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The geometry of quaternionic-contact manifolds and the Yamabe problem

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Contents

Introduction

In the thesis we consider a variety of problems related to the theory of quaternionic-contact (QC) manifolds. The QC geometry was first introduced by O. Biquard [Biq] to describe a type of geometric structure that appears naturally at the boundary at infinity of the quaternionic hyperbolic space. In general, a QC structure on a real (4n+3)-dimensional manifold M is a codimension three distribution H (to be called contact or horizontal distribution) which is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$, with values in $\mathbb{R}^3$, such that the three 2-forms $d\eta_i|_H$ (the exterior derivatives of η_i, restricted to the contact distribution) are the fundamental 2-forms of some quaternionic structure on H (see also Definition 5.1 below). It is a fundamental theorem of Biquard [Biq] that a QC structure on a real analytic $(4n+3)$-dimensional manifold M is always the conformal infinity of a unique quaternionic-Kähler metric defined in a "small" $(4n+4)$-dimensional neighborhood of M. This theorem generalizes an earlier result of LeBrun [LeB82] stating that a real analytic conformal 3-manifold is always the conformal infinity of a self-dual Einstein metric. From this point of view, the QC geometry is a natural generalization of the classical concept of a conformal 3-dimensional Riemannian geometry to the higher dimensions of the type $4n + 3$.

Furthermore, the QC geometry provides a natural setting for certain Yamabe-type problem concerning the extremals and the best constant of a special L^2 Sobolev-type embedding on the quaternionic Heisenberg group known as the Folland-Stein embedding theorem [FS74]. Obtaining a solution to this problem on the Heisenberg group is one of our main goals in the theses. To explain this in some more details, let us consider a 1-form η, with values in $\mathbb{R}^3$, that defines a QC structure H. This form is not uniquely determined by the contact distribution H but it is rather determined only up to a conformal factor and the action of the group $SO(3)$ on $\mathbb{R}^3$. Therefore H is equipped with a conformal class $[g]$ of metrics and a 3-dimensional quaternionic bundle $Q \subset End(H)$ over M. The associated 2-sphere bundle $S^2(Q) \to M$ is called the twistor space of the QC-structure. The transformations preserving the QC structure, i.e., the transformations of the type $\bar{\eta} = \mu\Psi \cdot \eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries, are called *quaternionic-contact conformal (QC conformal) transformations*. If the function μ is constant, we have *quaternionic-contact homothetic (QC homothetic) transformations*. To each metric in the fixed conformal class $[g]$ on H, one can associate a linear connection preserving the QC structure [Biq] which we call Biquard connection. This connection is invariant under QC homothetic transformations, but changes in a non-trivial way under QC conformal transformations. The scalar curvature, $Scal$, with respect to the Biquard connection is one of the most important differential invariants in the QC geometry with a fixed metric tensor $g \in [g]$. In this setting, the quaternionic-contact Yamabe problem is the problem of finding all metrics $g \in [g]$ for which the associated scalar curvature $Scal$ is constant.

Already by the very appearance of the new concept of QC geometry, in the year 2000, it was clear that there exist infinitely many examples of such manifolds. The argument for this came from a paper of LeBrun [LeB91] who managed to proof, by using the deformation theory of complex manifolds, the existence of an infinite dimensional family of special complete quaternionic-Kähler metrics on the unite Ball B^{4n+4}. LeBrun observed that, if multiplied by a function that vanishes along the boundary sphere S^{4n+3} to order two, each of his special metrics on the ball extends smoothly across the boundary sphere S^{4n+3} where its rank drops to four. It was discovered later by Biquard [Biq] that the arising structure on S^{4n+3} is essentially a QC structure and therefore, on the sphere, we have infinitely many (globally defined) such structures. Clearly, the whole construction is very non-explicit and the argument of LeBrun does not help much for the construction of any explicit examples of QC structures. In fact, the number of the explicitly known examples of QC manifolds remains so far very restricted. There is essentially only one generic method for obtaining such structures explicitly. It is based on the existence of a certain very special type of Riemannian manifolds, the so called 3-Sasaki-like spaces. These are Riemannian manifolds that admit a special triple R_1, R_2, R_3 of Killing vector fields, subject to some additional requirements (see Chapter **??** and the references therein for more details), which carry a natural QC structure defined by the orthogonal complement of the triple R_1, R_2, R_3. So far, there are no explicit examples of QC structures (not even locally) for which it is proven that they can not be generated by the above construction. Investigating the relationship between the 3-Sasakian spaces and the QC geometry will be one of our main tasks here.

The first chapter of the thesis (with title "Preliminaries") is intended to be an introduction to the subject, where we explain our motivation for studying quaternionic-contact geometry and recall the main results known in this area. The rest of the thesis is built on original material most of which was published already in separate papers.

The core of the thesis is Chapter 2; it is based on results published in [IMV14]. Here we develop the basic concepts in the QC geometry and proof a number of important results upon which the rest of the thesis is build. In theorems A and B of this chapter, we obtain a partial solution to the QC Yamabe problem on the quaternionic Heisenberg group. Theorem C presents our first result relating the Riemannian geometry of 3-Sasakian manifolds to the geometry of QC Einstein spaces.

In Chapter 3, we proceed with the investigations, started in Chapter 2, concerning the geometry of QC Einstein spaces. In Theorem 5.9 (Chapter 2), we show that the QC scalar curvature of a QC Einstein space of dimension at least eleven is constant but we leave the seven dimensional case open. In Theorem D (Chapter 3), we extend this result to cover also the more difficult 7 dimensional case. Furthermore, in this chapter, we show that, depending on the value of the QC scalar curvature, the QC Einstein spaces are "essentially" bundles over quaternionic-Kähler or hyper-Kähler manifolds. The results presented here are published in [**IMV16**].

In Chapter 4, we use the techniques developed in Chapter 2 to obtain a complete solution to the QC Yamabe problem on the seven dimensional quaternionic Heisenberg group (theorems E and F). The results here are published in [**IMV10**].

In Chapter 5, we determine the best (optimal) constant in the L^2 Folland-Stein inequality (Theorem G) on the quaternionic Heisenberg group (in all dimensions) and the non-negative extremal functions, i.e., the functions for which equality holds. The argument presented here is purely analytical. In this respect, even though the QC Yamabe functional is involved, the QC scalar curvature is used in the proof without much geometric meaning. Rather, it is the conformal sub-laplacian that plays a central role and the QC scalar curvature appears as a constant determined by the Cayley transform and the left-invariant sub-laplacian on the quaternionic Heisenberg group. The method employed here does not give all solutions of the QC Yamabe equation on the quaternionic-contact sphere but only these that realize the infimum of the QC Yamabe functional. Therefore, if considering the seven dimensional case, the result presented here is weaker than Theorem E of Chapter 3. All results here are published in [**IMV12**].

Preliminaries

1. Quaternionic-Kähler geometry

1.1. Quaternions. The algebra $\mathbb{H}$ of quaternions is by definition the vector space $\mathbb{R}^4$ endowed with a multiplication operation $(z, w) \mapsto zw$ which is associative, satisfies the left and right distributivity axioms, and for which the element

$$\mathbf{1} \stackrel{def}{=} (1, 0, 0, 0)$$

is the neutral element. Using the notation

$$\mathbf{i} = (0, 1, 0, 0), \qquad \mathbf{j} = (0, 0, 1, 0), \qquad \mathbf{k} = (0, 0, 0, 1),$$

the multiplication of the basis elements is defined by the following list of identities, called also the quaternionic identities:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}. \tag{1}$$

The algebra of quaternions is a division ring, that is, every nonzero element in $\mathbb{H}$ has an inverse. To see this, consider the conjugation $z \mapsto \bar{z}$ in $\mathbb{H}$: If

$$z = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \qquad a, b, c, d \in \mathbb{R},$$

then, by definition,

$$\bar{z} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}. \tag{2}$$

A simple computation shows that

$$\bar{z}z = a^2 + b^2 + c^2 + d^2 = |z|^2$$

and that $\overline{zw} = \overline{w}\,\overline{z}$. Therefore, the inverse element z^{-1} of a non-zero $z \in H$ is explicitly given by

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

1.2. Quaternionic structure on a vector space. Consider a real vector space V and a 3-dimensional subspace Q of the algebra of all real endomorphisms of V, $Q \subset End(V)$. We say that Q is a *quaternionic structure* on V, if there exists a basis I_1, I_2, I_3 of Q that satisfies the quaternionic identities

$$I_1^2 = I_2^2 = I_3^2 = -id, \qquad I_1 I_2 = -I_2 I_1 = I_3. \tag{3}$$

Note that if Q is a quaternionic structure on V then on Q we have canonically induced scalar product $\langle ., . \rangle$ and orientation. Indeed, the elements $I \in Q$ of unit length are precisely those that satisfy the equation $I^2 = -id$, and two elements $I, J \in Q$ are orthogonal with respect to $\langle ., . \rangle$ if and only if $IJ = -JI$.

Furthermore, it follows that a triple J_1, J_2, J_3 of elements of Q satisfies the quaternionic identities

$$J_1^2 = J_2^2 = J_3^2 = -id, \qquad J_1 J_2 = -J_2 J_1 = J_3,$$

if and only if the 3×3 matrix, obtained by expressing J_1, J_2, J_3 with respect to the initial basis I_1, I_2, I_3, is an element of the group $SO(3)$.

1.3. The group Sp(1). Observe that, since $\mathbb{H}$ is a non-commutative algebra, we must distinguish between left and right modules (vector spaces) over $\mathbb{H}$. Each finite dimensional left or right $\mathbb{H}$-vector space is, of course, isomorphic to one of the coordinate spaces $\mathbb{H}^n$ with its natural left or right multiplication operation.

We shell fix (once and for all) a real vector space isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$. Then, the right $\mathbb{H}$-multiplication on $\mathbb{H}^n$ defines a natural quaternionic structure $Q = span\{I_1, I_2, I_3\}$ on $\mathbb{R}^{4n}$ with $I_1(z) = z\bar{i},\ I_2(z) = z\bar{j},\ I_3(z) = z\bar{k}$ for all $z \in \mathbb{H}^n$. Obviously, Q is then a 3-dimensional Lie subalgebra of the Lie algebra $End(\mathbb{R}^{4n})$, isomorphic to the classical Lie algebra $so(3) = sp(1)$.

The unique connected Lie subgroup $G \subset GL(4n, \mathbb{R})$ with Lie algebra Q is explicitly given by

$$G = \{a_0 id + a_1 I_1 + a_2 I_2 + a_3 I_3 \mid a_s \in \mathbb{R},\ a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\}$$

Since, obviously, G is a simply-connected Lie group, it is necessarily isomorphic to the classical group $Sp(1)$. In the sequel, we shell often identify these two groups without any further comment.

Typically one introduces a quaternionic valued, positive definite Hermitian scalar product on $\mathbb{H}^n$ by the formula $\langle \bar{z}, w \rangle_{\mathbb{H}} = \sum_s \bar{z}_s w_s,\ z, w \in \mathbb{H}^n$. The real part of the latter is just the standard real scalar product in $\mathbb{R}^{4n}$, i.e. $\langle z, w \rangle_{\mathbb{R}} = Re(\langle \bar{z}, w \rangle_{\mathbb{H}})$. Since $Sp(1)$ clearly preserves the Hermitian form $\langle \bar{z}, w \rangle_{\mathbb{H}}$, we have $Sp(1) \subset SO(4n)$.

1.4. The groups GL(n,$\mathbb{H}$) and Sp(n). By definition, $GL(n, \mathbb{H})$ is the group of the non-degenerate quaternionic $n \times n$ matrices. Clearly, $GL(n, \mathbb{H})$ acts from the left on $\mathbb{H}^n$ and, by this action, the elements of $GL(n, \mathbb{H})$ represent the endomorphisms of $\mathbb{H}^n$ that commute with the right multiplication with $\mathbb{H}$. By the real isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$ which we assume fixed (cf. section 1.3), we can identify the group $GL(n, \mathbb{H})$ with a certain subgroup of $GL(4n, \mathbb{R})$ by

$$GL(n, \mathbb{H}) = \{A \in GL(4n, \mathbb{R}) \mid AJ = JA, J \in Q\}.$$

Take P to be the stabilizer in $GL(n, \mathbb{H})$ of the Hermitian form $\langle \bar{z}, w \rangle_{\mathbb{H}}$, i.e.

$$P = \{A \in GL(n, \mathbb{H}) \mid \bar{A}^t A = E\} = \{B \in SO(4n) \mid BJ = JB, \ J \in Q\}.$$

Then, clearly, the Lie group P is isomorphic to the classical group $Sp(n)$ and in the sequel we shell identify these two. Let us remark explicitly here that the notation which we have just introduced brings a certain ambiguity in the case $n = 1$. In this case, the just defined group P and the previously defined group G (in section 1.3) are both identified with the same classical group $Sp(1)$ but, in fact, P and G are two different subgroups of $SO(4)$. Despite the ambiguity, this is a convenient notation that has been adopted by many authors in the area and therefore, we shell use it as well. Usually, it is quite clear from the context which of the two different copies of $Sp(1)$ in $SO(4)$ we are working with at any particular moment.

