# PEER REVIEW

by Corresponding Member of BAS Oleg Krastev Mushkarov,
Institute of Mathematics and Informatics, BAS

on the thesis of Ivan Minchev Minchev "Geometry of quaternionic-contact manifolds and the Yamabe problem" for acquiring the scientific degree Doctor of Science , Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional field: 4.5 Mathematics.

I am presenting this Peer Review as an assessor of an Academic Board, formed with Order № РД 38-113/19. 02. 2020 г. of the Rector of Sofia University Prof. Anastas Gergikov and a decision of the Council of the Department of Mathematics and Informatics, Protocol №2, 17. 02. 2020. It is prepared according to the requirements of:

• The Law Act for Development of the Academic Staff in the Republic of Bulgaria (LADASRB);
• The Statutes for application of LADASRB;
• The Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in Sofia University;

**1.** The thesis of Ivan Minchev is devoted to questions of current interest in Differential geometry and Geometric analysis which are related to the study of quaternionic-contact(QC) structures. These structures were introduced by Olivie Biquard in 2000 as a tool for studying the conformal boundary at infinity of a quaternionic-Kähler manifold. Note that the manifolds endowed with such structures (QC-manifolds) and the strongly pseudoconvex CR-manifolds are model categories for sub-Riemannian manifolds with special holonomy groups.

The motivation for introducing QC-manifolds is the observation that the Einstein deformations of the standart invariant metrics of complex, quaternionic and octonionic symmetric spaces are in one to one correspondence with the metrics of Caratheodory-Karno type on their conformal boundaries at infinity. Recall that if $g$ is a Riemannian metric on a manifold $M$ with boundary $N$, then a conformal class $[h]$ of Riemannian metrics on $N$ is called *conformal infinity* of $g$ if there exists a function $\rho$, positive on $M$ and vanishing on $N$ of first order, and such that the metric $\rho^2 g$ can be extended continuously on $N$ to a metric in $[h]$. A standard example is the hyperbolic metric on the ball $\mathbb{B}^{n+1}$, whose conformal infinity is the conformal class of the standard metric on the sphere $\mathbb{S}^n$. Having in mind this and other examples, Biquard posed the general question for finding the Einstein metrics whose conformal infinities are metrics of Caratheodory-Karno type. This problem has been studied most fully in the complex case and for Kähler-Enstein manifolds it was solved completely by Cheng and Yau in 1980. For the ball(real hyperbolic space) the above problem was solved by Graham and Lee in 1991. A general result in dimension 4 was obtained by LeBrun in 1982.

He proved by using twistor methods that any conformal class on a 3-dimensional real-analytic manifold $N$ is conformal infinity of a self-dual Einstein metric, defined in a "small"4-dimensional neighbourhood of $N$. A natural generalization of this theorem in the quaternionic case was obtained by Biquard in 2000. He proved that for $n \geq 2$ any real-analytic QC-structure on a $(4n+3)$-dimensionaol manifold is conformal infinity of a uniquely determined quaternionic-Kähler metric, defined in a small $(4n+4)$-dimensional neighbourhood. In dimension 7 the problem was solved by Douchemin in 2006.

Recall that a QC-structure on a $(4n+3)$-dimensional manifold $M$ is a distribution $H$ of co-dimension 3 on $M$, which locally is the kernel of an $\mathbb{R}^3$-valued 1-form , such that the restrictions on $H$ of the exterior derivatives of its coordinate 1-forms are the fundamental forms of a quaternionic structure on $H$. This definition determines, generally speaking, the two main goals of the thesis. The first one is to study the relations between the geometric properties of quaternionic-Kähler and 3-Sasakian structures and their QC-structures. The second goal is to use analytic techniques from sub-Riemannian geometry to solve particularly the QC-Yamabe problem for the quaternionic Heisenberg groups of arbitrary dimension and to find the optimal constant in the $L^2-$Folland-Stein inequality on these groups.

Now I'll describe in more details the content and the main achievements of the thesis. It is divided into Introduction and 5 chapters.

