak{M}$ is not bounded below by $\mathrm{Const}(\log Z)^{-1} \left(\frac{2Z}{3}\right)^{\frac{2\gamma - c}{c}}$, the Lebesgue measure of $\mathcal{E}_o(Z)$ is bounded above by $\mathrm{Const}\, Z \exp\left(-\left(\frac{\log\left(\frac{2Z}{3}\right)}{c}\right)^{\frac{1}{4}}\right)$.

The dissertation represents the results of three articles in highly reputable peer reviewed scientific journals - Monatshefte für Mathematik (Impact Factor 0.735 for 2017, which is the nearest available to the publication year 2018), Proceedings of the Steklov Institute of Mathematics (Impact Factor 0,623 for 2017) and Annual of Sofia University. The first two articles are joint with Angel Kumchev, respectively, Doychin Tolev, while the third one is independent. To the best of my knowledge, the contributions of the co-authors of the joint articles are commensurate. The article from the Proceedings of the Steklov Institute of Mathematics appeared in 2017 and is cited in the same year in Acta Arithmetica. This is a clear evidence for the advanced level of the scientific contributions of the dissertation.

Zhivko Petrov is not supposed to satisfy Decree 26 from February 13, 2019 on the Rules for application of the Law on Development of the Academic Staff of Republic Bulgaria, since he was accepted as a graduate student in 2016 at the Section of Mathematical Analysis of the Department of Mathematics and Informatics of Sofia University "St. Kliment Ohridski" and he is to defend his thesis under the regulations from 2016. Nevertheless, his scientific contributions exceed more than four times the 30 points, required by the aforementioned Decree for a defence of a Ph.D. thesis. According to Web of Science, the journal "Monatshefte für Mathematik" is in quartile Q2 in 2017, which is the nearest available to the publication year 2018 and earns him 60 points. A publication in the Proceedings of the Steklov Institute of Mathematics from 2017 is estimated by 45 points, since this journal is in quartile Q3 in 2017. The independent article of Zhivko Hristow Petrov from the Annual of Sofia University will be refereed in Mathematical Reviews under number MR3901574 and that earns him more 18 points. Thus, the total of 123 points exceeds considerably the required 30 points for a Ph.D. thesis to be admitted to a defence.

The results of the dissertation are reported at the conference Journees Arithmetiques in France in 2017, more than five times at the Department of Mathematics and Informatics of Sofia University "St. Kliment Ohridski", as well as at the Section of Algebra and Logic of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences.

The summary of the thesis is well organized and reflects truthfully the scientific contributions of Zhivko Hristov Petrov. It formulates the problems under consideration, recalls their history and outlines the ideas of the proofs. The publications on the dissertation are clearly specified, as well as the talks and the citation of the corresponding results.

The aforementioned is an evidence for a thorough theoretical knowledge and an ability of Zhivko Petrov to perform an independent scientific research in analytic number theory. He excels with high level, both, in research and teaching. His accuracy and diligence has earned him the respect of all colleagues. A testimony for the high reputation of Zhivko Petrov is his election as a member of the committee on the election of a Dean and a Faculty Council of the Department of Mathematics and Informatics of Sofia University "St. Kliment Ohridski".

The dissertation under review solves significant mathematical problems and is an original contribution to mathematics. Through his Ph.D. study, Zhivko Petrov has gained a profound knowledge and professional competence for performing an independent scientific research. The thesis satisfies the requirements of the Law on Development of the Academic Staff of Republic Bulgaria, the Rules for its application, as well as the terms and conditions for acquirement of academic degrees and occupation of academic positions of Sofia University "St. Kliment Ohridski" and the Department of Mathematics and Informatics of Sofia University "St. Kliment Ohridski". According to the aforementioned facts, I strongly recommend the award of the educational and scientific degree "Doctor of Philosophy" to Zhivko Hristow Petrov.

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