Following the above notation, the product groups $Sp(n)Sp(1) \cong Sp(n) \times Sp(1)/\mathbb{Z}_2$ and $GL(n, \mathbb{H})Sp(1)$ can be described as a subgroups of $GL(4n, \mathbb{R})$ as follows:

$$Sp(n)Sp(1) = \{A \in SO(4n) \mid A^{-1}JA \in Q, \ J \in Q\}$$

$$GL(n, \mathbb{H})Sp(1) = \{A \in GL(4n, \mathbb{R}) \mid A^{-1}JA \in Q, \ J \in Q\}.$$

1.5. Riemannian holonomy. Let (M, g) be a connected Riemannian m-manifold, and let $p \in M$ be a chosen base point. The holonomy group of (M, g) at p is the subgroup of $End(T_pM)$ consisting of those transformations that are induced by parallel transport around piecewise-smooth loops based at p. The restricted holonomy group is similarly defined, using only loops representing $1 \in \pi_1(M, p)$. The latter is automatically a connected Lie group and may be identified with a Lie subgroup of $SO(m)$ by choosing an orthogonal frame for T_pM. Changing the base point only changes the subgroup by conjugation.

Excluding Riemannian products and symmetric spaces, very few subgroups of $SO(m)$ can be restricted holonomy groups, as was first pointed by Berger [Ber]. In fact, the full list is as follows: $SO(m)$, $U(m/2)$, $SU(m/2)$, $Sp(m/4)Sp(1)$ (in dimension $m \geq 8$); G_2 (in dimension 7); and $Spin(7)$ (in dimension 8).

1.6. Quaternionic-Kähler manifolds. A Riemannian manifold (M, g) of dimension $m = 4n \geq 8$ is called Quaternionic-Kähler (QK) if its group of holonomy is contained in $Sp(n)Sp(1)$. Equivalently, (M, g) is a QK manifold if there exists a pointwise quaternionic structure Q (cf. section 1.2) on the tangent bundle of M such that $\nabla Q \subset Q$ holds everywhere on M, with ∇ being the Levi-Civita connection of g. If for a given QK manifold, $F \to M$ denotes the principle $Sp(n)Sp(1)$-bundle generated by parallel transport of an arbitrary orthonormal frame, then Q may be described as the vector bundle associated to F, corresponding to the adjoint representation of the group $Sp(n)Sp(1)$ on $sp(1)$.

The above definition of a QK manifold explicitly excludes the 4-dimensional case. Indeed, since $SO(4) = Sp(1)Sp(1)$, nothing interesting can generally be said about 4-manifolds with this holonomy group. The proper definition here is: A Riemannian 4-manifold is said to be Quaternionic-Kähler if it is Einstein and half-conformally flat. Recall that a Riemannian 4-manifold is called half-conformally flat if there exists an orientation with respect to which the conformal curvature satisfies $*W = W$ (or alternatively $*W = -W$), where $*$ is the Hodge star. Historically, the four dimensional case was considered first (cf. [**Pen**, **AHS**]) and the results achieved there gave the motivation for introducing the concept of the higher dimensional QK manifolds.

2. Twistor construction

Let (M, g) be a QK manifold of dimension $4n \geq 4$ and take $F \to M$ to be the corresponding principle bundle over M with structure group $Sp(n)Sp(1)$. Consiser the stabilizer $Sp(n)U(1) \subset Sp(n)Sp(1)$ of the fixed endomorphism I_1 of $\mathbb{R}^{4n}$ (as defined in 1.3) and let Z be the quotient bundle $F/(Sp(n)U(1))$. Then, $\pi : Z \to M$ is a 2-sphere bundle over M and each element I of Z corresponds to an orthogonal complex structure (to be denoted identically)

$$I : T_pM \to T_pM, \quad I^2 = -1, \quad g(IX, IY) = g(X, Y),$$

where $p = \pi(I)$ and $X, Y \in T_pM$. Each fiber $Z_p = \pi^{-1}(p)$ is topologically a 2-sphere that can be described explicitly as

$$Z_p = \{I \in Q_p \mid I^2 = -id\}.$$

Using the Levi-Civita connection ∇ of the Riemannian metric g we can split the tangent bundle of Z into horizontal and vertical parts, $TZ = D \oplus V$; the vertical part V is the kernel of the differential π_* of the projection map $\pi : Z \to M$. Since $\pi_* : D_I \to T_pM$ is an isomorphism of real vector spaces, we can lift each element $I \in Z_p$ to be an endomorphism $J'_I : D_I \to D_I$, $(J'_I)^2 = -1$, so that $D \subset TZ$ becomes a complex vector bundle, where the multiplication with $\sqrt{-1}$ is given by J'. On the other hand the fibers of π are oriented metric 2-spheres and so may be considered as Riemann surfaces. Therefore, the vertical tangent space $V = \ker \pi_*$ carries also an endomorphism $J'' : V \to V$ with $(J'')^2 = -1$. This allows us to define an almost complex structure J on the whole of $TZ = D \oplus V$ by setting $J = J' \oplus J''$. It was discovered independently by Salamon [**Sal1**] and Bérard-Bergery [**Brd**] that the almost complex structure J is always integrable, i.e, that its Nijenhuis tensor vanishes and thus, by the Newlander-Nirenberg theorem, Z is in fact a complex manifold.

Furthermore, $D \subset TZ$ is a holomorphic subbundle and the projection $TZ \to TZ/D$ gives a holomorphic line-bundle-valued 1-form $\Theta \in \Gamma(Z, \Omega^1(\mathcal{L}))$, $\mathcal{L} := TZ/D$, which satisfies $\Theta \wedge (d\Theta)^n \neq 0$, i.e., Θ is a holomorphic contact structure on Z. The map $\sigma : Z \to Z$, given by $I \mapsto -I$ and corresponding to the antipodal map on each 2-sphere $\pi^{-1}(p)$, is an antiholomorphic involution ($\sigma^2 = 1$) without fixed points. By definition, the twistor space of the QK manifold (M, g) is given by the triple (Z, Θ, σ).

3. Inverse twistor construction

It is essential that the above twistor construction is actually invertible [**LeB89**, **PP**, **BE**]. Indeed, let (Z, Θ, σ) be a triple, where Z is a $(2n+1)$-dimensional complex manifold, Θ is a holomorphic contact 1-form on Z that takes values in some holomorphic line bundle $\mathcal{L}$ over Z, and $\sigma : Z \to Z$ is a fixed-point-free antiholomorphic involution compatible with the contact structure Θ. Following [**LeB91**], we define M^c to be the set of all genus zero compact complex curves $C \subset Z$ which have normal bundle isomorphic to the bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. Here $\mathcal{O}(k) \to \mathbb{CP}_1,\ k \in \mathbb{Z}$, is the line bundle of Chern class k. Since the group $H^1(\mathbb{CP}_1, \mathcal{O}(1) \otimes \mathbb{C}^{2n})$ vanishes, it follows, by a theorem of Kodaira [**Kod**], that the set M^c (if not empty) must be a $4n$-dimensional complex manifold with tangent space at any point $C \in M^c$ given by $H^0(\mathbb{CP}_1, \mathcal{O}(1) \otimes \mathbb{C}^{2n}) \cong \mathbb{C}^{4n}$. The subset $M \subset M^c$ of all $C \in M^c$ that are σ-invariant is a real analytic manifold which sits in M^c as a real slice. By Proposition 1 in [**LeB91**], the subset $S^c \subset M^c$ consisting of those $C \in M^c$ that are tangent to the contact distribution $D = \ker(\Theta)$ is (if not empty) a non-singular, complex hypersurface in M^c. Moreover, the set S of all σ-invariant $C \in S^c$ is a real slice of S^c, and it is a smooth close hypersurface in M. The real manifold $M - S$ carries a natural pseudo Riemannian metric of holonomy $Sp(n-l)Sp(1)$, $0 \leq l \leq n$, with non-vanishing scalar curvature (cf. Theorem 1.3 in [**LeB89**]) and twistor space given by the triple (Z, Θ, σ).

This inverse construction is also unique in the following sense: If (Z, Θ, σ) is the twistor space of some QK-manifold N, then N is naturally isometric to an open subset of $M - S$. Furthermore, the germ of the geometry at any point $p \in M - S$ determines the germ of Z along the corresponding curve C_p up to a biholomorphism.

4. Geometry of the boundary surface S

It was observed by Biquard [**Biq**] that the hypersurface $S \subset M$ (we are using the notation form the previous section 3), which is a $(4n-1)$-dimensional real analytic submanifold, carries a natural geometrical structure which Biquard introduced as quaternionic-contact (QC) geometry. This structure can be described in the following way: Take any point $p \in M^c$ and denote by C_p the corresponding compact complex curve in Z. By assumption $C_p \cong \mathbb{CP}_1$ (biholomorphic equivalence) and the normal bundle $N_p := TZ/TC_p$ over C_p is isomorphic to $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. The $\mathcal{L}$-valued contact form Θ determines an isomorphism

$$\Theta \wedge d\Theta^n : \Lambda^{2n+1}(TZ) \to \mathcal{L}^{n+1}. \tag{4}$$

If restricting only to the curve C_p, we have

$$TZ|_{C_p} \cong TC_p \oplus N_p$$

and therefore, via the isomorphism (4),

$$\Lambda^{2n+1}(TZ)|_{C_p} \cong TC_p \otimes \Lambda^{2n} N_p.$$

Since $TC_p \cong \mathcal{O}(2)$ and $\Lambda^{2n} N_p \cong \Lambda^{2n}\mathcal{O}(1) \cong \mathcal{O}(2n)$, we obtain

$$(\mathcal{L}|_{C_p})^{n+1} \cong \mathcal{O}(2n+2),$$

and thus $\mathcal{L}|_{C_p} \cong \mathcal{O}(2)$. Consequently, the bundle $(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \cong \mathcal{O}(-2) \otimes \mathcal{O}(2)$ is trivial over each separate curve C_p. But can one find a uniform trivialization that works for all the curves C_p, $p \in M^c$ simultaneously? To answer this, denote by $\mathbb{L}_p$ the 1-dimensional space of holomorphic sections of the bundle $(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \to C_p$. If $\mathbb{L} = \cup_{p \in M^c} \mathbb{L}_p$, then $\mathbb{L}$ is a holomorphic line bundle over M^c and the restriction of the contact form Θ to the tangent spaces TC_p determines a holomorphic section θ of $\mathbb{L}$, i.e., there is indeed a uniform trivialization for the bundles $(T^*C_p) \otimes (\mathcal{L}|_{C_p}) \to C_p$ only if we consider domains in M^c where θ is non-vanishing. Clearly, the hypersurface $S^c \subset M^c$ is precisely the zero locus of θ which is known to be non-degenerate by Proposition 1 in [LeB91].

In what follows, we shell often need to assume that there exists a square root $\mathcal{L}^{\frac{1}{2}}$ of the bundle $\mathcal{L}$. Since $\mathcal{L}|_{C_p} \cong \mathcal{O}(2) \cong \mathcal{O}(1)^2$, this assumption is certainly true if restricting to sufficiently small open subsets of M^c, although, even then, there will exist different possible choices for the square root $\mathcal{L}^{\frac{1}{2}}$. The final conclusions from the considerations below, however, will not depend on the local choices for $\mathcal{L}^{\frac{1}{2}}$ that we could make, and thus will remain true also without the assumption about the global existence of $\mathcal{L}^{\frac{1}{2}}$.