Chapter 1 of the thesis is an introduction to the subject. In paragraph 1 the author introduces the main algebraic and differential-geometric notions connected with quaternionic-Kähler manifolds, the construction of their twistor spaces and the natural complex and holomorphic contact structures on these spaces. The inverse twistor correspondence is also described which can be used to construct quaternionic-Kähler manifolds by means of algebraic geometry methods. The twistor methods are very important for this theory since they are on the basis of the proofs of the famous result of LeBrun for existence of infinite dimensional families of complete quaternionic-Kähler manifolds on the unit ball $B^{4n+4}$ and the above mentioned theorem of Biquard. At the end of this paragraph the author gives explicit descriptions of the twistor spaces of quaternionic projective space $\mathbb{HP}^n = Sp(n+1)/Sp(n)Sp(1)$ and quaternionic hyperbolic space $\mathbb{HH}^n = Sp(n;1)/Sp(n)Sp(1)$. The second paragraph is devoted to the basic properties of the QC-structures. The notion of *conformal infinity* is explained precisely and it is illustrated by the model example of quaternionic hyperbolic space and some deformations of its twistor spaces, considered by LeBrun. Another application of twistor methods is the construction of Biquard of integrable CR-structures on the twistor spaces of QC-manifolds. The rest of this paragraph is devoted to the main properties of Biquard connection (Theorem 2.4) which are used later. In paragraph 3 the author gives a detailed description of the quaternionic Heisenberg groups $\mathbf{G}(\mathbb{H}) = \mathbb{H}^n \times Im\mathbb{H}$ and their left-invariant QC-structures. These groups are a main object of investigation in the thesis in connection with the solution of the QC-Yamabe problem and finding the optimal constant in the $L^2$ Folland-Stein inequality on them.

Chapter 2 is based on the results published in [IMV14]. Here the author develops the needed differential-geometric and analytic techniques for obtaining a partial solution of QC-Yamabe problem in the conformally flat case, which is equivalent to that on the quaternionic Heisenberg groups of arbitrary dimension.

The classical Yamabe problem says that if $(M, g)$ is a compact Riemannian manifold of dimension $\geq 3$, then there exists a metric of constant scalar curvature in the conformal class of $g$. The solution of this problem, due to the works of Yamabe, Trudinger, Aubin, and Schoen, is a milestone in the developement of the theory of nonlinear PDE's. In the complex case the analog of Yamabe problem is for strongly pseudoconvex CR-manifolds. Here the Levi form plays the role of a metric, that of a conformal metric is played by a contact 1-form which vanishes on the Levi distribution(pseudohermitian structure), and the scalar curvature is that of the pseudohermitian structure introduced independently by Webster and Tanaka in 1978. In these terms the CR-Yamabe problem was posed and solved in the general case by Jerison and Lee in 1987. In 2001 Gamara and Yacoub completed their result for the remaining cases of 3-dimensional and conformally flat CR-manifolds. The QC-Yamabe problem is the quaternionic version of that for CR-manifolds. In this case one seeks a conformal change of the canonical $\mathbb{R}^3$-valued contact 1-form whose Biquard connection has constant scalar curvature. This problem was solved by Wang in 2005 in the sub-critical case when the Yamabe constant of the QC-manifolds is less than that of the canonical QC-structure of the quaternionic Heisenberg group of same dimension.

Biquard has shown that using a special conformal change the QC-Yamabe problem for a QC-manifold $(M^{4n+3}, H)$ is equivalent to the solution of the following nonlinear PDE

$$4\frac{n+2}{n+1} \triangle u - \; u \, Scal \;=\; -Cu^{\frac{n+2}{n+1}} \,, \tag{1}$$

known as QC-Yamabe equation. Here $\triangle$ is the horizontal sub-Laplacian of Biquard connection for a given metric of $H$; $Scal$ is the QC-scalar curvature, and $C$ is a positive constant. The QC-Yamabe equation for the $(4n{+}3)$-dimensional quaternionic Heisenberg group has the form

$$\triangle u = \sum_{\alpha=1}^{n} \left(T_\alpha^2 u \;+\; X_\alpha^2 u \;+\; Y_\alpha^2 u \;+\; Z_\alpha^2 u\right) \;=\; -\frac{C(n+1)}{4(n+2)}\, u^{\frac{n+2}{n+1}}. \tag{2}$$

Up to a constant this is the Euler-Lagrange equation for the extremals of the $L^2$-Folland-Stein inequality.

The main idea for solving equation (2) follows the approaches in the papers [LP] and [JL3] for solving the Riemannian and CR-Yamabe equations. An important first step is the reduction of this equation to a geometric system. To do this the author investigates the Biquard connection in paragraphs 4 and 5 of Chapter 2. The main results in this direction are Theorem 4.13 that the QC-Ricci tensor can be represented in terms of its torsion, Theorem 5.8 where some very useful identities for the horizontal divergences of the curvature and the torsion of Biquard connection are obtained, and Theorem 5.9 that any QC-Einstein metric has constant scalar curvature. These results are used in the proof of the following:

*Theorem C. Let $(M^{4n+3}, H, g)$ be a QC-manifold with positive QC-scalar curvature which is assumed to be constant when $n = 1$. The following conditions are equivalent:*

a). $(M^{4n+3}, H, g)$ *is a QC-Einstein manifold.*
b). $M^{4n+3}$ *is locally a 3-Sasakian manifold.*
c). *The torsion of Biquard connection vanishes.*