Consider the 2-dimensional holomorphic vector bundle $\mathcal{H}$ over M^c with fibers $\mathcal{H}_p = H^0(C_p, \mathcal{L}^{\frac{1}{2}})$ (note that $H^0(C_p, \mathcal{L}^{\frac{1}{2}}) \cong H^0(\mathbb{CP}_1, \mathcal{O}(1)) \cong \mathbb{C}^2$ for any fixed $p \in M^c$). The Wronskian

$$W : \Lambda^2(\mathcal{H}_p) \to H^0(C_p, T^*C_p \otimes \mathcal{L}) \cong \mathbb{L}_p$$
$$u \wedge v \mapsto u \otimes dv - v \otimes du$$

defines a non-degenerate $\mathbb{L}$-valued 2-form on the bundle $\mathcal{H} \to M^c$, i.e., it defines an $Sp(1, \mathbb{C})$-structure on $\mathcal{H}$, and thus also an $SO(3, \mathbb{C})$-structure on the bundle

$Sym^2(\mathcal{H}) \to M^c$ of symmetric 2-tensors on $\mathcal{H}$. Since

$$Sym^2(\mathcal{H}_p) = Sym^2\Big(H^0(C_p, \mathcal{L}^{\frac{1}{2}})\Big) = H^0(C_p, \mathcal{L}),$$

the bundle $Sym^2(\mathcal{H})$ does not depend on the choice of the square root $\mathcal{L}^{\frac{1}{2}}$ and therefore, it is well defined globally over M^c.

Now let $p \in S^c = \theta^{-1}(0)$. The tangent space T_pS^c is given by $\ker(d\theta) \subset T_pM^c$. Since $\Theta(TC_p) = 0$ the contact form Θ induces a linear map $\eta : H^0(C_p, N_p) \to H^0(C_p, \mathcal{L})$, i.e., a linear map $\eta : T_pM^c \to Sym^2(\mathcal{H}_p)$. If restricting to $\ker(d\theta)$ we obtain a rank three linear map $\eta : T_pS^c \to Sym^2(\mathcal{H}_p)$. Let $H^c_p \subset T_pS^c$ be the kernel of this map. Then H^c is a holomorphic codimension three distribution on the hypersurface S^c. If we take a local $SO(3,\mathbb{C})$ equivariant trivialization $Sym^2(\mathcal{H}) \cong \mathbb{C}^3$ then we obtain a triple η_1, η_2, η_3 of local one-forms on S^c by the composition

$$T_pS^c \to Sym^2(\mathcal{H}_p) \to \mathbb{C}^3.$$

It follows from the considerations in [**Biq**], III.2.C that there exist a non-degenerate holomorphic symmetric 2-tensor g on H^c and a triple of holomorphic endomorphism I_1, I_2, I_3 of H^c, satisfying the quaternionic identities such that $d\eta_s(X,Y) = 2g(I_sX, Y),\ X, Y \in H^c,\ s = 1,2,3$. Thus, if we go to the real slice $S \subset S^c$ (through the antiholomorphic involution σ), we obtain that (S, H) is an analytic quaternionic-contact manifold in the sense of definition 5.1.

5. QC structures

5.1. Definition and basic properties. Let M be a $4n+3$-dimensional $(n \geq 2)$ manifold and consider any codimension three distribution H on M. Denote by L the three dimensional quotient bundle $L = TM/H$, and by L^* its dual. For a fixed point $p \in M$, each element $\eta \in L^*_p$ is, by definition, a linear map $L_p \to \mathbb{R}$. If we take the composition of η with the projection $T_pM \to L_p$, we obtain an element of T^*_pM. Therefore, we have an identification

$$L^*_p \cong \{\lambda \in T^*_pM \ :\ \lambda|_H = 0\} \tag{5}$$

and the sections of L^* are simply the 1-forms on M that are vanishing along the distribution H.

Definition 5.1. *A quaternionic-contact structure (QC structure) on a $(4n+3)$-dimensional $(n \geq 2)$ manifold M is a codimension three distribution H with the property that locally, around each point $p \in M$, there exist:*

i) sections η_1, η_2, η_3 of L^*;

ii) sections I_1, I_2, I_3 of the bundle $End(H)$ satisfying the quaternionic identities

$$I_1^2 = I_2^2 = I_3^2 = -id_H \quad I_1I_2 = -I_2I_1 = I_3;$$

iii) a symmetric and positive definite section g of the bundle $H^* \otimes H^*$,
so that all these satisfy the identities

$$dη_s(X,Y) = 2g(I_sX,Y), \;\; s=1,2,3. \tag{6}$$

Any ordered list

$$(\eta_1,\eta_2,\eta_3,I_1,I_2,I_3,g) \tag{7}$$

of local sections with the same type and properties as in the above Definition 5.1 will be called *admissible set* for the QC structure H on M. The 1-form $\eta = (\eta_1,\eta_2,\eta_3)$ with values in $\mathbb{R}^3$ will be called simply – *contact form*. Neither the contact form η nor the admissible set $(\eta_1,\eta_2,\eta_3,I_1,I_2,I_3,g)$ (which we shall often abbreviate to (η_s,I_s,g), presuming that s is an index running from 1 to 3) are uniquely determined by the QC structure H. In fact, we have the following

Lemma 5.2. *Let (η_s,I_s,g) and (η'_s,I'_s,g') be two admissible sets for the same QC structure H on M, defined on some open set $U \subset M$. Then, there exists a positive function $f : U \to R$ and a matrix-valued function $\mathcal{A} = (a_{ij}) : U \to SO(3)$ so that*

$$(I'_1,I'_2,I'_3) = (I_1,I_2,I_3)\mathcal{A}, \quad (\eta'_1,\eta'_2,\eta'_3) = f(\eta_1,\eta_2,\eta_3)\mathcal{A}, \quad g' = f\,g.$$

Proof. By assumption, $H = \cap_{i=1}^3 \eta_i = \cap_{i=1}^3 \eta'_i$. Thus there exists a matrix-valued function $\mathcal{B} = (b_{ij}) : U \to GL(3)$ with $\eta'_s = \sum_{t=1}^3 b_{st}\eta_t, \; s=1,2,3$. Taking the exterior derivative of the above equations we obtain

$$(d\eta'_s)|_H = \sum_t b_{st}(d\eta_t)|_H. \tag{8}$$

Let us fix a symmetric and positive definite section h of the bundle $H^* \otimes H^*$ which we will use as a "background" metric on H. With respect to this metric, consider the restrictions of the 2-forms $(d\eta'_s)|_H$ to H as endomorphisms of H, i.e., sections of the bundle $End(H) = H^* \otimes H$. This identification depends on the choice of h. However, it is easy to see that the composition of two endomorphisms of the form $((d\eta'_s)|_H)^{-1} \circ (d\eta'_t)|_H$, is an endomorphism independent of the choice of h. For (i,j,k) a cyclic permutation of $(1,2,3)$ and $h = g'$ we have

$$((d\eta'_j)|_H)^{-1} \circ (d\eta'_i)|_H = I'_k. \tag{9}$$

The above equation holds for any choice of the "background" metric h on H, in particular, also for $h = g$. Using 8, we conclude that

$$I'_k \;=\; ((d\eta'_j)|_H)^{-1} \circ (d\eta'_i)|_H \;\in\; \mathrm{span}_{\mathbb{R}}\,\{id_H, I_1, I_2, I_3\}.$$

Note that $\mathrm{span}_{\mathbb{R}}\,\{id_H,I_1,I_2,I_3\} \subset End(H)$ is an algebra with respect to the usual composition of endomorphisms, which is isomorphic to the algebra of the quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\,\{1,i,j,k\}$. If an element of $\mathbb{H}$ has square -1 then this element belongs to $Im(\mathbb{H})$. Therefore, $I'_s \in \text{span}\{I_1, I_2, I_3\}$ and hence

$$\text{span}_{\mathbb{R}} \{I_1, I_2, I_3\} = \text{span}_{\mathbb{R}} \{I'_1, I'_2, I'_3\}.$$

Now, still identifying $H^* \otimes H$ with $End(H)$, using $h = g$, and recalling that the metric g is I_s- and I'_s-compatible, we observe that each of the endomorphisms $(d\eta'_k)_H$ anti-commutes with both I'_i and I'_j. This implies that, as an endomorphism, $(d\eta'_k)_H$ is proportional to I'_k, which gives $g' = f\, g$ for some $f > 0$. The fact that the matrix valued function

$$\mathcal{A} \stackrel{def}{=} \frac{1}{f}\mathcal{B}$$

takes values in $SO(3)$ follows from the requirement that both (I_1, I_2, I_3) and (I'_1, I'_2, I'_3) satisfy the quaternionic identities. □

Let us remark that the above lemma (which is a well known fact as a sort of mathematical folklore) reveals a property that is very particular for the QC geometry in contrast with the situation in the CR case. The lemma implies that with each QC manifold (M, H) we have the following list of naturally associated objects:

i) There is a 3-dimensional bundle

$$Q = \text{span}\{I_1, I_2, I_3\} \subset End(H)$$

over M which we shall call *quaternionic structure* of H. Note that Q is canonically endowed with a scalar product $\langle .,. \rangle$ and orientation in such a way that for any admissible set (7), the basis I_1, I_2, I_3 of Q is orthonormal and oriented.

ii) The conformal class $[g]$ of symmetric sections of $H^* \otimes H^*$ from (7) is well defined globally on M. Using the standard partition of unity argument, it is easily seen that in $[g]$ we can always pick a globally defined positive definite representative, which we call *metric* on the contact distribution H .

iii) The bundle L^* is canonically endowed with an $CSO(3)$ structure (i.e., a conformal structure). This is done by declaring that for each admissible set (7) the basis η_1, η_2, η_3 is orthogonal and oriented.

5.2. Quaternionic-contact structures in dimension 7. It turns out that for $n = 1$ the definition 5.1 is too weak and one needs some further assumptions in order to make it reasonable.

To clarify the problem here, consider an arbitrary orientable 4-dimensional distribution H on a 7-dimensional manifold M, and choose some volume form ϵ on H—here, by volume form, we mean a globally defined non-vanishing section ϵ of the bundle $\Lambda^4(H^*)$ over M. Since for any two $\phi, \psi \in \Lambda^2(H^*)$, the wedge product $\phi \wedge \psi$ is proportional to ϵ, we can define a bilinear symmetric 2-form B on $\Lambda^2(H^*)$

by the equation $\phi \wedge \psi = B(\phi, \psi)\epsilon$. Then, B is non-degenerate, of signature $(3,3)$, and it defines an inner-product on the bundle $\Lambda^2(H^*)$ over M. For any local frame η_1, η_2, η_3 of L^* (with $L = TM/H$), consider, pointwise, the subspace

$$\Gamma = span\{(d\eta_1)_H, (d\eta_2)_H, (d\eta_3)_H\} \ \subset \ \Lambda^2(H^*).$$

It is an easy observation that Γ depends only on the distribution H, but not on the particular choice of the frame η_1, η_2, η_3. Now, a simple calculation shows that the distribution H satisfies the conditions of Definition 5.1 if and only if Γ is 3-dimensional at each point of M (i.e., if Γ is non-degenerate) and the restriction of B to Γ is either positive or negative definite. Clearly, each of the above two conditions is generic (i.e., it defines an open subset in the set of all distributions on M with respect to some natural topology) which makes the set of distribution that satisfy it uncomfortably "large".

As shown in [D1], the proper definition in dimension 7 requires one to add an extra assumption to the conditions of definition 5.1; namely, the assumption about the existence of Reeb vector fields which we shall explain below.

5.3. Existence of Reeb vector fields. Assume H is a $4n$-dimensional distribution on a $(4n+3)$-dimensional manifold M that satisfies the requirements of definition 5.1 without the assumption $n \geq 2$, i.e., here, we allow also $dim(M) = 7$.

We fix some admissible set (η_s, I_s, g) for H. If V is any complementary to H distribution, i.e., such that

$$TM = H \oplus V,$$

then clearly there exists a unique frame (ξ_1, ξ_2, ξ_3) of V dual to $(\eta_1|_V, \eta_2|_V, \eta_3|_V)$.

In what follows it will be important to find a special complementary distribution $\bar{V}$ in such a way that the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ would satisfy in addition the relations

$$d\eta_s(\bar{\xi}_t, X) = -d\eta_t(\bar{\xi}_s, X) \quad s, t = 1, 2, 3, \ X \in H. \tag{10}$$

If such a complement $\bar{V}$ exist, then the vector fields of the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ will be called *Reeb vector fields* of the contact form $\eta = (\eta_1, \eta_2, \eta_3)$.

Following [Biq], consider the 3×3 matrix with entries the elements of H^*, given by

$$a_{st} = d\eta_s(\xi_t, .) + d\eta_t(\xi_s, .), \quad s, t = 1, 2, 3,$$

with respect to some starting complement V. One could think of the matrix a_{st} as the local representation of a certain section of the bundle $Q \otimes Q \otimes H^*$ given by the formula

$$\sum_{s,t=1}^{3} I_s \otimes I_t \otimes a_{st}. \tag{11}$$

Clearly, (11) remains unchanged if rotating the admissible set (η_s, I_s) by an $SO(3)$ matrix and therefore it depends only on the choice of the complement V but not on the choice of the admissible set. Now the problem of finding a dual basis with the properties (10) is equivalent to the problem of finding a complement $\bar{V}$ with vanishing associated section

$$\sum I_s \otimes I_t \otimes \bar{a}_{st} \tag{12}$$

of the bundle $Q \otimes Q \otimes H^*$.

Since the matrix a_{st} is symmetric, by definition, the (12) is actually an element of the bundle $Q \odot Q \otimes H^*$, where $Q \odot Q$ denotes the symmetric component of $Q \otimes Q$. The bundle $Q \odot Q \otimes H^*$ decomposes into exactly three irreducible components with respect to the natural $Sp(n)Sp(1)$ action. With the standard notation for the irreducible $Sp(n)Sp(1)$-representations, we have

$$Q \odot Q \otimes H^* = [\lambda^1 \sigma^5] \oplus [\lambda^1 \sigma^3] \oplus [\lambda^1 \sigma^1].$$

Of particular interest for us is the component $[\lambda^1 \sigma^5]$ that can be described explicitly by

$$[\lambda^1 \sigma^5] = \{\sum_{st} I_s \otimes I_t \otimes x_{st} \in Q \otimes Q \otimes H^* \ : \ x_{st} = x_{ts}, \ \sum_t I_t x_{st} = 0\}.$$

The other two components can be described similarly:

$$[\lambda^1 \sigma^3] = \{\sum_{st} I_s \otimes I_t \otimes (I_s y_t + I_t y_s) \in Q \otimes Q \otimes H^* \ : \ y_s \in H^*, \ \sum_s I_s y_s = 0\},$$

$$[\lambda^1 \sigma^1] = \{\sum_{st} I_s \otimes I_t \otimes (\delta_{st} y) \in Q \otimes Q \otimes H^* \ : \ y \in H^*\}.$$

Explicitly the $[\lambda^1 \sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ of a section $\sum I_s \otimes I_t \otimes a_{st}$, that has been associated to some complementary distribution V, is given by

$$b_{st} = a_{st} + \frac{1}{5} \sum_{r=1}^{3} (I_s I_r a_{tr} + I_t I_r a_{sr} - \delta_{st} a_{rr}).$$

In fact, by using some simple representation theoretic arguments or by a direct calculation, it is easy to show that the above component $\sum I_s \otimes I_t \otimes b_{st}$ is actually independent of the choice of the starting V, and therefore it is an object that characterizes the given QC-structure as a whole. With a bit more effort this section could be viewed as a component of the intrinsic torsion of the QC-structure but this will not be considered here further. For our purposes, significant is only the following:

Proposition 5.3. *There exists a complementary distribution $\bar{V}$ to H for which the associated dual frame $(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ satisfies equations (10) if and only if the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ vanishes for some (and hence for each) complementary distribution V.*

Proof. Indeed, assume that $b_{st} = 0$ $s, t = 1, 2, 3$ for some fixed V with associated dual frame (ξ_1, ξ_2, ξ_3). If we define the three vector fields $r_1, r_2, r_3 \in H$ by the formula

$$g(r_s, X) = -\frac{1}{5}\sum_p (I_p a_{sp} + \frac{1}{2} I_s a_{pp})(X), \;\; s = 1, 2, 3, \;\; X \in H,$$

then the dual frame $\bar{\xi}_s = \xi_s - \frac{1}{2} r_s$ will have the desired property (10). $\square$

If $\dim(M) > 7$, the $[\lambda^1\sigma^5]$-component $\sum I_\alpha \otimes I_\beta \otimes b_{\alpha\beta}$ vanishes automatically, by a result in [**Biq**], and therefore the existence of Reeb vector fields is provided without any further assumptions about the QC structure H.

In dimension 7, however, the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ may not vanish in general and there are examples of distributions in dimension 7 that satisfy all the requirements of definition 5.1 (but the assumption $n \geq 2$) for which there exist no Reeb vector fields at all (cf. [**D1**]).

5.4. Special agreement in dimension 7. If the dimension of the QC-manifold M is 7 then we will assume in addition to definition 5.1 that the $[\lambda^1\sigma^5]$-component $\sum I_s \otimes I_t \otimes b_{st}$ (cf. section 5.3) for some (and hence for each) complement V to H vanishes and thus the existence of the Reeb vector fields is provided.

5.5. Biquard connection. Let (M^{4n+3}, H) be a quaternionic-contact manifold and fix some admissible set (η_s, I_s, g) for H. As explained in section 5.1, the contact form $\eta = (\eta_1, \eta_2, \eta_3)$ in the admissible set is determined only up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$. The distribution H is equipped with a conformal class $[g]$ of metrics and a 3-dimensional quaternionic bundle Q. According to Lemma 5.2 the most general transformation of the contact form η has the type $\bar{\eta} = \mu\Psi \cdot \eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries. We call such transformations quaternionic-contact conformal (QC conformal). If the function μ is constant we say that the transformation is quaternionic-contact homothetic. Clearly, the contact forms $\bar{\eta}$ which one obtains by applying homothetic quaternionic-contact transformations are precisely those for which the corresponding metric $\bar{g}$ (in the corresponding to $\bar{\eta}$ admisible set) is a constant multiple of g. If the tensor g in the natural conformal class $[g]$ is fixed we will say that (M, H, g) is a quaternionic-contact metric manifold. To every metric in the conformal class $[g]$ on H one can associate a linear connection preserving the QC structure (M, H, g) (cf. [**Biq**]) which we shall call the Biquard connection. This connection is invariant under QC homothetic transformations but changes in a non-trivial way under QC conformal transformations. If following the analogy with the 3-dimensional conformal geometry mentioned in **??**, one should think of the Biquard connection as an analog of the Levi-Civita connection.

Next we explain briefly the construction of the Biquard connection (for more details see [**Biq**]). Let us first consider the general situation of a manifold M with an arbitrary distribution $H \subset TM$ and a general vector bundle E over M. A partial connection on E along H is, by definition, a bilinear map $\nabla_X \sigma$ defined for vector fields X with values in H and sections σ of E such that $\nabla_{fX}\sigma = f\nabla_X\sigma$ and $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$ for every smooth function f on M.

If g is a metric on H then (as shown in [**Biq**, Lemma II.1.1]) for any supplementary distribution V of H in TM, there is a unique partial connection ∇ on H along H such that

- (i) $\nabla_X g = 0, \quad X \in H;$
- (ii) for any two sections X, Y of H, the torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ satisfies the identity $T(X,Y) = -[X,Y]_V$, where the subscript V means "the component in V".

Now let (M, H, g) be a quaternionic-contact metric manifold with admissible set (η_s, I_s) and Reeb vector fields ξ_1, ξ_2, ξ_3. Chose V to be the span of ξ_1, ξ_2, ξ_3. Each endomorphism f of H extends naturally to an endomorphism of TM by setting $f(\xi) = 0,\ \xi \in V$. With this agreement we may consider the endomorphism in the basis $\{I_1, I_2, I_3\}$ of Q as endomorphisms of the tangent bundle TM. We define the fundamental 2-forms $\omega_1, \omega_2, \omega_3$ of Q by

$$\omega_s(A,B) = g(I_s A, (B)_H),\ \ A, B \in TM,\ \ s = 1,2,3, \tag{13}$$

where the subscript H means "the component in H". Definition 5.1 imply that

$$\omega_s(A,B) = \begin{cases} \frac{1}{2} d\eta_s(A,B), & A, B \in H \\ 0, & A \in V,\ B \in TM \end{cases} \qquad \text{for } s = 1,2,3. \tag{14}$$

.

Take ∇ to be the partial connection on H along H that corresponds to the fixed supplementary distribution V. Then according to [**Biq**, Proposition II.1.7], the partial connection ∇ preserves the bundle Q, i.e., we have

- (iii) $\nabla_X Q \subset Q,\ X \in H.$

More precisely, for each cyclic permutation (i, j, k) of the numbers $(1, 2, 3)$ and any $X \in H$ there are the relations

$$\begin{aligned} \nabla_X \omega_i &= -\alpha_j(X)\omega_k + \alpha_k(X)\omega_j; \\ \nabla_X I_i &= -\alpha_j(X) I_k + \alpha_k(X) I_j, \end{aligned} \tag{15}$$

where $\alpha_i(X) := d\eta_k(\xi_j, X)$.

Let us introduce an inner-product $\langle ., . \rangle$ on the bundle V by setting the Reeb vector fields ξ_1, ξ_2, ξ_3 to be an orthonormal basis of V. Given a section ξ of V and a section X of H, set $\nabla_X \xi = [X, \xi]_V$. By [**Biq**, Proposition II.1.9], the latter formula defines a partial connection on V along H such that $\nabla \langle ., . \rangle = 0$.

The natural assignment

$$\xi_s \to I_s, \quad s = 1, 2, 3, \tag{16}$$

determines a bundle isomorphism $\varphi : V \to Q$. The isomorphism φ has the property that

(iv) $\nabla_X \varphi = 0$ for $X \in H$.

Indeed, by (15), we have

$$\nabla_X(\varphi(\xi_t)) = \nabla_X I_t = -\sum_{s=1}^3 d\eta_t(\xi_s, X) I_s = \sum_{s=1}^3 d\eta_s(\xi_t, X) I_s = \sum_{s=1}^3 \eta_s([X, \xi_t]_V) I_s = \sum_{s=1}^3 \eta_s(\nabla_X \xi_t)\varphi(\xi_s) = \varphi(\nabla_X \xi_t)$$

Set

$$P = \{A \in End(H) \mid A \text{ is skew-symmetric and } AI = IA \text{ for every } I \in Q\}.$$

This is a subbundle of $End(H)$ of rank $2n^2 + n$, orthogonal to Q and such that the commutator $[A_1, A_2]$ of two endomorphisms $A_1, A_2 \in P$ is also in P. Clearly, every fibre of P (resp. Q) is isomorphic to the Lie algebra $sp(n)$ (resp. $sp(1)$).

It is shown in [**Biq**, Lemma II.2.1] that there is a unique partial connection ∇ on H along V such that

(v) $\nabla_\xi g = 0, \quad \xi \in H$;
(vi) $\nabla_\xi Q \subset Q, \quad \xi \in H$;
(vii) setting $T(\xi, X) = \nabla_\xi X - \nabla_X \xi - [\xi, X]$ for $\xi \in V$ and $X \in H$, every endomorphism

$$T_\xi : H \in X \to T(\xi, X) = \nabla_\xi X - [\xi, X]_H \in H$$

is an element of $(P \oplus Q)^\perp \subset End(H)$.

Note, that we have a bundle isomorphism $\{(P \oplus Q)^\perp \subset End(H)\} \cong \{(sp(n) \oplus sp(1))^\perp \subset gl(4n)\}$.

Since $\nabla_\xi Q \subset Q$ for every $\xi \in V$, we can transfer ∇_ξ from Q to V via the isomorphism $\varphi : V \to Q$. In this way get a partial connection on V along V with the property

$(viii)$ $\nabla_\xi \varphi = 0, \quad \xi \in Q.$

Combining the partial connections we have defined, we obtain a connection ∇ on TM having the properties (i)-$(viii)$. We shall call ∇ the Biquard connection of the metric QC-structure (M, H, g). This connection is uniquely determined by its properties (i)-$(viii)$.

By using the isomorphism $\varphi : V \to Q$ we can transfer to V the metric and the orientation of Q. Then any frame ξ_1, ξ_2, ξ_3 associated to an admissible set of the QC-structure is orthonormal and positively oriented. Putting together the metric of V and the metric g of H we obtain a metric on $TM = H \oplus V$ for which H and V are orthogonal. This metric on TM will be also denoted by g and it is also parallel with respect to the Biquard connection, $\nabla g = 0$.

Summing up the facts from the above discussion, we have obtained the following result which is originally due to O. Biquard:

Theorem 5.4. [Biq] *Let (M, H, g) be a quaternionic-contact metric manifold of dimension $4n + 3 \geq 7$. Then, there exists a unique connection ∇ with torsion T on M and a unique supplementary distribution V to H in TM, such that:*

i) ∇ preserves the decomposition $H \oplus V$ and the metric g;
ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;
iii) ∇ preserves the $Sp(n)Sp(1)$-structure on H, i.e., $\nabla g = 0$ and $\nabla Q \subset Q$;
iv) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;
v) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of endomorphisms of H, i.e., $\nabla \varphi = 0$.