The main goal of paragraph 6 is to describe the conformal deformations preserving the QC-Einstein condition for Biquard connection. The main result is the following:

*Theorem A. Let* $\Theta = \frac{1}{2h}\Theta_0$ *be a conformal deformation of the standard QC-structure* $\Theta_0$ *on the quaternionic Heisenberg group. If* $\Theta$ *is a QC-Einstein structure too, then up to a left translation the function* $h$ *is given by*

$$h(q, \omega) \; = \; c\, \left[ \left(1 \; + \; \nu\, |q|^2\right)^2 \; + \; \nu^2 \; |\omega|^2 \right], \; (q, \omega) \in \mathbb{H}^n \times Im\mathbb{H},$$

*where* $c$ *and* $\nu$ *are positive constants. Conversely, all functions* $h$ *as above have this property.*

In paragraph 7 the author studies some special classes of functions which preserve the quaternionic structures of quaternonic spaces, their hypersurfaces and QC-manifolds. The real parts of these functions are called *quaternionic pluriharmonic* and they are related to the conformal deformations preserving the QC-Ricci tensors. Another important class of functions are the so-called anti-CRF functions whose coordinate functions satisfy the horizontal Cauchy-Riemann-Fueter equations (Proposition 7.17). The real parts of these functions play a very important role in the further investigations in the thesis. Their analytic properties are proved in Theorem 7.20.

The purpose of paragraph 8 is to study the QC-vector fields whose infinitesimal automorphisms preserve the QC-structures. In Proposition 8.8 it is shown that they depend on three functions satisfying some compatibility conditions. This result is used in the proof of Theorem 8.10 which characterizes in terms of Reeb vector fields the QC-structures that are homothetic to 3-Sasakian structures.

The results obtained in paragraph 6-8 are used in paragraph 9 to prove one of the main results in the thesis. This is Theorem B which solves the Yamabe problem for the quaternionic Heisenberg group under an additional geometric condition.

*Theorem B. Let* $\eta = f\,\eta_0$ *be a conformal deformation of the standard contact 1-form* $\eta_0$ *on the sphere* $\mathbb{S}^{4n+3}$*. Suppose that* $\eta$ *has constant QC-scalar curvature. If the vertical distribution determined by* $\eta$ *is integrable, then up to a constant multiple* $\eta$ *is obtained from* $\eta_0$ *by a conformal QC-automorphism of the sphere. In the case* $n > 1$*, the same conclusion holds also for functions* $f$*, whose real parts are quaternionic anti-CRF functions.*

The main results of Chapter 3 are published in the paper [IMV16]. Here the author continuous the investigations on the geometry of QC-Einstein structures and their relations with the 3-Sasakian structures. Paragraph 10 is devoted to the 7-dimensional case. The main result is Theorem D which combined with Theorem 5.9 shows that the QC-scalar curvature of any QC-Einstein manifold is constant. An important consequence of this result is Theorem C that any QC-Einstein manifold with non-zero QC-scalar curvature is locally QC-homothetic to a 3-Sasakian manifold. The proof of Theorem D uses an QC-analog of the conformal Weyl tensor introduced by Ivanov and Vassilev[IV1] and a generalization of Theorem A for the poitwise QC-conformal case. In Paragraph 11 the author introduces the so-called *vertical connection* of a QC-manifold, which is the main tool for studying QC-Einstein manifolds. More precisely, Theorem 11.3 characterizes these manifolds as QC-manifolds with flat vertical connection. Using this connection the author obtains in Paragraph 12 the structure equations of QC-Einstein manifolds in terms of the contact 1-form and its exterior derivative (Theorem 12.1). The main result of the last paragraph of this chapter is Proposition 13.3 which says that in the general case the QC-scalar flat QC-Einstein manifolds are bundles over hyper-Kähler manifolds. It is proved also that any QC-Einstein maniflod with non-zero QC-scalar curvature has two different Einstein Riemannian metrics which generalizes a well-known result of Boyer, Galicki, and Mann [BGN] for 3-Sasakian manifolds.

The purpose of Chapter 4 is to find all solutions of the QC-Yamabe problem on the 7-dimensional sphere $\mathbb{S}^7$, i.e. all contact 1-forms of the canonical QC-structure with constant QC-scalar curvature. The results of this chapter are published in [IMV10].

Recall that the above problem is solved in Theorem B for any dimension under the additional assumption for integrability of the vertical distribution. In this chapter it is shown that in dimension 7 this condition can be removed. More precisely, the following theorem is true:

*Theorem E. Let $\eta$ be a conformal deformation of the standart contact 1-form $\eta_0$ on the unit sphere $S^7$. If $\eta$ has constant QC-scalar curvature, then up to a multiplicative constant, $\eta$ is obtained from $\eta_0$ by means of a conformal QC-authomorphism of the sphere. In particular, the QC-Yamabe constant $\lambda(S^7)$ is equal to $48\,(4\pi)^{1/5}$ and it is attained only for the images of $\eta_0$ by conformal QC-authomorphisms of the sphere.*

An important motivation for studying the QC-Yamabe problem on the sphere is its connection with the problem for determining the optimal constant and the extremals of $L^2$-Folland-Stein inequality. Using Theorem E this problem is solved completely in dimension 7.