5.6. Further properties of the Biquard connection. Any endomorphism Ψ of H can be naturally decomposed, with respect to some admissible set (η_s, I_s), into four parts (this we call $Sp(n)$-invariant decomposition of Ψ)

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. Explicitly,

$$4\Psi^{+++} = \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3, \quad 4\Psi^{+--} = \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3,$$

$$4\Psi^{-+-} = \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3, \quad 4\Psi^{--+} = \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.$$

The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}. \tag{17}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger \;=\; I_1 \otimes I_1 \;+\; I_2 \otimes I_2 \;+\; I_3 \otimes I_3 \tag{18}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_3 = \frac{tr(\Psi)}{4} Id_{|H}$.

The supplementary "vertical" (sub-)space V is the linear span of the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields. We shall extend g to a metric on M by requiring

$$span\{\xi_1, \xi_2, \xi_3\} = V \perp H \text{ and } g(\xi_s, \xi_k) = \delta_{sk}. \tag{19}$$

The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on TM, $\nabla g = 0$. We shall also extend the quternionic structure by setting $I_{s|V} = 0$.

Suppose the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ have been fixed. The restriction of the torsion of the Biquard connection to the vertical space V satisfies [Biq]

$$T(\xi_i, \xi_j) = \lambda \xi_k - [\xi_i, \xi_j]_{|H}, \tag{20}$$

where λ is a smooth function on M, which will be determined in Corollary **??**. Here (i, j, k) is any cyclic permutation of $1, 2, 3$. Further properties of the Biquard connection are encoded in the properties of the torsion endomorphism

$$T_\xi = T(\xi, .) : H \to H, \quad \xi \in V.$$

Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part b_ξ,

$$T_\xi = T^0_\xi + b_\xi,$$

we summarize the description of the torsion due to O. Biquard in the following Proposition.

Proposition 5.5. [Biq] *The torsion T_ξ is completely trace-free,*

$$trT_\xi = \sum_{a=1}^{4n} g(T_\xi(e_a), e_a) = 0, \quad trT_\xi \circ I = \sum_{a=1}^{4n} g(T_\xi(e_a), Ie_a) = 0, \quad I \in Q, \tag{21}$$

where $e_1 \dots e_{4n}$ is an orthonormal basis of H. The symmetric part of the torsion has the properties:

$$T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \qquad i = 1,2,3. \tag{22}$$

In addition, we have

$$\begin{aligned} I_2(T^0_{\xi_2})^{+--} &= I_1(T^0_{\xi_1})^{-+-}, \qquad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \\ I_1(T^0_{\xi_1})^{--+} &= I_3(T^0_{\xi_3})^{+--}. \end{aligned} \tag{23}$$

The skew-symmetric part can be represented in the following way

$$b_{\xi_i} = I_i u, \quad i = 1,2,3, \tag{24}$$

where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3.

If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

Let $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be the curvature tensor of the Biquard connection. The QC Ricci curvature Ric, the QC Ricci forms ρ_s and the QC scalar curvature $Scal$ are defined respectively by

$$Ric(A,B) = \sum_{a,b=1}^{4n} g(R(e_b, A)B, e_b), \ \ A, B \in TM,$$

$$\rho_s(A,B) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(A,B)e_a, I_s e_a), \quad Scal = \sum_{a,b=1}^{4n} g(R(e_b, e_a)e_a, e_b),$$

where $e_1, ..., e_{4n}$ is an orthonormal basis of H. The restriction of the Ricci curvature Ric to H is a symetric 2-tensor ([**Biq**]) which could be $Sp(n)Sp(1)$-invariantly decomposed in exactly three components.

6. Quaternionic Heisenberg group

A very basic example of a QC manifold is provided by the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. These group can be modeled on the product space $\mathbb{H}^n \times \mathrm{Im}\mathbb{H}$ $(n \geq 1)$ with a group law given by

$$(q', \omega') \ = \ (q_o, \omega_o) \circ (q, \omega) \ = \ (q_o \ + \ q, \omega \ + \ \omega_o \ + \ 2 \ \mathrm{Im} \ q_o \, \bar{q}),$$

where $q, \ q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \mathrm{Im}\,\mathbb{H}$. In coordinates, with the obvious notation, a basis of left invariant horizontal vector fields $T_\alpha, X_\alpha = I_1 T_\alpha, Y_\alpha = I_2 T_\alpha, Z_\alpha =$

$I_3T_\alpha, \alpha = 1\dots, n$ is given by

$$T_\alpha = \partial_{t_\alpha} + 2x^\alpha\partial_x + 2y^\alpha\partial_y + 2z^\alpha\partial_z \qquad X_\alpha = \partial_{x_\alpha} - 2t^\alpha\partial_x - 2z^\alpha\partial_y + 2y^\alpha\partial_z$$
$$Y_\alpha = \partial_{y_\alpha} + 2z^\alpha\partial_x - 2t^\alpha\partial_y - 2x^\alpha\partial_z \qquad Z_\alpha = \partial_{z_\alpha} - 2y^\alpha\partial_x + 2x^\alpha\partial_y - 2t^\alpha\partial_z\,.$$

The central (vertical) vector fields ξ_1, ξ_2, ξ_3 are

$$\xi_1 = 2\partial_x \quad \xi_2 = 2\partial_y \quad \xi_3 = 2\partial_z\,.$$

We have the following commutator relations

$$[I_j\,T_\alpha, T_\alpha] = 2\xi_j \qquad [I_j\,T_\alpha, I_i\,T_\alpha] = 2\xi_k, \tag{25}$$

where (i, j, k) is any cyclic permutation of the indices $(1, 2, 3)$.

With respect to the local coordinates $(q', \omega) \subset \boldsymbol{G}\,(\mathbb{H})$, the standart 3-contact form $\tilde\Theta = (\tilde\Theta_1,\ \tilde\Theta_2,\ \tilde\Theta_3)$ is given by

$$2\tilde\Theta = d\omega - q'\cdot d\bar q' + dq'\cdot\bar q'. \tag{26}$$

The kernel of the contact form $\tilde\Theta$ is given by the distribution H which is easily seen to satisfy all the conditions of Definition 5.1 and thus defines a QC structure on $\boldsymbol{G}\,(\mathbb{H})$. Since the distribution H and the contact form $\tilde\Theta$ are left-invariant, they are preserved by the natural left-invariant connection on $\boldsymbol{G}\,(\mathbb{H})$. Let g be the left invariant metric on H which is determined by $\tilde\Theta$. Then, the central vector fields ξ_1, ξ_2, ξ_3 coincide with the correponding Reeb vector fields (cf. 5.3). If V is the linear span of these Reeb vector fields. Then V is a left-invariant distribution on $\boldsymbol{G}\,(\mathbb{H})$ and $T\boldsymbol{G}\,(\mathbb{H}) = H \oplus V$. Moreover, the left-invariant connection on $\boldsymbol{G}\,(\mathbb{H})$ is easily seen to coincide with the the Biquard connection of the quaternionic-contact metric structure $(\boldsymbol{G}\,(\mathbb{H}), H, g)$.

The translations $\tau_{(q_o,\omega_o)}$ on $\boldsymbol{G}\,(\mathbb{H})$—mapping each (q, ω) to a point $(q+q_o, \omega+\omega_o)$ of the group—and the dilations δ_λ, $\lambda > 0$—defined by $\delta_\lambda\,(q, \omega) = (\lambda q, \lambda^2\omega)$—are transformation of the group that preserve the QC distribution H. Such transformations are called conformal QC automorphisms of the group. If $u(q, \omega)$ is any function, then under the action of the translations and the dilations it is transformed to another function by the formulas:

$$\tau_{(q_o,\omega_o)}\bar u\;(q, \omega) \overset{def}{=} \bar u(q_o + q, \omega + \omega_o), \tag{27}$$

$$u_\lambda \overset{def}{=} \lambda^{(n_h-2)/2}\; u \circ \delta_\lambda, \tag{28}$$

where $n_h = 4n + 6$ denotes the homogeneous dimension of the group.

We shall often identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H}\ :\ \mathrm{Re}\, p' = |q'|^2\},$$

by the mapping $(q', \omega) \mapsto (q', |q'|^2 - \omega)$. Since $dp' = q' \cdot d\bar{q}' + dq' \cdot \bar{q}' - d\omega$, under the identification of $\boldsymbol{G}\,(\mathbb{H})$ with Σ we have also $2\tilde{\Theta} = -dp' + 2dq' \cdot \bar{q}'$. Taking into account that $\tilde{\Theta}$ is purely imaginary, the last equation can be written in the following form

$$4\,\tilde{\Theta} = (d\bar{p}' - dp') + 2dq' \cdot \bar{q}' - 2q' \cdot d\bar{q}'.$$

6.1. Folland-Stein inequality. On the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, there is a natural left-invariant measure known as the Haar measure dH of the group. Using this and the left-invariant metric g on $\boldsymbol{G}\,(\mathbb{H})$, we have the following classical result due to Folland and Stein [FS74]:

THEOREM 6.1 (Folland-Stein). *There exists a constant $S > 0$ such that for each $u \in C_o^\infty$ (i.e., for each smooth function u with compact support), we have the inequality*

$$\left(\int_{\boldsymbol{G}\,(\mathbb{H})} |u|^{2^*}\, dH\right)^{1/2^*} \leq S \left(\int_{\boldsymbol{G}\,(\mathbb{H})} |(\nabla u)_H|^2\, dH\right)^{1/2}, \tag{29}$$

where $(\nabla u)_H$ denotes the H-component of the gradient of u with respect to the splitting $T\boldsymbol{G}\,(\mathbb{H}) = H \oplus V$, and 2^* stands for the constant $\frac{2n_h}{n_h-2} = 1 + \frac{n+2}{n+1}$, where $n_h = 4n + 6$ is the so called homogeneous dimension of the group.

This theorem raises the following very natural question—known as the QC Yamabe problem—about the sharpness of the above inequality:

* What is the best possible choice for the constant S in the above inequality and for which functions u does this inequality become an equality?

Following the analogy with the classical Sobolev inequality one shows (see [**GV1, Va2, IMV10**]) that the above question reduces to the solvability of the following second order differential equation on the quaternionic Heisenberg group:

$$\triangle u = -Cu^{2^*-1}, \tag{30}$$

where C is a positive constant and $\triangle$ is the horizontal sub-Laplacian defined in terms of the Biquard connection ∇ (which in this case is just the flat, left-invariant, connection of $\boldsymbol{G}\,(\mathbb{H})$) by the formula

$$\triangle u = \sum_{s=1}^{4n} (\nabla du)(e_s, e_s), \quad e_1, \ldots, e_{4n} \text{ is any } g\text{-orthonormal basis of } H. \tag{31}$$

In general, the equation

$$\triangle u = -Cu^q \tag{32}$$

is a sort of a non-linear eigenvalue problem for the operator $\triangle$ on $\boldsymbol{G}\,(\mathbb{H})$, whose analytic properties depend on the value of the exponent q. For $q = 1$, we have the linear eigenvalue problem; if the value of q is close to 1, the analytic behavior of (32) is very similar to the linear case and the problem is easily solved. For large q, all the methods based on linear theory are useless. The value $2^* - 1$ of the Yamabe equation (30) appears to be critical for the exponent q in the senses that if q is less, then (32) is easy to solve and if q is more, it might be impossible to solve at all. This accounts for the complexity of the QC Yamabe equation.

6.2. QC sphere. The unit sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$ $(n \geq 1)$ is given in the usual way:

$$S^{4n+3} \;=\; \Big\{(q,p) \in \mathbb{H}^n \times \mathbb{H} \;\Big|\; |q|^2 + |p|^2 = 1\Big\}.$$

The canonical QC structure on S^{4n+3} can be described as follows: By differentiating the sphere equation $p \cdot \bar{p} + q \cdot \bar{q} = 1$, we obtain that at any fixed point $x = (q,p) \in S^{4n+3}$, the tangent space of the sphere is

$$T_x S^{4n+3} \;= \Big\{(dq, dp) \in \mathbb{H}^n \times \mathbb{H} \;\Big|\; \mathrm{Re}(dq \cdot \bar{q} + dp \cdot \bar{p}) = 0\;\Big\}$$

Then, the canonical contact 1-form $\tilde{\eta}$ with values in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$ on S^{4n+3} is defined by

$$\tilde{\eta} = \mathrm{Im}(dq \cdot \bar{q} + dp \cdot \bar{p}).$$

The kernel of $\tilde{\eta}$ gives a QC structure in S^{4n+3} since it coincides with the canonical distribution H^{can} on S^{4n+3} that have been introduced in Section **??** as the conformal infinity of the quternionic hyperbolic space.

6.3. Cayley transform. The Cayley transform, $\mathcal{C}$, is a natural identification between the sphere S^{4n+3} with one point deleted and the quternionic Heisenberg group. It plays a roll in the QC geometry that is very close to this of the classical stereographic projection in the conformal Riemannian geometry.

If idenitfyign $\boldsymbol{G}\,(\mathbb{H})$ with Σ as above, we have that, by definition, $\mathcal{C}$ identifies each point $(q,p) \in S^{4n+3}, p \neq 1$, with a point $(q',p') \in \Sigma$,

$$(q',p') \;=\; \mathcal{C}\,\Big((q,p)\Big), \qquad q' \;=\; (1+p)^{-1}\,q, \qquad p' \;=\; (1+p)^{-1}\,(1-p).$$

The inverse map $(q,p) \;=\; \mathcal{C}^{-1}\Big((q',p')\Big)$ is given by

$$q \;=\; 2(1+p')^{-1}\,q', \qquad p \;=\; (1+p')^{-1}\,(1-p').$$

The above equations are consistent with our definitions since

$$\text{Re}\ p' \;=\; \text{Re}\ \frac{(1+\bar p)(1-p)}{|1+p\,|^2} \;=\; \text{Re}\ \frac{1-|p|}{|1+p\,|^2} \;=\; \frac{|q|^2}{|1+p\,|^2} \;=\; |q'|^2.$$

Writing the Cayley transform in the form: $(1+p)q' \;=\; q, \quad (1+p)p' \;=\; 1-p$, gives

$$dp\cdot q' \;+\; (1+p)\cdot dq' \;=\; dq, \qquad\qquad dp\cdot p' \;+\; (1+p)\cdot dp' \;=\; -dp,$$

from where we find

$$\begin{aligned} dp' \;&=\; -2(1+p)^{-1}\cdot dp\cdot(1+p)^{-1} \\ dq' \;&=\; (1+p)^{-1}\cdot[dq \;-\; dp\cdot(1+p)^{-1}\cdot q]. \end{aligned} \tag{33}$$

The Cayley transform is a quaternionic-contact diffeomorphism between the quaternionic Heisenberg group with its standard QC structure $\tilde\Theta$ and the sphere minus a point with its standard QC structure H^{can}; a fact which can be seen as follows. Equations (33) imply the following identities

$$\begin{aligned} 2\,\mathcal{C}^*\,\tilde\Theta \;=\; &-(1+\bar p)^{-1}\cdot d\bar p\cdot(1+\bar p)^{-1} \;+\; (1+p)^{-1}\cdot dp\cdot(1+p)^{-1} \\ &+\; (1+p)^{-1}\,[dq \;-\; dp\cdot(1+p)^{-1}\cdot q]\cdot\bar q\cdot(1+\bar p)^{-1} \\ &\qquad -\; (1+p)^{-1}\,q\cdot[d\bar q \;-\; \bar q\cdot(1+\bar p)^{-1}\cdot d\bar p\,]\cdot(1+\bar p)^{-1} \end{aligned} \tag{34}$$

$$\begin{aligned} =\; &(1+p)^{-1}\Big[dp\cdot(1+p)^{-1}\cdot(1+\bar p) \;-\; |q|^2\,dp\cdot(1+p)^{-1}\Big](1+\bar p)^{-1} \\ &+\; (1+p)^{-1}\Big[-(1+p)\cdot(1+\bar p)^{-1}\cdot d\bar p \;+\; |q|^2\,(1+p)^{-1}d\bar p\Big](1+\bar p)^{-1} \\ &+(1+p)^{-1}\Big[dq\cdot\bar q \;-\; q\cdot d\bar q\Big](1+\bar p)^{-1} \;=\; \frac{1}{|1+p\,|^2}\,\lambda\,\tilde\eta\,\bar\lambda, \end{aligned}$$

where $\lambda \;=|1+p\,|\,(1+p)^{-1}$ is a unit quaternion and $\tilde\eta$ is the standard contact form on the sphere.

Since $|1+p| \;=\; \frac{2}{|1+p'|}$, we have $\lambda \;=\; \frac{1+p'}{|1+p'\,|}$ and equation (34) can be put in the form

$$\lambda\;\cdot(\mathcal{C}^{-1})^*\,\tilde\eta\;\cdot\bar\lambda \;=\; \frac{8}{|1+p'\,|^2}\,\tilde\Theta.$$

We see that $(\mathcal{C}^{-1})^*\tilde\eta$ and $\tilde\Theta$ correspond to the same QC structure on Σ.

6.4. The QC Yamabe problem. Following the classical scheme, we can translate question $(*)$ from Section 6.1 to an equivalent problem on the QC sphere S^{4n+3} via Cayley transform

$$\mathcal{C} : S^{4n+3} - \{Point\} \longrightarrow \boldsymbol{G}(\mathbb{H})$$

(cf. [K] and [CDKR1]). It turns out that $[*]$, with a standard argumentation (cf. [IMV10]), reduces to the following question, known as the Yamabe problem on the Sphere:

$**$ What are the representatives g of the natural conformal class $[g]$ of the QC structure H^{can} on S^{4n+3} for which the QC scalar curvature $Scal$ of the associated Biquard connection is a non-zero constant?

More generally, given a QC manifold (M, H), the QC Yamabe problem is the problem of finding a global representative g of constant QC scalar curvature in the natural conformal class of metrics $[g]$ on H. This problem reduces to the solvability of the equation ([Biq]),

$$\mathcal{L}u \;:=\; 4\frac{n+2}{n+1}\,\triangle u - \;u\,Scal \;=\; -Cu^{2^*-1}\;, \tag{35}$$

known as the QC Yamabe equation. Here $\triangle$ is the horizontal sub-Laplacian, defined by (31), with respect to the Biquard connection ∇ of some fixed (arbitrary) metric g on H; $Scal$ is the QC-scalar curvature of g and C is a positive constant.

In the case of the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$, the QC Yamabe equation takes the form

$$\mathcal{L}u \;\equiv\; \sum_{\alpha=1}^{n}\left(T_\alpha^2 u \;+\; X_\alpha^2 u \;+\; Y_\alpha^2 u \;+\; Z_\alpha^2 u\right) \;=\; -\frac{C(n+1)}{4(n+2)}\,u^{2^*-1}, \tag{36}$$

which is, up to scaling, the Euler-Lagrange equation describing the extremals in the L^2 Folland-Stein embedding theorem 6.1.

More generally, on a compact QC manifold M, the QC Yamabe equation characterizes the extremals of the Yamabe functional Υ,

$$\Upsilon(u) \;\overset{def}{=}\; \int_M 4\frac{n+2}{n+1}\,|\nabla u|^2 \;+\; Scal\cdot u^2\,dv_g, \qquad \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0, \tag{37}$$

where dv_g is the volume form on M associated to g. Note that according to [GV2] the extremals of the above variational problem are C^∞ functions, so we will not consider regularity questions here.

The Yamabe constant $\lambda(M, H)$ of a compact QC manifold is given, by definition, as the infimum of the Yamabe functional,

$$\lambda(M, H) \stackrel{def}{=} \inf\{\Upsilon(u) : \int_M u^{2^*} \, dv_g \;=\; 1,\; u > 0\}.$$

If $\lambda(M, H)$ is less than the Yamabe constant $\lambda(S^{4n+3}, H^{can})$ of the standard QC sphere (cf. Section 6.2), the existence of solutions of the Yamabe equation is shown in [W]. The proof of this is a straightforward generalization of the argument known from [JL2] concerning the CR case. Thefore, it is only relevant to studdy the problem on manifolds with the same Yambe constant as the sphere.

Overview of the results in the thesis

7. Chapter 2 of thesis

In this chapter, we develop the differential geometry of quaternionic-contact manifolds with an emphasis on the QC Yamabe problem (cf. Section 6.4). The origin and the form of this problem are very similar to those arising in the classical theory concerning the Riemannian [LP] and CR [JL1, JL2, JL3, JL4] cases. Both the Riemannian and the CR Yamabe problems have been a very fruitful subject in geometry and analysis and have been completely sloved. An important step in the achieved solution was the understanding of the conformally flat case, given by the correponding Heisenberg group; in the Riemannian case, this is just $\mathbb{R}^n$ (0-dimensional center); in the CR case, it is the complex Heisenberg group (1-dimensional center), whereas here we are dealing with the quaternionic Heisenberg group (three dimensional center).

In general, the quaternionic-contact Yamabe problem is about the possibility of finding, in the natural conformal class $[g]$, associated to given QC manifold (M, H), a representative of constant QC scalar curvature. The question reduces to the solvability of a certain nonlinear differential equation known as the QC Yamabe equation. In fact, if taking the conformal factor in the form $\bar\eta = u^{1/(n+1)}\eta$, the QC Yamabe equation reduces to

$$4\frac{n+2}{n+1}\,\triangle u - \;u\,Scal \;=\; -u^{2^*-1}\,\overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, defined by 31, whereas $Scal$ and $\overline{Scal}$ are the QC scalar curvatures corresponding to the two contact forms η and $\bar\eta$ respectively; the number 2^* is given by $\frac{2n_h}{n_h-2}$, where $n_h = 4n+6$ is the so called homogeneous dimension of the problem. In the case of the quaternionic Heisenberg group, this is, up to scaling, the Euler-Lagrange equation describing the extremals in the L^2 Folland-Stein embedding theorem, cf. Section 6.1.

If the Yamabe constant

$$\lambda(M) = \lambda(M,H) \overset{def}{=} \inf\{\Upsilon(u)\,:\, \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0\}$$

is strictly less than that of the standard QC sphere S^{4n+3} (cf. Section 6.4), the existence of solutions is shown in [W], see also [JL2]. Therefore, it is only relevant to study the problem on manifolds with the same Yambe constant as the sphere.

In this chapter, we provide a partial solution to the Yamabe problem on the standard QC sphere or, equivalently, on the quaternionic Heisenberg group. Let us observe that [GV2] solves the same problem in a more general setting, but under the assumption that the solution is invariant under a certain group of rotation. If one is on the flat models, i.e., the groups of Iwasawa type [CDKR1] the assumption in [GV2] is equivalent to the a-priori assumption that, up to a translation, the solution is radial with respect to the variables in the first layer. The proof goes on by using the moving plane method and showing that the solution is radial also in the variables from the center, after which a very non-trivial identity is used to determine all cylindrical solutions. In this chapter, the a-priori assumption is of a different nature (see further below) and the partial solution that we obtain here serves as an intermediate step for the results presented in the subsequent chapters. The strategy, following the steps of [LP] and [JL3], is to solve the Yamabe problem on the quaternionic sphere by replacing the non-linear Yamabe equation by an appropriate geometrical system of equations which can be solved.

Our first observation is that if $n > 1$ and the QC Ricci tensor is trace-free (QC Einstein condition) then the QC scalar curvature is constant (Theorem 5.9). Studying conformal deformation of QC structures preserving the QC Einstein condition, we describe explicitly all global functions on the quaternionic Heisenberg group that deform conformally the standard flat QC structure to another QC Einstein structure. Our main result here is the following Theorem.

Theorem A. *Let*

$$\Theta = \frac{1}{2h}\tilde{\Theta}$$

be a conformal deformation of the standard QC structure $\tilde{\Theta}$ *on the quaternionic Heisenberg group* $\boldsymbol{G}(\mathbb{H})$. *If* Θ *is also QC Einstein, then up to a left translation the function* h *is given by*

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\left(x^2 + y^2 + z^2\right)\right],$$

where c *and* ν *are positive constants. All functions* h *of this form have this property.*

The crucial fact which allows the reduction of the Yamabe equation to a system preserving the QC Einstein condition is Proposition 9.2 from the disertation which asserts that, under some "extra" conditions, QC structures with constant QC scalar curvature obtained by conformal deformations of a QC Einstein metric on a compact manifold must be again QC Einstein. The prove of this relies on detailed analysis of the Bianchi identities for the Biquard connection. Using the quaternionic Cayley transform combined with Theorem A we obtain a partial solution for the QC Yamabe problem on the sphere:

THEOREM B. *Let $\eta = f\,\tilde{\eta}$ be a conformal deformation of the standard QC structure $\tilde{\eta}$ on the sphere S^{4n+3}. Suppose η has constant QC scalar curvature.* **If the vertical space of η is integrable** *then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic-contact automorphism. In the case $n > 1$, the same conclusion holds when the function f is a real part of anti-CRF function.*

The solutions (conformal factors) we find agree with those conjectured in [GV1]. The above theorem is only a partial solution to the problem because of the "extra" assumption (printed in bold in the theorem) about the integrability of the vertical space of η. As we show later, in Chapter 4 of the thesis, this condition could actually be dropped, if the dimension is 7, but the argument for this is more involved.