*Theorem F. The optimal constant of $L^2$-Folland-Stein inequality for the 7-dimensional Heisenberg group $\mathbb{H}\times Im\mathbb{H}$ is $S_2 = \frac{2\sqrt{3}}{\pi^{3/5}}$. Every nonnegative extremal of this inequality is obtained by a translation and a dilation of the function*

$$v(q,\omega) \;=\; \frac{2^{11}\sqrt{3}}{\pi^{3/5}}[(1+|q|^2)^2 \;+\; |\omega|^2]^{-2},\;\; (q,\omega)\in\mathbb{H}\times Im\mathbb{H}.$$

The proof of Theorem E is based on Theorem 15.4 where a special divergence formula is obtained for a 7-dimensional QC-manifoldq whose structure is conformally equivalent to a 3-Sasakian structure. It is an analog of similar formulas in Riemannian and CR-geometry.

The results in the last chapter of the thesis are published in [IMV12]. Its purpose is to determine the optimal constant and the positive extremals of $L^2$-Folland-Stein inequality on quaternionic Heisenberg groups of arbitrary dimension. Note that the methods used in this chapter are analytic and are based on the properties of the conformal sub-Laplacian. In dimensions $\geq 11$ however, this approach allows to determine not all solutions of the QC-Yamabe equation, but only the minimizers of the QC-Yamabe functional.

*Theorem G. a) The optimal constant of $L^2$-Folland-Stein inequality on the $(4n+3)$-dimensional quaternionic Heisenberg group is*

$$S_2 = \frac{\left[2^{-2n}\ \omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

*where $\omega_{4n+3} = 2\pi^{2n+2}/(2n+1)!$ is the volume of the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4}$. The positive extremals of the inequality are the functions*

$$F(q,\omega) = \gamma\left[(1+|q|^2)^2 + |\omega|^2\right]^{-(n+1)}, \ (q,\omega) \in \mathbb{H}^n \times Im\mathbb{H}, \qquad \gamma = const, \tag{3}$$

*and that obtained from $F$ by translation and homothety.*

*b) The QC-Yamabe constant of the unit QC-sphere* is

$$\lambda(S^{4n+3}, H^{can}) = 16\,n(n+2)\ \left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}. \tag{4}$$

The proof of Theorem G uses techniques developed in [BFM] and [FL] for finding the optimal Moser-Trudinger inequality for the CR-sphere and the optimal logarithmique Hardy-Littlewood-Sobolev inequalities on Heisenberg groups.

In conclusion, I would like to point out that to obtain the results in the dissertation, the author has overcome a number of technical and conceptual difficulties and he practically uses the whole apparatus of differential geometry and geometric analysis.

**2.** The most important results in the thesis are included in 4 papers published in the reputable math journals Memoirs of AMS (IF-1.727), Journal of European Mathematical Society(IF-1.353), Math Research Letters(IF-0.716) and Annali della Scuola Normale Superiore di Pisa(IF-0.683). These papers are written jointly with S. Ivanov and D. Vassilev and my opinion is that the contribution of Ivan Minchev is equivalent to the one of the co-authors. This is confirmed also by his statement which was sent to me further. He has provided information for 13 citations of two of the above papers in journals with a high impact-factor.

**3.** I have the following technical remarks:

1. The papers [CDKR1], [D1], [Sal1], [Va2] in the Bibliography of the Abstract and [CDKR1] in the Bibliography of the thesis are written without the names of the authors.

2. The title of the thesis in Declaration of authorship is not written correctly.

**4.** The abstract correctly reflects the main results and the scientific contributions of the thesis.

**Conclusion.** The thesis of Ivan Minchev contains important theoretical generalizations and solutions of difficult problems of current interest in differential geometry and geometric analysis of quaternionic-contact manifolds which are significant contributions in these contemporary mathematical fields. This together with the provided scientometric indicators shows that the thesis satisfies all requirements of the Law Act for Development of the Academic Staff in the Republic of Bulgaria, the Statutes for application of LADASRB, and the Statutes for the conditions and regulations for acquiring academic degrees and occupying academic posts in Sofia University. So, I recommend with conviction to the honorable Jury to vote "Yes" for the award of Ivan Minchev Minchev the degree "Doctor of Science" , Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional field: 4.5 Mathematics.

05.05.2020 г. Signed:

(Oleg Mushkarov)