Studying the geometry of the Biquard connection, our main geometrical tool towards understanding the geometry of the Yamabe equation, we show that the QC Einstein condition is equivalent to the vanishing of the torsion of the Biquard connection. In our third main result here, we give a local characterization of such spaces as 3-Sasakian manifolds:

THEOREM C. *Let (M^{4n+3}, H, g) be a QC manifold with positive QC scalar curvature $Scal > 0$, assumed to be constant if $n = 1$. The next conditions are equivalent:*

a) (M^{4n+3}, H, g) is a QC Einstein manifold.

b) M^{4n+3} is locally 3-Sasakian, i.e., locally there exists an $SO(3)$-matrix Ψ with smooth entries, such that, the local contact form $\frac{16n(n+2)}{Scal}\Psi \cdot \eta$ is 3-Sasakian.

c) The torsion of the Biquard connection is identically zero.

In particular, a QC Einstein manifold of positive QC scalar curvature, assumed in addition to be constant if $n = 1$, is an Einstein manifold of positive Riemannian scalar curvature.

Organization of the chapter:

Section 4 of the thesis. [1]

Here we develop some important properties and formulae concerning the Biquard connection that will be important for the future investigations in the thesis. We show that the torsion of the Biquard connection is essentially determined by two tensors—T^0, U, and one function—$tr(\tilde{u})$, where

$$T^0(X,Y) \overset{def}{=} g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X, Y), \qquad (38)$$

$$U(X,Y) \overset{def}{=} g(uX,Y), \quad X,Y \in H.$$

[1]This is the first section of Chapter 2 of the dissertation

It is easily observed that T^0 and U are $Sp(n)Sp(1)$-invariant and traceless symmetric tensors with the properties:

$$T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0,$$
$$3U(X,Y) - U(I_1X, I_1Y) - U(I_2X, I_2Y) - U(I_3X, I_3Y) = 0.$$

Our main result here shows that the Ricci curvature is completely determined by the torsion of the Bquard connection:

THEOREM 4.13. *Let (M^{4n+3}, H, g) be a quaternionic-contact $(4n+3)$-dimensional manifold, $n > 1$. For any $X, Y \in H$ the QC Ricci tensor and the QC scalar curvature satisfy*

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y) \\ &\qquad + (2n+4)\frac{tr(\tilde{u})}{n} g(X,Y), \\ Scal &= (8n+16)tr(\tilde{u}). \end{aligned} \tag{39}$$

For $n = 1$, we have

$$Ric(X,Y) = 4T^0(X,Y) + 6\frac{tr(\tilde{u})}{n} g(X,Y).$$

Section 5 of the thesis.

Here we write explicitly the Bianchi identities and use them to obtain a differential system relating divergences of some important tensors:

THEOREM 5.8. *The horizontal divergences of the curvature and torsion tensors satisfy the system $B\, b = 0$, where*

$$\mathbf{B} = \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$$\mathbf{b} = \Big(\nabla^* T^0, \quad \nabla^* U, \quad \mathbb{A}, \quad d\,Scal\,|_H, \quad \textstyle\sum_{j=1}^3 Ric\,(\xi_j, I_j\,.) \Big)^t,$$

with T^0 and U defined in (38) and

$$\mathbb{A}(X) = g(I_1[\xi_2, \xi_3] + I_2[\xi_3, \xi_1] + I_3[\xi_1, \xi_2], X).$$

Using this theorem we derive the following important result in the dissertation:

Theorem 5.9. *The QC scalar curvature of a QC Einstein quaternionic-contact manifold* **of dimension at least 11** *is a global constant. In addition, the vertical distribution V of a QC Einstein structure is integrable. On a seven dimensional QC Einstein manifold the constancy of the QC scalar curvature is equivalent to the integrability of the vertical distribution. In both cases the Ricci tensors are given by*

$$\rho_{t|H} = \tau_{t|H} = -2\zeta_{t|H} = -\frac{Scal}{8n(n+2)}\omega_t \quad s,t = 1,2,3.,$$
$$Ric(\xi_s, X) = \rho_s(X, \xi_t) = \zeta_s(X, \xi_t) = 0, \quad s,t = 1,2,3.$$

Later on, in Chapter 3 of the dissertation, we show that the printed in bold condition, excluding the 7-dimensional case in the theorem, can be actually removed. The argument for this is developed further below in the thesis.

Using Theorem 5.8, in this section, we also proof Theorem C.

Section 6 of the thesis.

Here we describe the conformal transformations preserving the QC Einstein condition. Note that a conformal quaternionic contact automorphism of a QC manifold is a diffeomorphism Φ which satisfies

$$\Phi^*\eta = \mu\ \Psi \cdot \eta,$$

for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries; $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$. Let us note that the Biquard connection does not change under rotations as above, i.e., the Biquard connection of $\Psi \cdot \eta$ coincides with this of η. In particular, when studying conformal transformations we can consider only transformations with $\Phi^*\eta = \mu\ \eta$. We find all conformal transformations preserving the QC Einstein condition on the quaternionic Heisenberg group or, equivalently, on the QC sphere, and prove Theorem A.

Section 7 of the thesis.

This section concerns a special class of functions, which we call anti-regular, defined respectively on the quaternionic space, real hyper-surface in it, or on a quaternionic-contact manifold, cf. Definitions 7.6 and 7.15 from the thesis, as functions preserving the quaternionic structure. The anti-regular functions play a role somewhat similar to this played by the CR functions, but the analogy is not complete. The real parts of such functions will be also of interest in connection with conformal transformation preserving the QC Einstein tensor and should be thought of as generalization of pluriharmonic functions. Let us stress explicitly that regular quaternionic functions have been studied extensively, see [S] and many subsequent papers, but they are not as relevant for the considered geometrical structures. Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [**CL1**, **CL2**, **LZ**] in a different context, namely in connection with minimal surfaces and quaternionic maps between quaternionic Kähler manifolds. The notion of hypercomplex contact structures will appear in this section again since on such manifolds the real part of anti-CRF functions have some interesting properties, as we show in

THEOREM 7.20. *If $f : M \to \mathbb{R}$ is the real part of an anti-CRF function*

$$f + iw + ju + kv$$

on a (4n+3)-dimensional ($n > 1$) hyperhermitian contact manifold (M, η, Q), then the following equivalent conditions hold true.

i) The next equalities hold

$$DD_{I_i} f = \lambda\omega_i - 4(\xi_j f)\omega_k \qquad mod\ \eta. \tag{40}$$

ii) For any $X, Y \in H$ we have the equality

$$\begin{aligned}(\nabla_X df)(Y) + (\nabla_{I_1X} df)(I_1Y) + (\nabla_{I_2X} df)(I_2Y) + (\nabla_{I_3X} df)(I_3Y) \\ = \lambda g(X,Y) + df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_3(I_2Y) \\ - df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_3(Y) + df(Y)\alpha_3(I_3X) \\ + df(I_1Y)\alpha_3(I_2X) - df(I_2Y)\alpha_3(I_1X) - df(I_3Y)\alpha_3(X).\end{aligned} \tag{41}$$

iii) The function f satisfies the second order system of partial differential equations

$$\begin{aligned}\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) = \lambda g(T_\beta, T_\alpha) \\ + df(\nabla_{T_\beta}T_\alpha) + df(\nabla_{I_1T_\beta}I_1T_\alpha) + df(\nabla_{I_2T_\beta}I_2T_\alpha) + df(\nabla_{I_3T_\beta}I_3T_\alpha) \\ + df(T_\beta)\alpha_3(I_3T_\alpha) + df(I_1T_\beta)\alpha_3(I_2T_\alpha) - df(I_2T_\beta)\alpha_3(I_1T_\alpha) - df(I_3T_\beta)\alpha_3(T_\alpha) \\ + df(T_\alpha)\alpha_3(I_3T_\beta) + df(I_1T_\alpha)\alpha_3(I_2T_\beta) - df(I_2T_\alpha)\alpha_3 I_1(T_\beta) - df(I_3T_\alpha)\alpha_3(T_\beta)\end{aligned} \tag{42}$$

$$\begin{aligned}\Re(iD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_1T_\beta}\overline{D}_{T_\alpha} f), \quad \Re(jD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_2T_\beta}\overline{D}_{T_\alpha} f), \\ \Re(jD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_3T_\beta}\overline{D}_{T_\alpha} f).\end{aligned} \tag{43}$$

The function λ is determined by

$$\lambda \;=\; 4\left[(\xi_1 w) + (\xi_2 u) + (\xi_3 v)\right]. \tag{44}$$

Section 8 of the thesis.

In this section, we study infinitesimal conformal automorphisms of QC structures (QC vector fields) and show that they depend on three functions satisfying some differential conditions thus establishing a '3-hamiltonian' form of the QC vector fields (Proposition 8.8 from dissertation). The formula becomes very simple expression on a 3-Sasakian manifolds (Corollary 8.9 from dissertation). We characterize the vanishing of the torsion of Biquard connection in terms of the existence of three vertical vector fields whose flow preserves the metric and the quaternionic structure. We show that among them, 3-Sasakian manifolds are exactly those admitting three transversal QC vector fields:

THEOREM 8.10. *Let (M, H, g) be a QC manifold with positive QC scalar curvature, assumed constant in dimension seven. The following conditions are equivalent.*

i) Each of the Reeb vector fields is a QC vector field.

ii) The QC structure is homothetic to a 3-Sasakian structure. In particular, the Reeb vector fields are infinitesimal isometries.

Section 9 of the thesis.

In the final section of this chapter, we complete the proof of Theorem B.

8. Chapter 3 of thesis

An extensively studied class of QC structures is provided by the 3-Sasakian spaces. In Therem C of Chapter 2, we have shown that a QC manifold is locally 3-Sasakian iff it is QC Einstein with poitive and constant QC scalar curvature. Furthermore, as a consequence of the Bianchi identities, in Theorem 5.9 (Chapter 2), we have shown that the QC scalar curvature of a QC Einstein manifold of dimension at least eleven is constant while the seven dimensional case was left open. In this chapter, we extend this two results starting with:

THEOREM D. *The QC scalar curvature of a 7-dimensional QC Einstein manifold is always a constant.*

The main application of Theorem D is the removal of the a-priori assumption of constancy of the QC scalar curvature in some previous results concerning seven dimensional QC Einstein manifolds. As a consequence of this theorem we show that on each 7-dimensional QC Einstein manifold, the associated vertical distribution V is always integrable (Corollary 10.3 of the thesis) and the corresponding fundamental 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is necessarily closed (Corollary 10.4 of the thesis).

Recall that the complete, and regular 3-Sasakian and nS-spaces (called negative 3-Sasakian here) have a canonical fibering with fiber $Sp(1)$ or $SO(3)$, and base a quaternionic-Kähler manifold. In this chapter we show that if the QC scalar curvature is strictly positive (resp. strictly negative), the QC Einstein manifolds are "essentially" $SO(3)$ bundles over quaternionic-Kähler manifolds with positive (resp. negative) scalar curvature. We show also that in the "regular" case, a QC Einstein manifold of zero QC scalar curvature fibers over a hyper-Kähler manifold (cf. Proposition 13.3 of the thesis).

Organization of the chapter:

Section 10 of the thesis.

Here we prove Theorem D. In the proof, we use the concept of QC conformal curvature introduced in [IV1] that characterizes the QC conformally flat structures in any dimension. We use also a result of Kulkarni [Kul] on the algebraic properties of curvature tensors in four dimensions, and an extension of Theorem A (Chapter 2) describing explicitly the different QC Einstein structures defined locally on the quaternionic Heisenberg group which are also point-wise QC conformal to the flat one.

Section 11 of the thesis.

Here we introduce a special (vertical) connection $\tilde{\nabla}$ on the 3-dimensional canonical vector bundle V over M which is associated to a fixed metric g of the QC structure. We prove the following:

THEOREM 11.3. *A QC manifold M is QC Einstein iff the connection $\tilde{\nabla}$ is flat.*

The vertical connection $\tilde{\nabla}$ is used in this chapter as a technical tool for revealing the various properties of QC Einstein spaces.

Section 12 of the thesis.

Using the vertical connection $\tilde{\nabla}$, here, we develop certain differential equations describing the QC Einstein spaces via the contact form $\eta = (\eta_1, \eta_2, \eta_3)$ and its exterior derivative $d\eta = (d\eta_1, d\eta_2, d\eta_3)$:

THEOREM 12.1. *Let M be a QC manifold. The following conditions are equivalent:*

a) M is a QC Einstein manifold;

b) locally, the given QC structure is defined by 1-form (η_1, η_2, η_3) such that for some constant S we have

$$d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k; \tag{45}$$

c) locally, the given QC structure is defined by 1-form (η_1, η_2, η_3) such that the corresponding connection 1-forms vanish on H, $\alpha_s = -S\eta_s$.

Section 13 of the thesis.

Here we consider the relation between the QC Einstein spaces (M, H, g) and the geometry of the naturally associated Riemannian matric h on M, defined by requiring that $\text{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and that

$$h|_H = g, \qquad h|_V = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3. \tag{46}$$

By Theorem C (Chapter 2 of the thesis), if the QC scalar curvature of M is positive, then (M, h) is locally a 3-Sasakian space. The case of negative QC scalar curvature can be treated similarly to show that M is a negative locally 3-Sasakian space. If the QC scalar curvature vanishes, Lemma 13.2 of the thesis shows that the Reeb vector fields of the QC structure are Killing vector fields for h but unlike the 3-Sasakian case, the Lie brackets of each pair must be zero.

The main result of this section is Propsition 13.3[2], showing that the QC Einstein spaces of vanishing QC scalar curvature are "essentially" bundles over hyper-Kähler manifolds.

Here we show also that on each QC Einstein manifold of vanishing QC scalar curvature there is a pair of naturally associated Riamannian Einstein metrics. Note that in the 3-Sasakian case (i.e., the case of positive QC scalar curvature) the corresponding result is well known, cf. [BGN].

9. Chapter 4 of thesis

The QC Yamabe problem on S^7 is about the determinantion of all contact 1-forms η of the canonical QC structure on the sphere that have constant QC scalar curvature. In Chapter 2 of the thesis, we conjectured that these are precisely the forms that can be obtained as pull-back $\phi^*(\tilde{\eta})$ of the standard contact form $\tilde{\eta}$, where ϕ is a conformal quaternionic-contact automorphism of the sphere. In Theorem B (Chapter 2 of the thesis), we have shown a weaker result, namely, that the same conclusion holds provided the vertical space of η is integrable. The purpose of this chapter is to remove this extra assumption and to prove the conjecture when the dimension is seven:

Theorem E. *Let $\tilde{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the unit sphere S^7. If η has constant QC scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic-contact automorphism. In particular, the Yamabe constant $\lambda(S^7)$ of the sphere is*

[2]The number corresponds to the notation in the thesis

$48\,(4\pi)^{1/5}$ *and this minimum value is achieved only by* $\tilde{\eta}$ *and its images under conformal quaternionic-contact automorphisms.*

An important motivation for studying the QC Yamabe problem on the sphere comes from its connection with the determination of the norm and extremals of the related Folland-Stein embedding on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, cf. Theorem 6.1. Using Theorem E, we obtain:

Theorem F. *Let*

$$\boldsymbol{G}\,(\mathbb{H}) \;=\; \mathbb{H}\times Im\,\mathbb{H}$$

be the seven dimensional quaternionic Heisenberg group. The best constant in the L^2 *Folland-Stein embedding theorem is*

$$S_2 \;=\; \frac{2\sqrt{3}}{\pi^{3/5}}$$

An extremal is given by the function

$$v \;=\; \frac{2^{11}\sqrt{3}}{\pi^{3/5}}[(1+|q|^2)^2 \;+\; |\omega|^2]^{-2},\;\; (q,\omega)\in \boldsymbol{G}\,(\mathbb{H})$$

Any other non-negative extremal is obtained from v *by translations* (27) *and dilations* (28).

Our result confirms the Conjecture made after [**GV1**, Theorem 1.1]. In [**GV1**, Theorem 1.6], a similar result is obtained in all dimensions, but with the extra assumption of partial-symmetry. Here with a completely different method, we show that the symmetry assumption is superfluous in the case of the first quaternionic Heisenberg group.

A key step in the present result is the establishment of a suitable divergence formula, Theorem 15.4 [3], see [**JL3**] for the CR case and [**Ob**], [**LP**] for the Riemannian case. With the help of this divergence formula we show that the 'new' structure is also QC Einstein, thus we reduce the Yamabe problem on S^7 from solving the non-linear Yamabe equation to a geometrical system of differential equations describing the QC Einstein structures conformal to the standard one. Invoking the (quaternionic) Cayley transform, which is a contact conformal diffeomorphism (cf. Section 6.3), we turn the question to the corresponding system on the quaternionic Heisenberg group. On the latter, all global solutions were explicitly described in Theorem A (Chapter 2 of the thesis) and this is enough to conclude the proof of the result.

[3]The number corresponds to the notation in the thesis

Organization of the chapter:

Section 14 of the thesis.

In this section we develop various formulas concerning conformal deformations of the canonical QC structure of the sphere S^7. In general, any conformal deformation of the associated metric to a QC structure yields a deformation of the Reeb vector fields and the vertical distribution V. The Biquard connection changes, as well, in a non-trivial way under such deformations.

Section 15 of the thesis.

Here we construct the divergence formula (47), which is the key step from the proof of Theorems E and F.

THEOREM 15.4. *Suppose (M^7, η) is a quaternionic-contact structure conformal to a 3-Sasakian structure $\tilde{\eta}$, $\tilde{\eta} = \frac{1}{2h}\eta$. If*

$$Scal_{\eta} = Scal_{\tilde{\eta}} = 16n(n+2),$$

then with f given by

$$f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2,$$

the following identity holds

$$\nabla^*\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big) = f|T^0|^2 + h\,\langle \mathcal{Q}V, V\rangle. \tag{47}$$

Here, $\mathcal{Q}$ is a positive semi-definite matrix and

$$V = (D_1, D_2, D_3, A_1, A_2, A_3)$$

with A_s, D_s defined, correspondingly, in (48) and (49).

Explicitly, we have:

$$A_i = I_i[\xi_j, \xi_k], \tag{48}$$

$$\begin{aligned} D_1(X) &= -h^{-1}T^{0+--}(X, \nabla h) \\ D_2(X) &= -h^{-1}T^{0-+-}(X, \nabla h) \\ D_3(X) &= -h^{-1}T^{0--+}(X, \nabla h). \end{aligned} \tag{49}$$

The matrix $\mathcal{Q}$ is given by

$$\mathcal{Q} := \begin{bmatrix} 2 & 0 & 0 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 2 & 0 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ 0 & 0 & 2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

The eigenvalues of $\mathcal{Q}$ are

$$\{0, \quad 0, \quad 2\,(2+\sqrt{2}), \quad 2\,(2-\sqrt{2}), \quad 10, \quad 10\}.$$

Section 16 of the thesis.

Here we complete the proof of Theorems E and F. By integrating the divergence formula (47) and applying Proposition 9.1 (Chapter 2 of the thesis), we conclude that the integral obtained from the LHS of (47) is zero and therefore the integral from the RHS must vanish. This observation produces an equation which implies that "new" contact form η must have vanishing torsion and therefore it is QC Einstein (cf. Theorem 4.13). We complete the proof by applying the Cayley transform (cf. Section 6.3) and Theorem A (Chapter 2 of the thesis).

10. Chapter 5 of thesis

In this chapter we determine the best (optimal) constant in the L^2 Folland-Stein inequality (cf. Theorem 6.1) on the quaternionic Heisenberg group (in all dimensions) and the non-negative extremal functions, i.e., the functions for which equality holds:

Theorem G.

a) Let $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}^n \times Im\,\mathbb{H}$ be the quaternionic Heisenberg group. The best constant in the L^2 Folland-Stein embedding inequality (29) is

$$S_2 = \frac{\left[2^{-2n}\ \omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

where $\omega_{4n+3} = 2\pi^{2n+2}/(2n+1)!$ *is the volume of the unit sphere* $S^{4n+3} \subset \mathbb{R}^{4n+4}$. *The non-negative functions for which* (29) *becomes an equality are given by the functions of the form*

$$F \;=\; \gamma\left[(1+|q|^2)^2 \;+\; |\omega|^2\right]^{-(n+1)}, \qquad \gamma = const, \tag{50}$$

and all functions obtained from F *by translations* (??) *and dilations* (??).

b) The QC Yamabe constant of the standard QC structure of the sphere is

$$\lambda(S^{4n+3}, H^{can}) \;=\; 16\, n(n+2)\ \left[((2n)!)\, \omega_{4n+3}\right]^{1/(2n+3)}. \tag{51}$$

The proof relies on a realization of Branson, Fontana and Morpurgo [BFM], used also by Frank and Lieb [FL], that the old idea of Szegö [Sz], see also Hersch [He], can be used to find the sharp form of (logarithmic) Hardy-Littlewood-Sobolev type inequalities on the Heisenberg group. The argument presented here is purely analytical. In this respect, even though the QC Yamabe functional is involved, the QC scalar curvature is used in the proof without much geometric meaning. Rather, it is the conformal sub-laplacian that plays a central role and the QC scalar curvature appears as a constant determined by the Cayley transform and the left-invariant sub-laplacian on the quaternionic Heisenberg group. This method does not give all solutions of the QC Yamabe equation on the quaternionic-contact sphere but only these that realize the infimum of the QC Yamabe functional. Therefore, if considering the seven dimensional case, the result presented here is clearly weaker than Theorem E of Chapter 3.

Organization of the chapter:

Section 17 of the thesis.

In this section we obtain a variety of preliminary result needed for the proof of the main results in the chapter. In Lemma[4] 17.4 we show that the first eigenvalue of the sub-Laplacian on the sphere $S^{4n+3} \subset \mathbb{H}^n \times \mathbb{H}$ is $\lambda_1 = 2n$ and that the restriction of the real coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ to the sphere are the λ_1-eigenfunctions.

Section 18 of the thesis.

In this section we proof Theorem G. The proof is split into a number of steps formulated as separate lemmas. The first step is given by Lemma[4] 18.1, showing that for each L^1 function v on the sphere that satisfies

$$\int_{S^{4n+3}} v \; Vol_{\tilde{\eta}} = 1,$$

[4]The number corresponds to the notation in the thesis

there exists a conformal QC automorphism ψ of the sphere so that

$$\int_{S^{4n+3}} \psi\, v\; Vol_{\tilde{\eta}} = 0.$$

The second step is Lemma[5] 18.2 which asserts that one is always allowed to assume that the infimum of the Yamabe functional is achieved on a function u that satisfies the additional condition of being "well centered", i.e., that it satisfies

$$\int_{S^{4n+3}} P\, u^{2^*}(P)\, Vol_{\tilde{\eta}} = 0, \qquad P \in \mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}.$$

The third step is Lemma[5] 18.3, showing that if u is a "well centered" local minimum for the Yamabe functional, then $u \equiv const$.

By using this three lemmas, the proof of Theorem G is obtained as follows: Let F be any minimizer (local minimum) of the Yamabe functional on the quternionic Heisenberg group, and let g be the corresponding function on the sphere. By Lemma 18.2, the function $g_0 = \phi^{-1}(g \circ \psi^{-1})$ is a "well centered" minimizer for the Yamabe functional on the sphere. Then, by Lemma 18.3, we must have $g_o = const$. Looking back at the corresponding function on the group, we see that

$$F_0 = \gamma\left[(1+|q'|^2)^2 + |\omega'|^2\right]^{-(n_h-2)/4}$$

for some $\gamma = const > 0$. Furthermore, the proof of Lemma 18.1 shows that F_0 is obtained from F by a conformal QC transformation. Correspondingly, any positive minimizer (local maximum) of problem is given up to a conformal QC transformation of the sphere by the function

$$F = \gamma\left[(1+|q'|^2)^2 + |\omega'|^2\right]^{-(n_h-2)/4}, \qquad \gamma = const > 0.$$

[5]The number corresponds to the notation in the thesis

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